Math 351

Review problems for Exam #2 November 12, 2007

Exam #2 will be given during the normal class period on Monday, November 19. It will cover material from Sections 4.5, 4.6, 5.1 - 5.3, 6.3, 9.1, 9.4, 7.1 - 7.4. This set of review problems is about twice as long as the exam. As usual, \( \mathbb{Z} \) denotes the ring of integers, \( \mathbb{Q} \) denotes the field of rational numbers, and \( \mathbb{C} \) denotes the field of complex numbers.

#1 Let \( f(x) \in \mathbb{R}[x] \) have degree 7. Prove that \( f(x) \) is a reducible polynomial in \( \mathbb{R}[x] \). You may want to use the fact that every irreducible polynomial in \( \mathbb{C}[x] \) has degree 1.

#2 Let \( f(x) \) and \( g(x) \) be polynomials in \( \mathbb{Z}[x] \). Let \( p \) be a prime integer. Prove that if \( p \) divides every coefficient of \( f(x)g(x) \) then either \( p \) divides every coefficient of \( f(x) \) or \( p \) divides every coefficient of \( g(x) \).

#3 Let \( f(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x] \) and suppose that \( \frac{r}{s} \neq 0 \) is a root of \( f(x) \) where \( r, s \in \mathbb{Z} \) and \( r \) and \( s \) are relatively prime. Prove that \( r|a_0 \) and \( s|a_n \).

#4 Let \( f(x) \in \mathbb{Z}[x] \) and assume that \( f(x) \) is an irreducible polynomial in \( \mathbb{Z}[x] \). Prove that \( f(x) \) is an irreducible polynomial in \( \mathbb{Q}[x] \). You may want to use the results of problems #2 and #3.

#5 Show (by constructing an example) that there is a field with 8 elements.

#6 Let \( F \) be a field and \( f(x) \in F[x] \). Let \( p(x) \in F[x] \) be a polynomial of degree \( \geq 1 \). Prove that \( f(x) + (p(x)) \) is a unit in \( F[x]/(p(x)) \) if and only if \( f(x) \) and \( p(x) \) are relatively prime.

#7 (a) State the definition a prime ideal in a ring \( R \).

(b) Prove that an ideal \( I \) in a commutative ring with identity \( R \) is a prime ideal if and only if \( R/I \) is an integral domain.

(c) State the definition of a maximal ideal in a ring \( R \).

(d) Prove that if \( R \) is a commutative ring with identity, then an ideal \( I \) in \( R \) is maximal if and only if \( R/I \) is a field.

#8 Show that \( \mathbb{Z}[\sqrt{-2}] \) is a Euclidean domain with \( \delta(a + b\sqrt{-2}) = a^2 + 2b^2 \).

#9 Let \( R \) be an integral domain. Define \( S = \{(a, b)|a, b \in R, b \neq 0\} \). Define \( (a, b) \sim (c, d) \) if \( ad = bc \). Show that \( \sim \) is an equivalence relation.

#10 Let \( R = \{a + b\sqrt{3}|a, b \in \mathbb{Z}\} \). Then \( R \) is an integral domain (why?) and so \( R \) has a quotient field \( F \). What is \( F \)?

#11 Let \( G \) be a group, \( g, h, k \in G \) and \( gh = gk \). Prove that \( h = k \). Conclude that the multiplicative inverse of \( g \) is unique.
#12 Compute the product

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
4 & 6 & 1 & 2 & 3 & 7 & 5 \\
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 3 & 4 & 1 & 5 & 6 & 7 \\
\end{pmatrix}
\]

in the symmetric group on 7 elements.

#13 Let \( g = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7) (2 \ 5 \ 7 \ 6 \ 4 \ 1 \ 3) \) in the symmetric group on 7 elements.

(a) Find \( g^{-1} \).

(b) Find the order of \( g \).

#14 Let \( G \) be a group with identity element \( e \). Suppose \( g^2 = e \) for all \( g \in G \). Prove that \( G \) is commutative.

#15 Let \( G \) be a commutative group with identity element \( e \) and let \( n \in \mathbb{Z}, n \geq 1 \). Let \( H = \{g \in G | g^n = e\} \). Prove that \( H \) is a subgroup of \( G \).