#1 Suppose that $A$ is a 5 by 5 matrix and

$$B = A + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $\det(A) = 1$ and $\det(B) = 3$, what is $\det(2A + B)$. Why?

**Solution:** Let $a_i$ denote the $i$-th row of $A$ and $b_i$ denote the $i$-th row of $B$. Thus $b_1 = a_1, b_2 = a_2 + [1, -1, 2, 0, 1], b_3 = a_3, b_4 = a_4, b_5 = a_5$, and we may write

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, \quad B = \begin{bmatrix} a_1 \\ b_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Let $C = \left(\frac{2}{3}\right)A + \left(\frac{1}{3}\right)B$. Thus

$$C = \begin{bmatrix} \frac{2}{3}a_1 + \frac{1}{3}b_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Then

$$\det(C) = \left(\frac{2}{3}\right)\det(A) + \left(\frac{1}{3}\right)\det(B) = \left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)3 = \frac{5}{3}. $$

Now $2A + B = 3C = (3I)C$ and so

$$\det(2A + B) = \det(3I)\det(C) = 3^5 \left(\frac{5}{3}\right) = 3^4(5) = 405.$$
Let the 4 by 7 matrix $A$ have columns $a_1, \ldots, a_7$. Suppose the reduced row echelon form of $A$ is
\[
\begin{bmatrix}
1 & 2 & 0 & 0 & -1 & 0 & 3 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
Suppose further that $a_2 = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 6 \end{bmatrix}$, $a_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, and $a_5 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ -3 \end{bmatrix}$. Find $A$.

**Solution:** Let $R$ denote the reduced row echelon form of $A$ and let $r_i$ denote the $i$-th column of $R$. Then we know that if $b_1, \ldots, b_7 \in F$ we have $b_1 a_1 + \ldots + b_7 a_7 = 0$ if and only if $b_1 r_1 + \ldots + b_7 r_7 = 0$. Now $r_2 = 2 r_1, r_5 = -r_1 + 2 r_3 + r_4, r_6 = r_4$, and $r_7 = 3 r_1 + r_3 + 3 r_4$. Hence $a_2 = 2 a_1$ and so
\[
a_1 = \left( \frac{1}{2} \right) a_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}.
\]
Also $a_5 = -a_1 + 2 a_3 + a_4$ and so
\[
a_4 = a_1 - 2 a_3 + a_5 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} 12 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -1 \\ -4 \end{bmatrix}.
\]
Finally,
\[
a_6 = a_4 = \begin{bmatrix} -2 \\ -2 \\ -1 \\ -4 \end{bmatrix},
\]
and
\[
a_7 = 3 a_1 + a_3 + 3 a_4 = 3 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ -2 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ -11 \\ -2 \\ -1 \end{bmatrix}.
\]
Thus
\[
A = \begin{bmatrix}
1 & 2 & 1 & -2 & -1 & -2 & -2 \\
-2 & -4 & 1 & -2 & 2 & -2 & -11 \\
0 & 0 & 1 & -1 & 1 & -1 & -2 \\
3 & 6 & 2 & -4 & -3 & -4 & -1
\end{bmatrix}.
\]
3 A 9 by 9 diagonalizable matrix \( A \) has three eigenvalues: 1, 2 and 3. If

\[
\text{rank}(A - I) = 7
\]

and

\[
\text{rank}(A - 2I) = 5,
\]

what is the multiplicity of the eigenvalue 3? Why?

**Solution:** Since the matrix is diagonalizable, the sum of the dimensions of the eigenspaces must equal 9. Now the 1-eigenspace, \( E_1 \), is equal to \( N(A - I) \) and so its dimension is the nullity of \( A - I \) which is equal to \( 9 - \text{rank}(A - I) = 9 - 7 = 2 \). Similarly, the dimension of \( E_2 \) is \( 9 - \text{rank}(A - 2I) = 9 - 5 = 4 \). Then \( 2 + 4 + \dim(E_3) = 9 \) and so \( \dim(E_3) = 3 \). This is the (geometric) multiplicity of the eigenvalue 3.

#4 Let \( A \) be an \( m \) by \( n \) matrix. Write \( A = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \) where \( A_i \) denotes the \( i \)-th column of \( A \). Let \( A_k = \begin{bmatrix} a_1 & \ldots & a_k \end{bmatrix} \), i.e., the matrix consisting of the first \( k \) columns of \( A \). Set \( s_i(A) = \text{rank}(A_i) \) for \( 1 \leq i \leq n \), and let \( s(A) \) denote the \( n \)-tuple \([ s_1(A) , \ldots , s_n(A) ]\).

(a) Let \( P \) be an invertible \( m \) by \( m \) matrix. Prove that \( s(PA) = s(A) \).

(b) Let \( R \) be the reduced row echelon form of \( A \). Prove that \( s(R) = s(A) \).

(c) Say that a column of \( A \) is a basic column if the corresponding column of \( R \) contains the initial nonzero entry of some row. Show how to determine the basic columns from the \( n \)-tuple \( s(A) \).

(d) Show that the column \( a_i \) of \( A \) is a linear combination of the columns \( a_j \) such that \( j \leq i \) and \( a_j \) is basic.

(e) Explain why a matrix \( A \) has only one reduced row echelon form.

**Solution:**

(a) We know from the definition of matrix multiplication that the \( i \)-th column of \( PA \) is \( Pa_i \). Therefore \( (PA)_k = P(A_k) \) and so, \( s_k(PA) = \text{rank}((PA)_k) = \text{rank}(P(A_k)) = \text{rank}(A_k) = A_k \).

(b) Since \( R = PA \) for some invertible matrix \( P \), this follows from part (a).

(c) The \( k \)-th column of \( R \) is basic if and only if it is not contained in the span of the first \( k - 1 \) columns. This occurs if and only if either \( k = 1 \) and \( s_1(R) \neq 0 \) or if \( k > 1 \) and \( s_k(R) > s_{k-1}(R) \). In view of part (b), this means that the \( k \)-th column is basic if and only if either \( k = 1 \) and \( s_1(A) \neq 0 \) or if \( k > 1 \) and \( s_k(A) > s_{k-1}(A) \).

(d) We know that for scalars \( b_1, \ldots , b_n \) we have \( b_1a_1 + \ldots + b_na_n = 0 \) if and only if \( b_1r_1 + \ldots + b_nr_n \). Since \( r_i \) is a linear combination of the columns \( r_j \) such that \( j \leq i \) and \( r_j \) is basic, the same result holds for the \( a_i \).

(e) Suppose \( A \) has reduced row echelon forms

\[
R = \begin{bmatrix} r_1 & r_2 & \ldots & r_n \end{bmatrix}
\]
and
\[ T = [t_1 \ t_2 \ \ldots \ t_n]. \]

Then by (c) the basic columns of \( R \) are the same as the basic columns of \( T \). Furthermore, any column of \( A \) is a linear combination of basic columns of \( A \). Therefore the corresponding column of \( R \) is the same linear combination of the basic columns of \( R \) and the corresponding column of \( T \) is the same linear combination of the basic columns of \( T \). Thus every column of \( R \) is equal to the corresponding column of \( T \) and so the two matrices are equal.

#5 Let
\[
A = \begin{bmatrix}
1 & 3 & -1 & -1 & -1 \\
1 & 2 & 0 & 1 & -1 \\
2 & 5 & -1 & 0 & -2 \\
2 & 3 & 1 & 4 & -1
\end{bmatrix}.
\]

(a) Find the reduced row echelon form for \( A \)
(b) Find a basis for the null space \( N(L_A) \)
(c) Find a basis for the row space of \( A \)
(d) Find a basis for the column space of \( A \).

Solution:

(a) \( R = \begin{bmatrix}
1 & 0 & 2 & 5 & 0 \\
0 & 1 & -1 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \) is the reduced row echelon form.

(b) The free variables are \( x_3 \) and \( x_4 \). Suppose \( R \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = 0 \). Then
\[
\begin{bmatrix}
x_1 + 2x_3 + 5x_4 \\
x_2 - x_3 - 2x_4 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

and so
\[
x_1 = -2x_3 - 5x_4 \\
x_2 = x_3 + 2x_4 \\
x_3 = x_3
\]
\[ x_4 = x_4 \]
\[ x_5 = 0. \]

Then
\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 
\end{bmatrix} = \begin{bmatrix}
  x_3 + 2x_4 \\
  x_3 \\
  x_4 \\
  0 
\end{bmatrix} = x_3 \begin{bmatrix}
  -2 \\
  1 \\
  0 \\
  0 
\end{bmatrix} + x_4 \begin{bmatrix}
  2 \\
  1 \\
  0 \\
  1 
\end{bmatrix}.
\]

Thus
\[
\begin{bmatrix}
  -2 \\
  1 \\
  0 \\
  0 
\end{bmatrix}, \begin{bmatrix}
  -5 \\
  2 \\
  0 \\
  0 
\end{bmatrix}
\]
is a basis for \( N(L_A) \).

(c) The set of nonzero rows of the reduced row echelon form of \( A \) is (one) basis for the row space of \( A \). Thus \( \{[1 \ 0 \ 2 \ 5 \ 0], [0 \ 1 \ -1 \ -2 \ 0], [0 \ 0 \ 0 \ 0 \ 1] \} \) is a basis for the row space of \( A \).

(d) The set of basic columns of \( A \) (that is, those columns corresponding to the columns of \( R \) containing the initial nonzero element of some row) is one basis for the column space of \( A \). Thus \( \{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \\ -1 \end{bmatrix} \} \) is a basis for the column space of \( A \).

\#6 Let \( A = \begin{bmatrix}
  -3 & 0 & -5 \\
  0 & 2 & 0 \\
  1 & 0 & 3 
\end{bmatrix} \).

(a) Find all eigenvalues for \( A \) and find a basis for each eigenspace.

(b) Find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( P^{-1}AP = D \).

**Solution:**

(a) \[ \det \begin{bmatrix}
  -3 - \lambda & 0 & -5 \\
  0 & 2 - \lambda & 0 \\
  1 & 0 & 3 - \lambda 
\end{bmatrix} = \]

\[ (2 - \lambda)(\lambda^2 - 9 + 5) = (2 - \lambda)((\lambda^2 - 4) = -(\lambda - 2)^2(\lambda + 2). \]
Thus the eigenvalues are 2 and $-2$. Now $E_2 = N(A - 2I) = N(\begin{bmatrix} -5 & 0 & -5 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix})$.

Thus \{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \} \) is a basis for $E_2$. Also $E_{-2} = N(A - (-2)I) = N(A + 2I) = N(\begin{bmatrix} -1 & 0 & -5 \\ 0 & 4 & 0 \\ 1 & 0 & 5 \end{bmatrix})$. Thus \{ \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \} \) is a basis for $E_{-2}$.

(b) $P = \begin{bmatrix} -1 & 0 & -5 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is one choice for $O$ and $D$.

(a) Compute $\det A$ if

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 1 & 4 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & -1 & -4 \end{bmatrix}$$

(b) Compute $\det B$ if

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 \\ 0 & 3 & 7 & 3 & 0 \\ 0 & 0 & 4 & 13 & 4 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix}$$

(c) Let $a_1, \ldots, a_n \in F$. Compute

$$\det \begin{bmatrix} a_1^{(n-1)} & a_2^{(n-1)} & \ldots & a_n^{(n-1)} \\ a_1^{(n-2)} & a_2^{(n-2)} & \ldots & a_n^{(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_n \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$
(d) Let $a_0, \ldots, a_{n-1} \in F$. Find the characteristic polynomial of

$$
\begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & a_0 \\
1 & 0 & 0 & \ldots & 0 & a_1 \\
0 & 1 & 0 & \ldots & 0 & a_2 \\
0 & 0 & 1 & \ldots & 0 & a_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & a_{n-1}
\end{bmatrix}.
$$

Solution:

(a) $\det A = \det \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & 2 & 6 \\ 0 & -1 & 2 & 3 \\ 0 & 2 & 0 & -2 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & 2 & 3 \\ 0 & 2 & 2 & 6 \\ 0 & 2 & 0 & -2 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & -2 \end{bmatrix} = -24.$

(b) $\det B = \det \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 4 & 13 & 4 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & -15 \end{bmatrix} = -15.$

(c) Subtract $a_1$ times the second row from the first row. Then subtract $a_1$ times the third row from the second row. Continue in this way, finally subtracting $a_1$ times the $n$-th row from the $n - 1$st row to get

$$
\begin{bmatrix}
\begin{bmatrix} a_1^{(n-1)} & a_2^{(n-1)} & \ldots & a_n^{(n-1)} \\ a_1^{(n-2)} & a_2^{(n-2)} & \ldots & a_n^{(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_n \\
1 & 1 & \ldots & 1
\end{bmatrix}
\end{bmatrix} =
$$
Expanding along the first column shows that this is

\[
\begin{vmatrix}
0 & (a_2 - a_1)a_2^{(n-2)} & \cdots & (a_n - a_1)a_n^{(n-2)} \\
0 & (a_2 - a_1)a_2^{(n-3)} & \cdots & (a_n - a_1)a_n^{(n-3)} \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_2 - a_1 & \cdots & a_n - a_1 \\
1 & 1 & \cdots & 1
\end{vmatrix}.
\]

Factoring out the common factors from each column gives

\[
(-1)^{n+1}(a_2 - a_1)(a_3 - a_1)\cdots(a_n - a_1)det
\begin{vmatrix}
(a_2 - a_1)a_2^{(n-2)} & \cdots & (a_n - a_1)a_n^{(n-2)} \\
(a_2 - a_1)a_2^{(n-3)} & \cdots & (a_n - a_1)a_n^{(n-3)} \\
\vdots & \vdots & \ddots & \vdots \\
a_2 - a_1 & \cdots & a_n - a_1 \\
1 & 1 & \cdots & 1
\end{vmatrix} =
\begin{vmatrix}
(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n)det
\end{vmatrix}.
\]

Continuing in this way gives

\[
\begin{vmatrix}
(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n)det
\end{vmatrix} = (a_1 - a_2)\cdots(a_1 - a_n)(a_2 - a_3)\cdots(a_2 - a_n)\cdots(a_{n-1} - a_n).
\]
(d) Expanding along the first row gives

\[
\begin{vmatrix}
    -\lambda & 0 & 0 & \ldots & 0 & a_0 \\
    1 & -\lambda & 0 & \ldots & 0 & a_1 \\
    0 & 1 & -\lambda & \ldots & 0 & a_2 \\
    0 & 0 & 1 & \ldots & 0 & a_3 \\
    \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & 0 & \ldots & 1 & a_{n-1} - \lambda
\end{vmatrix}
\]

\[
\begin{vmatrix}
    1 & -\lambda & 0 & \ldots & 0 \\
    0 & 1 & -\lambda & \ldots & 0 \\
    0 & 0 & 1 & \ldots & 0 \\
    \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & 0 & \ldots & 1
\end{vmatrix}
\]

\[
(-\lambda \text{det} \begin{vmatrix}
    -\lambda & 0 & 0 & \ldots & 0 & a_1 \\
    1 & -\lambda & 0 & \ldots & 0 & a_2 \\
    0 & 1 & -\lambda & \ldots & 0 & a_3 \\
    0 & 0 & 1 & \ldots & 0 & a_4 \\
    \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & 0 & \ldots & 1 & a_{n-1} - \lambda
\end{vmatrix}
+ (-1)^{1+n} \text{det} \begin{vmatrix}
    1 & -\lambda & 0 & \ldots & 0 \\
    0 & 1 & -\lambda & \ldots & 0 \\
    0 & 0 & 1 & \ldots & 0 \\
    \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & 0 & \ldots & 1
\end{vmatrix}
\]

Since the matrix in the second summand is upper triangular with diagonal entries 1, its determinant is 1. Thus the characteristic polynomial of the given matrix is

\[
(-\lambda \text{det} \begin{vmatrix}
    -\lambda & 0 & 0 & \ldots & 0 & a_1 \\
    1 & -\lambda & 0 & \ldots & 0 & a_2 \\
    0 & 1 & -\lambda & \ldots & 0 & a_3 \\
    0 & 0 & 1 & \ldots & 0 & a_4 \\
    \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & 0 & \ldots & 1 & a_{n-1} - \lambda
\end{vmatrix}
+ (-1)^{1-n} a_0.
\]

Continuing in this way shows that the characteristic polynomial is

\[
(-1)^n (\lambda^n - a_{n-1}\lambda^{n-1} - \ldots - a_1\lambda - a_0).
\]

#8 Let \( A \) be an \( m \) by \( n \) matrix over \( \mathbb{R} \) and let \( R \) be the reduced row echelon form of \( A \). Suppose that the columns of \( A \) are \( a_1, \ldots, a_n \) and that the columns of \( R \) are \( r_1, \ldots, r_n \). Let \( k_1, \ldots, k_n \in \mathbb{R} \). Prove that

\[
k_1 a_1 + \ldots + k_n a_n = 0
\]

if and only if

\[
k_1 r_1 + \ldots + k_n r_n = 0.
\]
Solution: Write \( k = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \). Then \( k_1a_1 + \ldots + k_na_n = Ak \) and \( k_1r_1 + \ldots + k_nr_n = Rk \).

But \( R = PA \) for some invertible \( n \times m \) matrix \( P \). Now if \( Ak = 0 \) then \( Rk = (PA)k = P(Ak) = 0 \) and if \( Rk = 0 \) then \( Ak = (P^{-1}R)k = P^{-1}(Rk) = 0 \).

#9 Let \( T \) be the linear operator on \( P_3(\mathbb{R}) \) defined by

\[
T(f) = 3f - xf' + f''.
\]

(Here \( f = f(x) \in P_3(\mathbb{R}) \), \( f' \) denotes the derivative of \( f \), and \( f'' \) denotes the second derivative of \( f \).) Let \( W \) be the \( T \)-cyclic subspace of \( P_3(\mathbb{R}) \) generated by \( x^3 \).

(a) Find a basis for \( W \).
(b) Find the characteristic polynomial of \( T_W \), the restriction of \( T \) to \( W \).

Solution:
(a) \( T(x^3) = 3x^3 - x(3x^2) + 6x = 6x \) and so \( T^2(x^3) = T(6x) = 18x - x(6) + 0 = 12x \).

Thus \( T^2(x^3) \in \text{span}\{x^3, T(x^3)\} \). Since \( \{x^3, T(x^3)\} = \{x^3, 6x\} \) is linearly independent it is a basis for \( W \).
(b) \( T^2(x^3) = 2T(x^3) \) and therefore \( t^2 - 2t \) is the characteristic polynomial of \( T_W \).

#10 State the definitions of the following terms.
(a) An eigenvalue (respectively eigenvector, eigenspace) of a linear transformation from \( V \) to \( V \).
(b) An eigenvalue (respectively eigenvector, eigenspace) of an \( n \times n \) matrix \( A \).
(c) The direct sum of subspaces \( V_1, \ldots, V_k \) of a vector space \( V \).
(d) The determinant of an \( n \times n \) matrix \( A \).
(e) The characteristic polynomial of an \( n \times n \) matrix \( A \).
(f) Similar

Solution:
(a) A scalar \( \alpha \in F \) such that \( T(v) = \alpha v \) for some nonzero \( v \in V \) is called an eigenvalue for \( T \) and such a \( v \) is called an eigenvector belonging to \( \alpha \). The \( \alpha \)-eigenspace, denoted \( E_\alpha \), is \( \{v \in V | T(v) = \alpha v\} \).
(b) A scalar \( \alpha \in F \) such that \( Av = \alpha v \) for some nonzero column vector \( v \in F^n \) is called an eigenvalue for \( A \) and such a \( v \) is called an eigenvector belonging to \( \alpha \). The \( \alpha \)-eigenspace, denoted \( E_\alpha \), is \( \{v \in F^n | Av = \alpha v\} \).
(c) The sum, \( V_1 + ... + V_k \) of the subspaces \( V_1, ..., V_k \) is
\[
\{ v_1 + ... + v_k | v_1 \in V_1, ..., v_k \in V_k \}.
\]
The sum \( V_1 + ... + V_k \) is said to be a direct sum (and written \( V_1 \oplus ... \oplus V_k \)) if \( V_i \cap (V_1 + ... + v_{i-1} + V_{i+1} + ... + V_k) = \{0\} \) for all \( i, 1 \leq i \leq k \).

(d) The determinant of the 1 by 1 matrix \([a]\) is \( a \). Assume that determinants of \( n - 1 \) by \( n - 1 \) matrices have been defined and that \( A = [a_{ij}] \) is an \( n \) by \( n \) matrix. Then
\[
\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \tilde{A}_{1j}
\]
where \( \tilde{A}_{1j} \) is the matrix obtained from \( A \) by deleting the first row and the \( j \)-th column.

(e) The characteristic polynomial of the \( n \) by \( n \) matrix \( A \) is \( \det(A - \lambda I) \) where \( I \) denotes the \( n \) by \( n \) identity matrix.

(f) Two \( n \) by \( n \) matrices \( A \) and \( B \) are similar if there is an invertible \( n \) by \( n \) matrix \( P \) such that \( B = PAP^{-1} \).

#11 Prove that similar matrices have the same characteristic polynomials and (hence) the same eigenvalues. Give an example to show that they do not necessarily have the same eigenvectors.

**Solution:**

Suppose \( B = PAP^{-1} \) where \( P \) is invertible. Then
\[
\det(B - \lambda I) = \det(PAP^{-1} - \lambda I) = \det(P(A - \lambda I)P^{-1}) = \det(P)\det(A - \lambda I)\det(P^{-1}) =
\]
\[
\det(P)\det(A - \lambda I)\det(P)^{-1} = \det(P)\det(P)^{-1}\det(A - \lambda I) = \det(A - \lambda I).
\]

Let \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \). Then \( A \) and \( B \) are similar since \( B = PAP^{-1} \) where \( P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). However, the 0-eigenspace of \( A \) is \( N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = F\begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and the 0-eigenspace of \( B \) is \( N\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = F\begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

#12 Let \( A \) be an \( m \) by \( n \) matrix and \( B \) be an \( n \) by \( p \) matrix.

(a) Is the row space of \( AB \) contained in the row space of \( A \)? Why or why not?
(b) Is the row space of \( AB \) contained in the row space of \( B \)? Why or why not?
(c) Is the column space of \( AB \) contained in the column space of \( A \)? Why or why not?
(d) Is the column space of \( AB \) contained in the column space of \( B \)? Why or why not?
(e) Prove that \( \text{rank}(AB) \leq \text{rank}(A) \) and \( \text{rank}(AB) \leq \text{rank}(B) \).

**Solution:**

(a) No. In fact, the row space of \( A \) consists of (row) vectors in \( F^n \) and the row space of \( AB \) consists of vectors in \( F^p \), so if \( n \neq p \) an inclusion is impossible. Even if \( n = p \) the inclusion does not hold. For example, if \( A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \) then the row space of \( AB \) is \( F \begin{bmatrix} 0 & 1 \end{bmatrix} \) while the row space of \( A \) is \( F \begin{bmatrix} 1 & 0 \end{bmatrix} \).

(b) Yes. Let \( E_{ij} \) denote the matrix with entry 1 in the \((i,j)\) position and 0 in every other position. Then the \(i\)th row of \( E_{ij}B \) is equal to the \(j\)th row of \( B \) and all other rows of \( E_{ij}B \) are 0. Thus the row space of \( E_{ij}B \) is contained in the row space of \( B \). Since \( A \) is a linear combination of the \( E_{ij} \) it follows that the row space of \( AB \) is contained in the row space of \( B \).

(c) The column space of \( AB \) is the row space of \( (AB)^t = B^tA^t \). Now the row space of \( B^tA^t \) is contained in the row space of \( A^t \) which is the column space of \( A \). Thus the column space of \( AB \) is contained in the column space of \( A \).

(d) The example of (a) shows that the answer is no.

(e) We know that the rank of \( A \) is equal to the dimension of the row space. Thus (b) gives \( \text{rank}(AB) \leq \text{rank}(B) \). We also know that the rank of \( A \) is equal to the dimension of the column space. Thus (c) gives \( \text{rank}(AB) \leq \text{rank}(A) \).

#13 Suppose \( A \) is a 5 by 7 matrix and \( B \) is a 7 by 5 matrix. Suppose further that \( \text{det}(AB) = 3 \). What is \( \text{det}(BA) \)? Why?

**Solution:** We have \( \text{rank} A \leq 5 \) (since \( A \) has only 5 rows). Thus by (e) of the previous problem, \( \text{rank}(BA) \leq 5 \). But \( BA \) is a 7 by 7 matrix. Hence \( BA \) is not invertible and so its determinant is equal to 0.

#14 Let

\[
A = \begin{bmatrix}
1 & 1 & -1 \\
0 & 2 & 1 \\
0 & 0 & 3
\end{bmatrix}.
\]

(a) Find all eigenvalues for \( A \) and for each eigenvalue find a basis for the corresponding eigenspace.

(b) Find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \). (This is equivalent to \( P^{-1}AP = D \).)

(c) Using your answer to (b), find the general solution of the following system of linear differential equations:

\[
y_1' = y_1 + y_2 - y_3
\]
\[ y'_2 = 2y_2 + y_3 \]
\[ y'_3 = 3y_3 \]

**Solution:**  
(a) The eigenvalues are 1, 2, 3. The 1-eigenspace has basis \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \); the 2-eigenspace has basis \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \); the 3-eigenspace has basis \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

(b) We may take \( P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \) and \( D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \).

(c) Let \( y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \) be the general solution to the system and let \( x = P^{-1}y \). Then \( Ay = y' \) and \( Dx = P^{-1}APx = P^{-1}APP^{-1}y = P^{-1}y' = (P^{-1}y)' = x' \). Thus

\[
x = \begin{pmatrix} C_1e^t \\ C_2e^{2t} \\ C_3e^{3t} \end{pmatrix}
\]

and

\[
y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1e^t \\ C_2e^{2t} \\ C_3e^{3t} \end{pmatrix}.
\]

#15 A 3 by 3 matrix \( A \) has eigenvalues 1, 2, and 3. What are the eigenvalues of the matrix \( B = A^2 - I \)? Why?

**Solution:** Suppose \( v \) is an eigenvector for the matrix \( A \) corresponding to the eigenvalue \( i \). Then

\[
A^2v = A(Av) = A(iv) = i(Av) = i(iv) = i^2v
\]

and

\[(A^2 - I)v = a^2v - v = i^2v - v = (i^2 - 1)v.\]

Thus the eigenvalues of \( A^2 - I \) are \( 1^1 - 1 = 0, 2^2 - 1 = 3, \) and \( 3^2 - 1 = 8.\)

#16 In each part state whether or not the given matrix is diagonalizable and give your reason.
(a) \( R = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \)

(b) \( P = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \)

(c) \( Q = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \)

**Solution:**

(a) The characteristic polynomial is \((1 - \lambda)(2 - \lambda)(4 - \lambda)\). Since there are three distinct roots (and hence 3 eigenvalues), the matrix is diagonalizable.

(b) The characteristic polynomial is \((2 - \lambda)^2(3 - \lambda)\) and \(E_2 = N(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix})\). Thus \(E_2\) has dimension 1, so the geometric multiplicity of the eigenvalue 2 is not equal to its algebraic multiplicity. Hence \(P\) is not diagonalizable.

(c) The characteristic polynomial is \((2 - \lambda)^2(3 - \lambda)\) and \(E_2 = N(\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})\). Thus \(E_2\) has dimension 2, so the geometric multiplicity of the eigenvalue 2 is equal to its algebraic multiplicity. Hence \(Q\) is diagonalizable.