Math 300 - Review problems for Exam #1 - February 12, 2009

#1 Suppose A and B are true while P and Q are false. State whether or not each of the following is true and justify your answer.
(a) $(A \wedge P) \Rightarrow (P \wedge Q)$;
(b) $(A \vee \sim Q \vee \sim B) \Rightarrow (P \vee \sim Q)$.

Solution: (a) This is true, since $A \wedge P$ is false.
(b) This is true, since $P \vee \sim Q$ is true.

#2: Make truth tables for each of the following propositional forms:
(a) $(P \vee Q) \wedge (\sim P \vee \sim Q)$;
(b) $((P \wedge Q) \vee (P \wedge \sim R)) \vee (P \wedge R)$.

Solution:
(a)

$$\begin{bmatrix} P & Q & \sim P & \sim Q & P \vee Q & \sim P \vee \sim Q & (P \vee Q) \wedge (\sim P \vee \sim Q) \\ T & T & F & F & T & F & F \\ T & F & F & T & T & T & T \\ F & T & T & F & T & T & T \\ F & F & T & T & F & T & F \end{bmatrix}.$$

(b)

$$\begin{bmatrix} P & Q & R & \sim R & P \wedge Q & P \wedge \sim R & P \wedge R & ((P \wedge Q) \vee (P \wedge \sim R)) \vee (P \wedge R) \\ T & T & T & F & T & F & T & T \\ T & T & F & T & T & T & F & T \\ T & F & T & F & F & F & T & T \\ T & F & F & T & F & T & F & T \\ F & T & T & F & F & F & F & F \\ F & T & F & T & F & F & F & F \\ F & F & T & F & F & F & F & F \\ F & F & F & T & F & F & F & F \end{bmatrix}.$$

#3 Prove that $P \Leftrightarrow Q$ is equivalent to $(P \wedge Q) \vee (\sim P \wedge \sim Q)$.

Solution: We will compare the truth tables.

$$\begin{bmatrix} P & Q & P \Leftrightarrow Q \\ T & T & T \\ T & F & F \\ F & T & F \\ F & F & T \end{bmatrix}.$$

$$\begin{bmatrix} P & Q & \sim P & \sim Q & P \wedge Q & \sim P \wedge \sim Q & (P \wedge Q) \vee (\sim P \wedge \sim Q) \\ T & T & F & F & T & F & T \\ T & F & F & T & F & F & F \\ F & T & T & F & F & F & F \\ F & F & T & T & F & T & T \end{bmatrix}.$$

Since the last columns are identical in the two tables, the statements are equivalent.

#4 Is each of the following a tautology, a contadiction, or neither?

(a) $(P \vee \sim Q) \Rightarrow Q$

(b) $(P \wedge Q) \vee \sim (P \vee Q) \vee (P \Rightarrow Q) \vee (Q \Rightarrow P)$.

Solution: (a) is true if P and Q are both true, but is false if P is true and Q is false. Thus it is neither a tautology or a contradiction.

(b) Since $P \Rightarrow Q$ is true whenever Q is true and $Q \Rightarrow P$ is true whenever Q is false, $(P \Rightarrow Q) \vee (Q \Rightarrow P)$ is a tautology and so (b) is a tautology.

#5 Which of the following statements are true (where the universe is the set of all real numbers)? Why?

(a) $(\forall x)(\exists y)((x^2+1)y = 1)$;

(b) $(\exists x)(\forall y)((x^2+1)y = 1)$;

(c) $(\forall x)(\exists y)((x+1)y = 1)$;

(d) $(\exists x)(\forall y)((x+1)y = 1)$;

(e) $(\exists N)((N \text{ } is \text{ } an \text{ } integer) \wedge (N > 0) \wedge (\frac{1}{N}) < .001))$;

(f) $(\exists N)(\forall M)((N \text{ } is \text{ } an \text{ } integer) \wedge ((M > N) \Rightarrow (\frac{1}{M}) < .001))$;

(g) $(\exists M)(\forall N)((N \text{ } is \text{ } an \text{ } integer) \wedge ((M > N) \Rightarrow (\frac{1}{M}) < .001))$;

Solution:

(a) Taking $y = \frac{1}{x^2+1}$ shows that this is true.

(b) This is false, for unless $y = \frac{1}{x^2+1}$ the equality does not hold.

(c) This is false. If $x = -1$ there is no such y.

(d) This is false, for unless $y = \frac{1}{x+1}$ the equality does not hold.

(e) This is true, for $\frac{1}{N} < .001$ is equivalent to $1000 < N$ (as we see by multiplying by $1000N$). Thus, for example, we may take $N = 1001$.

(f) This is true, for $\frac{1}{M} < .001$ is equivalent to $1000 < M$ (as we see by multiplying by $1000M$). Thus, for example, we may take $N = 1000$.

(g) This is false. For example, for given any M there is some integer N greater than M. #6 Prove each of the following:

(a) If n is an integer, the 24 divides $x(x+1)(x+2)(x+3)$.

(b) For every natural number N and every nonzero real number r there is a natural number M such that for all natural numbers $m > M$

$$\frac{1}{m} < \frac{r}{N}.$$

Solution:

(a) Since $x, x+1, x+2, x+3$ are four consecutive integers, two of them must be divisible by 2, at least one must be divisible by 3 and one must be divisible by 4. Thus 24 divides the product.

(b) Since $\frac{1}{m} < \frac{r}{N}$ is equivalent to $\frac{N}{r} < m$ (as we see my multiplying by $\frac{mN}{r}$ we see that we may find such an m.

$7 (a) Give a direct proof that if x is an even integer and y is an odd integer, then xy is an even integer.

(b) Give a proof by contradiction to show that if a and b are integers and ab is odd, then a and b are both odd.

Solution:

(a) Let x and y be integers. If x is even, then $x = 2k$ for some integer k. Then $xy = (2k)y = 2(ky)$. Since ky is an integer, $2(ky) = xy$ is even.

(b) Let a and b be integers. Assume tht ab is odd and that a and b are not both odd. Then one of a and b is even, so by part (a) ab is even. This is a contradiction, proving the assertion.

#8 Let $A = \{1, 2, 3, 4, 5\}, B = \{2, 4, 6, 8\}, C = (1, 5)$, and $D =$ the set of natrual numbers. Find:

(a) $A \cap B$;
(b) $A \cup B$;
(c) $A \cap \tilde{C}$;
(d) $C \cap D$;
(e) the power set of $B \cap C$.
(f) the power set of $\emptyset$.

Solution:

(a) $A \cap B = \{2, 4\}$;
(b) $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$;
(c) $A \cap \tilde{C} = \{1, 5\}$;
(d) $C \cap D = \{2, 3, 4\}$;
(e) $B \cap C = \{2, 4\}$ so $\mathcal{P}(B \cap C) = \{\emptyset, \{2\}, \{4\}, \{2, 4\}\}$;
(f) $\{\emptyset\}$.

#9 Let A, B, C be sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, either $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$ and if $x \in C$, then $x \in A \cap C$. Thus $x \in (A \cap B) \cup (A \cap C)$.

Now let $x \in (A \cap B) \cup A \cap C$. Then either $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $x \in B$, we have $x \in B \cap C$ and so $x \in A \cup (B \cap C)$. If $x \in A \cap C$, then $x \in A$ and $x \in C$. Since $x \in C$, we have $x \in B \cap C$ and so $x \in A \cup (B \cap C)$.

#10 Give an example of a nested family of sets $\{A_1, A_2, ..., \}$ such that

(a) $\cap_{i=1}^{\infty} A_i = (2, 3]$;

(b) $\cap_{i=1}^{\infty} A_i = [2, \infty)$.

Solution:

(a) For example, $A_i = (2, 3 + \frac{1}{i})$.

(b) For example, $A_i = (2 - \frac{1}{i}, \infty)$.

#11 Prove that $\sqrt{5}$ is irrational.

Solution: First note that if n is an integer and 5 divides n^2, then 5 divides n. To see this, note that we may write $n = 5q + r$ for integers q and r with $0 \leq r < 5$. Thus $r = 0, 1, 2, 3$, or 4. Then $n^2 = (5q + r)^2 = 25q^2 + 10rq + r^2 = 5(5q^2 + 2q) + r^2$. Thus 5 divides n^2 if and only if 5 divides r^2. But $1^2 = 1, 2^2 = 4, 3^2 = 9$, and $4^2 = 16$ are not divisible by 5. Thus $r = 0$ so $n = 5q$ is divisible by 5.

We now prove that $\sqrt{5}$ is irrational by contradiction. Assume it is rational. Then $\sqrt{5} = \frac{a}{b}$ for integers a, b with $b \neq 0$ and such that both a and b are not divisible by 5. Then, multiplying both sides by b we have $\sqrt{5}b = a$, and squaring both sides, we have $5b^2 = a^2$. Thus 5 divides a^2 and so, by our preliminary result, 5 divides a. Thus $a = 5k$ for some integer k and so $5b^2 = (5k)^2 = 25k^2$. Dividing both sides by 5 gives $b^2 = 5k^2$ and so 5 divides b^2. Again using our preliminary result, we see that 5 divides b. Thus we have that 5 divides both a and b, contradicting our assumption.