Math 300 - Review problems for Exam #1 - February 12, 2009

#1 Suppose $A$ and $B$ are true while $P$ and $Q$ are false. State whether or not each of the following is true and justify your answer.

(a) $(A \land P) \Rightarrow (P \land Q)$;
(b) $(A \lor \sim Q \lor \sim B) \Rightarrow (P \lor \sim Q)$.

**Solution:** (a) This is true, since $A \land P$ is false.
(b) This is true, since $P \lor \sim Q$ is true.

#2: Make truth tables for each of the following propositional forms:

(a) $(P \lor Q) \land (\sim P \lor \sim Q)$;
(b) $((P \land Q) \lor (P \land R)) \lor (P \land R)$.

**Solution:**

(a) 

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#3 Prove that $P \iff Q$ is equivalent to $(P \land Q) \lor (\sim P \land \sim Q)$.

**Solution:** We will compare the truth tables.

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<th>$P \iff Q$</th>
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  T & F & F & T & F & F & F \\
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  F & F & T & T & F & T & T
\end{bmatrix}.$$  

Since the last columns are identical in the two tables, the statements are equivalent.

#4 Is each of the following a tautology, a contradiction, or neither?

(a) \((P \lor \sim Q) \Rightarrow Q\)

(b) \((P \land Q) \lor (P \lor Q) \lor (P \Rightarrow Q) \lor (Q \Rightarrow P)\).

**Solution:** (a) is true if \(P\) and \(Q\) are both true, but is false if \(P\) is true and \(Q\) is false. Thus it is neither a tautology or a contradiction.

(b) Since \(P \Rightarrow Q\) is true whenever \(Q\) is true and \(Q \Rightarrow P\) is true whenever \(Q\) is false, \((P \Rightarrow Q) \lor (Q \Rightarrow P)\) is a tautology and so (b) is a tautology.

#5 Which of the following statements are true (where the universe is the set of all real numbers)? Why?

(a) \((\forall x)(\exists y)((x^2 + 1)y = 1)\);

(b) \((\exists x)(\forall y)((x^2 + 1)y = 1)\);

(c) \((\forall x)(\exists y)((x + 1)y = 1)\);

(d) \((\exists x)(\forall y)((x + 1)y = 1)\);

(e) \((\exists N)((N \text{ is an integer}) \land (N > 0) \land (\frac{1}{N}) < .001))\);

(f) \((\exists N)(\forall M)((N \text{ is an integer}) \land ((M > N) \Rightarrow (\frac{1}{M}) < .001))\);

(g) \((\exists M)(\forall N)((N \text{ is an integer}) \land ((M > N) \Rightarrow (\frac{1}{M}) < .001))\);

**Solution:**

(a) Taking \(y = \frac{1}{x^2 + 1}\) shows that this is true.

(b) This is false, for unless \(y = \frac{1}{x^2 + 1}\) the equality does not hold.

(c) This is false. If \(x = -1\) there is no such \(y\).

(d) This is false, for unless \(y = \frac{1}{x + 1}\) the equality does not hold.

(e) This is true, for \(\frac{1}{N} < .001\) is equivalent to \(1000 < N\) (as we see by multiplying by 1000\(N\)). Thus, for example, we may take \(N = 1001\).

(f) This is true, for \(\frac{1}{M} < .001\) is equivalent to \(1000 < M\) (as we see by multiplying by 1000\(M\)). Thus, for example, we may take \(N = 1000\).

(g) This is false. For example, for given any \(M\) there is some integer \(N\) greater than \(M\).

#6 Prove each of the following:

(a) If \(n\) is an integer, the 24 divides \(x(x + 1)(x + 2)(x + 3)\).
(b) For every natural number $N$ and every nonzero real number $r$ there is a natural number $M$ such that for all natural numbers $m > M$

$$\frac{1}{m} < \frac{r}{N}.$$ 

**Solution:**

(a) Since $x, x+1, x+2, x+3$ are four consecutive integers, two of them must be divisible by 2, at least one must be divisible by 3 and one must be divisible by 4. Thus 24 divides the product.

(b) Since $\frac{1}{m} < \frac{r}{N}$ is equivalent to $\frac{N}{r} < m$ (as we see by multiplying by $\frac{mN}{r}$ we see that we may find such an $m$.

$7$ (a) Give a direct proof that if $x$ is an even integer and $y$ is an odd integer, then $xy$ is an even integer.

(b) Give a proof by contradiction to show that if $a$ and $b$ are integers and $ab$ is odd, then $a$ and $b$ are both odd.

**Solution:**

(a) Let $x$ and $y$ be integers. If $x$ is even, then $x = 2k$ for some integer $k$. Then $xy = (2k)y = 2(ky)$. Since $ky$ is an integer, $2(ky) = xy$ is even.

(b) Let $a$ and $b$ be integers. Assume that $ab$ is odd and that $a$ and $b$ are not both odd. Then one of $a$ and $b$ is even, so by part (a) $ab$ is even. This is a contradiction, proving the assertion.

#8 Let $A = \{1, 2, 3, 4, 5\}, B = \{2, 4, 6, 8\}, C = \{1, 5\}$, and $D = \text{the set of natural numbers}$. Find:

(a) $A \cap B$;
(b) $A \cup B$;
(c) $A \cap \tilde{C}$;
(d) $C \cap D$;
(e) the power set of $B \cap C$.
(f) the power set of $\emptyset$.

**Solution:**

(a) $A \cap B = \{2, 4\}$;
(b) $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$;
(c) $A \cap \tilde{C} = \{1, 5\}$;
(d) $C \cap D = \{2, 3, 4\}$;
(e) $B \cap C = \{2, 4\}$ so $\mathcal{P}(B \cap C) = \{\emptyset, \{2\}, \{4\}, \{2, 4\}\}$;
(f) $\{\emptyset\}$. 
#9 Let $A, B, C$ be sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

**Solution:** Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, either $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$ and if $x \in C$, then $x \in A \cap C$. Thus $x \in (A \cap B) \cup (A \cap C)$.

Now let $x \in (A \cap B) \cup A \cap C$. Then either $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $x \in B$, we have $x \in B \cap C$ and so $x \in A \cup (B \cap C)$. If $x \in A \cap C$, then $x \in A$ and $x \in C$. Since $x \in C$, we have $x \in B \cap C$ and so $x \in A \cup (B \cap C)$.

#10 Give an example of a nested family of sets $\{A_1, A_2, \ldots, \}$ such that
(a) $\cap_{i=1}^{\infty} A_i = [2, 3]$;
(b) $\cap_{i=1}^{\infty} A_i = [2, \infty)$.

**Solution:**
(a) For example, $A_i = (2, 3 + \frac{1}{i})$.
(b) For example, $A_i = (2 - \frac{1}{i}, \infty)$.

#11 Prove that $\sqrt{5}$ is irrational.

**Solution:** First note that if $n$ is an integer and 5 divides $n^2$, then 5 divides $n$. To see this, note that we may write $n = 5q + r$ for integers $q$ and $r$ with $0 \leq r < 5$. Thus $r = 0, 1, 2, 3$, or 4. Then $n^2 = (5q + r)^2 = 25q^2 + 10rq + r^2 = 5(5q^2 + 2q) + r^2$. Thus 5 divides $n^2$ if and only if 5 divides $r^2$. But $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, and $4^2 = 16$ are not divisible by 5. Thus $r = 0$ so $n = 5q$ is divisible by 5.

We now prove that $\sqrt{5}$ is irrational by contradiction. Assume it is rational. Then $\sqrt{5} = \frac{a}{b}$ for integers $a, b$ with $b \neq 0$ and such that both $a$ and $b$ are not divisible by 5. Then, multiplying both sides by $b$ we have $\sqrt{5}b = a$, and squaring both sides, we have $5b^2 = a^2$. Thus 5 divides $a^2$ and so, by our preliminary result, 5 divides $a$. Thus $a = 5k$ for some integer $k$ and so $5b^2 = (5k)^2 = 25k^2$. Dividing both sides by 5 gives $b^2 = 5k^2$ and so 5 divides $b^2$. Again using our preliminary result, we see that 5 divides $b$. Thus we have that 5 divides both $a$ and $b$, contradicting our assumption.