A smaller monoidal center for quantum group representations

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Recent developments in noncommutative algebra and related areas

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Contents

Quantum Groups

Drinfeld, Jimbo (∼1986–87): Quantum groups

$$\mathrm{U}_q(\mathfrak{g}) = \mathrm{U}_q(\mathfrak{n}_+) \otimes \mathrm{U}_q(\mathfrak{t}) \otimes \mathrm{U}_q(\mathfrak{n}_-)$$

$\mathfrak{g}$ — semisimple Lie algebra $\quad$ q — a parameter

$\mathfrak{n}_\pm$ — nilpotent parts $\quad$ $\mathfrak{t}$ — Cartan part

- Deformations of universal enveloping algebras $\mathrm{U}(\mathfrak{g})$
- Hopf algebras — representations form monoidal categories
- Reshetikhin–Turaev (∼1990) construct a 3d TQFT via representation theory of $\mathrm{U}_q(\mathfrak{g})$

The Monoidal Center

$\mathcal{M}$ — $\mathbb{k}$-linear monoidal category, tensor product $\otimes$

- Categorical analogue of a $\mathbb{k}$-algebra — $\mathbb{k}$ a field
- Study categorical representation theory

The monoidal center

$$\mathcal{Z}(\mathcal{M}) = \mathbf{Hom}_{\mathcal{M}\boxtimes\mathcal{M}^{\mathrm{op}}}(\mathcal{M}, \mathcal{M})$$

Note (Equivalent description)

An object $\phi\colon \mathcal{M} \to \mathcal{M}$ is determined by:

- the image $\phi(\mathbb{k}) = V$ of the tensor unit $\mathbb{k}$
- and natural isomorphisms

$$c_M\colon V \otimes M \xrightarrow{\sim} M \otimes V,$$

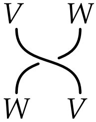
for any object M in $\mathcal{M}$ which are $\otimes$-compatible

$\longrightarrow$ original definition of Majid, Joyal–Street ($\sim$1989)

BRAIDINGS

Theorem
The center $\mathcal{Z}(\mathcal{M})$ is a braided monoidal category.

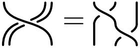
Braiding: Natural isomorphisms



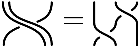
$$= \Psi_{V,W} \colon V \otimes W \xrightarrow{\sim} W \otimes V$$

satisfying

$$\Psi_{V,W\otimes X} = (\mathrm{Id}_W \otimes \Psi_{V,X})(\Psi_{V,W} \otimes \mathrm{Id}_X) \iff$$



$$\Psi_{V\otimes W,X} = (\Psi_{V,X} \otimes \mathrm{Id}_W)(\mathrm{Id}_V \otimes \Psi_{W,X}) \iff$$



Example

$\mathcal{M} = H-\mathbf{Mod}$, $\quad H$ — (finite-dimensional) $\Bbbk$-Hopf algebra
$\to \mathcal{M}$ monoidal category, via *coproduct* map $\Delta\colon H \to H \otimes_{\Bbbk} H$

Question: What is the center $\mathcal{Z}(\mathcal{M})$ in this case?

Answer: Modules over the Drinfeld double $\mathrm{Drin}(H)$

$\mathrm{Drin}(H) = H \otimes_{\Bbbk} H^*$ as a $\Bbbk$-vector space
H, H^* Hopf subalgebras.

Example

Theorem (Drinfeld)

$U_q(\mathfrak{g})$ *is a quotient of the double* $\mathrm{Drin}(U_q(\mathfrak{b}_+))$ *of its Borel part* $U_q(\mathfrak{b}_+)$*.*

Note

$\mathrm{Drin}(U_q(\mathfrak{b}_+))$ is defined on $U_q(\mathfrak{n}_+) \otimes U_q(\mathfrak{t}) \otimes U_q(\mathfrak{t})^* \otimes U_q(\mathfrak{n}_-)$

$$\implies \mathcal{Z}(U_q(\mathfrak{b}_+)\text{–}\mathbf{Mod}) \text{ is too large}$$

Rest of the talk:

Describe a smaller, more tailored, version of the monoidal center

Braided Hopf Algebras

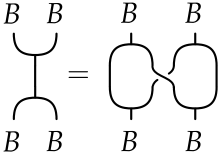
Idea: Take the Drinfeld double of $\mathrm{U}_q(\mathfrak{n}_+) \subseteq \mathrm{U}_q(\mathfrak{b}_+)$

Problem: $\mathrm{U}_q(\mathfrak{n}_+)$ is *not* a Hopf algebra in $\mathbf{Vect}_{\mathbb{k}}$
Solution: $\mathrm{U}_q(\mathfrak{n}_+)$ is a Hopf algebra in $\mathbf{Vect}_{\mathbb{k}}^q$

$\mathbf{Vect}_{\mathbb{k}}^q$: $\mathbb{Z}$-graded vector spaces with braiding

$$\Psi_{V,W}(v \otimes w) = q^{\deg(v)\deg(w)} w \otimes v$$

Bialgebra condition involves braiding



Module Categories

H — Hopf algebra in $\mathcal{B}$ (braided monoidal category)
$\mathcal{M} = H\text{–}\mathbf{Mod}(\mathcal{B})$ — category of left H-modules in $\mathcal{B}$

Monoidal functors

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\mathrm{T}"] \arrow[rr, bend left, "\mathrm{Id}"] & H\text{–}\mathbf{Mod}(\mathcal{B}) \arrow[r, "\mathrm{F}"] & \mathcal{B}
\end{tikzcd}$$

trivial module

forgetful functor

$+$ **system of isomorphisms** $\quad \sigma_{V,B} \colon V \otimes \mathrm{T}(B) \xrightarrow{\sim} \mathrm{T}(B) \otimes V$

$\Longrightarrow$ $\mathcal{M}$ is a **$\mathcal{B}$-augmented monoidal category**

BALANCED BIMODULES

To define the monoidal center: *use representation theory of monoidal categories*

$\mathcal{M}$ — $\mathcal{B}$-augmented monoidal category

$\mathcal{M}$**-bimodule:** $\mathcal{V}$ a category + monoidal functor

$$\triangleright \colon \mathcal{M} \boxtimes \mathcal{M}^{\mathrm{op}} \longrightarrow \mathbf{End}_{\Bbbk}(\mathcal{V})$$

Definition

$\mathcal{V}$ is $\mathcal{B}$-balanced if $\triangleright$ is a $\mathcal{B}$**-balanced functor**, compatible with tensor products

Example

The regular bimodule ($\mathcal{M}$ itself) is $\mathcal{B}$-balanced

THE RELATIVE MONOIDAL CENTER

Theorem (L.)

There exists a relative tensor product: a monoidal category $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{M}^{\mathrm{op}}$ such that $\mathcal{B}$-balanced bimodules correspond to all monoidal functors

$$\triangleright\colon \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{M}^{\mathrm{op}} \longrightarrow \mathbf{End}_{\Bbbk}(\mathcal{V})$$

Definition

The relative monoidal center is

$$\mathcal{Z}_{\mathcal{B}}(\mathcal{M}) = \mathbf{Hom}_{\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{M}^{\mathrm{op}}}(\mathcal{M}, \mathcal{M})$$

$\longrightarrow$ a monoidal subcategory of

$$\mathcal{Z}(\mathcal{M}) = \mathbf{Hom}_{\mathcal{M}\boxtimes\mathcal{M}^{\mathrm{op}}}(\mathcal{M}, \mathcal{M})$$

Categorical Morita Equivalence

Theorem (Ostrik, Etingof–Gelaki–Nikshych–Ostrik)

For $\mathcal{M}$ a finite tensor category,

$$\mathcal{Z}(\mathcal{M})-\mathbf{Mod} \simeq \mathcal{M} \boxtimes \mathcal{M}^{\mathrm{op}}-\mathbf{Mod}$$

Corollary (L.)

For $\mathcal{M}$ a finite tensor category,

$$\mathcal{Z}_{\mathcal{B}}(\mathcal{M})-\mathbf{Mod} \simeq \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{M}^{\mathrm{op}}-\mathbf{Mod}$$

Equivalent Description 1

Description 1: $\mathcal{Z}(\mathcal{M})$ objects $\longleftrightarrow$ pairs (V, c)

$$V \in \mathcal{M} \text{ object}, \qquad c_M \colon V \otimes M \xrightarrow{\sim} M \otimes V \text{ natural in } M$$

$$(V, c) \in \mathcal{Z}_{\mathcal{B}}(\mathcal{M}) \quad \Longleftrightarrow \quad \mathrm{F}(c_{\mathrm{T}(B)}) = \Psi_{\mathrm{F}(V),B}$$

braiding of $\mathcal{B}$

$\rightsquigarrow$ isomorphisms $c_{\mathrm{T}(B)}$ on objects B of $\mathcal{B}$ descent to braiding

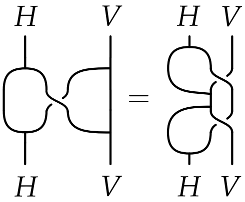
EQUIVALENT DESCRIPTION 2

Now assume $\mathcal{M} = H-\mathbf{Mod}(\mathcal{B})$

Description 2: $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ consists of objects V in $\mathcal{B}$ with

action $a = \colon H \otimes V \to V$ coaction $\delta = \colon V \to H \otimes V$

satisfying



Yetter–Drinfeld condition

Equivalent Description 3

$\mathcal{B} = K\text{-}\mathbf{Mod}$ — K **quasi-triangular** $\Bbbk$-Hopf algebra
$\mathcal{M} = H\text{-}\mathbf{Mod}(\mathcal{B})$ — H finite-dimensional over $\Bbbk$

Description 3:

$$\mathcal{Z}_{\mathcal{B}}(\mathcal{M}) \simeq \mathrm{Drin}_K(H, H^*)\text{-}\mathbf{Mod}$$

$\mathrm{Drin}_K(H, H^*)$ quasi-triangular Hopf algebra on $H \otimes K \otimes H^*$
double bosonization of Majid (∼1995)

Theorem (Majid ∼1997)

$$\mathrm{Drin}_{\mathrm{U}_q(\mathfrak{t})}(\mathrm{U}_q(\mathfrak{n}_+), \mathrm{U}_q(\mathfrak{n}_-)) \cong \mathrm{U}_q(\mathfrak{g})$$

as Hopf algebras

Difference: $\mathcal{Z}(\mathcal{M}) \hookrightarrow \mathrm{Drin}(\mathrm{U}_q(\mathfrak{b}_+))\text{-}\mathbf{Mod}$, extra copy of $\mathrm{U}_q(\mathfrak{t})$

Quantum Groups Example

$$\mathcal{M} = \mathrm{U}_q(\mathfrak{n}_+)\text{–}\mathbf{Mod}(\mathcal{B}), \qquad \mathcal{B} = \mathbf{Vect}_{\Bbbk}^{q}$$

$\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ is equivalent to the category of $\mathrm{U}_q(\mathfrak{g})$-modules V satisfying

- V is a weight module, i.e. $V = \bigoplus_{i \in \mathbb{Z}} V_i$, where $K_j \cdot v_i = q^{i \cdot j} v_i$ for any $v_i \in V_i$
- V is locally finite, i.e. $\dim(\mathrm{U}_q(\mathfrak{n}_+) \cdot v) < \infty$ for any $v \in V$

$\rightsquigarrow$ $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ contains an analogue of category $\mathcal{O}$: $\mathcal{O}_q$ defined by Andersen–Mazorchuk

$\rightsquigarrow$ braided monoidal category

OUTLOOK

- Implications for quantum group representations?
- Under what assumptions is $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ **spherical**, **modular**?
- Construction of 3**D TQFT with defects** using $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ and $\mathcal{B}$-balanced bimodules as defects?

Thank you for your attention!

SOME REFERENCES

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015.

[Lau17] R. Laugwitz, *Comodule Algebras and 2-Cocycles over the (Braided) Drinfeld Double*, ArXiv e-prints (August 2017), available at `math.QA/1708.02641`.

[Lau18] ______, *The Relative Monoidal Center and Tensor Products of Monoidal Categories*, ArXiv e-prints (March 2018), available at `math.QA/1803.04403`.

[Maj91] S. Majid, *Representations, duals and quantum doubles of monoidal categories*, Proceedings of the Winter School on Geometry and Physics (Srní, 1990), 1991, pp. 197–206.

[Maj99] ______, *Double-bosonization of braided groups and the construction of $U_q(\mathfrak{g})$*, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 1, 151–192.