# THE TOPOLOGY OF KNITTING

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## Contents

## 1. Introduction

Knitting is not often thought to be very mathematical. However, upon reflection one will determine that math is at the heart of knitting. It uses knots as its building blocks are therefore is inextricably connected to knot theory. The other obvious way that knitting may be "mathematical" is through gauging. Different yarns and different knitters result affects the size of a given pattern. Thus, before most projects are started a sample gauge is done, in order to adjust the pattern properly for the desired size. This of course will involve numbers and proportions, so even if you ask veteran knitters if knitting is mathematical, they will give this example. However, many knitters don't even realize that the thought process behind knitting is at its core mathematical. Inevitably a knitter will be in a situation where they did something wrong, and they must deduce what they must have done, and what they could do to correct it, from what they see. Similarly, if a knitter wants to make something, how should they do it? How can they put the pieces together to get what they want?

Another piece of evidence supporting the mathematics of knitting is the amount of interest in the combination of the subjects. There are many mathematicians who knit (as evidence by the many Moebius scarves and Klein Bottle hats). There have also been a couple of AMS special sessions on the mathematics of knitting. We will spend most of our time reviewing sarah-marie belcastro's article "'Every topological surface can be knit: a proof."'

First we begin with some preliminary definitions.

## 2. Definitions

**Definition 1.** *An* **m-manifold** *is a Hausdorff space $X$ with a countable basis such that each point $x \epsilon X$ has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^{\geqslant}$.*

*A 2-manifold is called a* **surface***.*

Let us also define what we mean by knitting. Knitting typically uses two sticks (i.e. **needles**) and uses one needle to hold stitches, and the other needle to manipulate the stitches. Thus the stitches are transferred from one needle to the other, and the roles of the needles are switched once an entire row has been worked through. There are a couple stitches which all the building blocks of all the other stitches, namely the **knit stitch** and the **pearl stitch**. These two stitches are very closely related. They are in fact inverses; if a knit stitch is made and the needles are switched, that same stitch looks like a pearl stitch. It will not be necessary for us to describe all the different variations of knitting (e.g. knitting with circular needles, knitting with double-pointed needles, other more complicated stitches, etc.).

We also **cast on** when we create new stiches at the beginning of a project, and we **cast off** when we knot off our stitches at the end of a project.

Our desired result is relatively intuitive. When we knit we are creating surfaces from scratch and thus we would think that our options are varied enough that we may make whatever surface we like. In fact, it seems that it may be a useful method of modelling surfaces. Surfaces, even though they are two-dimensional, require three or four dimensions to view them without self-intersections. So we will prove that we can knit any topological space but this does not follow trivially.

We continue with more definitions:

**Definition 1.** *A surface is* **orientable** *if it is two-sided; a surface is* **nonorientable** *if it is one-sided.*
*Orientable surfaces are classified by the number of holes they contain; nonorientable surfaces are classified by the number of twists they contain. The number of holes or twists of a surface is called its* **genus**. *(Note: I don't know why the spacing is so weird here).*

It is useful for us now to mention that not all knitting techniques are created equally. A good example is that of a moebius strip. There are multiple methods for knitting a moebius strip, and these methods fall into the following categories:

- those with an extrinsic twist and
- those with an intrinsic twist.

Here, as with many other subjects, it is better to come from within. That is to say, the intrinsic twist is preferrable to the extrinsic twist. The extrinsic twist makes a moebius strip much like one would with a strip of paper. A strip is knitted, and then the knitter twists one end of the strip and connects it to the other end of the strip. Alternatively, an instrinsic twist has the twist knitted into the strip as part of the pattern. It may seem obvious that the intrinsic twist is easier, and this is certainly the case. Twisting a strip as is required for a moebius strip is rarely, if ever, used in knitting, and thus is very foreign and strange to most knitters. Therefore, the extrinsic twist is much easier and therefore popular. However, the instrinsic twist is preferrable because it holds the shape of the twist much better. Instead of being superficially forced upon the pattern, the twist is build into the pattern and thus is anchored in all the stitches. The difference between an extrinsic twist moebius strip and an intrinsic twist moebius strip is obvious to those with a trained eye; the extrinsic twists will often be smaller, and are unstable (they will ripple through the strip quite easily).

Below is first an mobius strip with an intrinsic twist, then one with an extrinsic twist.

It is also important to note the practical restrictions of knitting a nonorientable surface (like a moebius strip). We remind ourselves of the difference between an embedding and an immersion:

**Definition 1.** *Let $f : X \mapsto Y$ be an injective continuous map where $X$ and $Y$ are topological spaces. Let $Z$ be the image set $f(X)$, considered as a subspaced of $Y$; then the function $f' : X \mapsto Z$ obtained by restricting the range of $f$ is bijective. If $f'$ is a homeomorphism (i.e. if both $f$ and $f^{-1}$ are continuous) of $X$ with $Z$, then $f$ is an* **imbedding** *of $X$ in $Y$.*

*Contrastingly, an* **immersion** *is a differentiable map between differentiable manifolds whose derivative is injective everywhere.*

An immersion is a local imbedding, and an imbedding is a local immersion, however the two are not the same. We note that nonorientable surfaces can be immersed but not imbedded in $\mathbb{R}^3$. Thus they cannot be accurated knitted (i.e. in $\mathbb{R}^3$) without self-intersections or boundaries. We deal with this by allowing minimal boundaries and no self-intersections.

We will not only prove that all topological surfaces can be knit, but we will provide a process by which to do so. First we must give yarn its topological properties. We consider yarn to be a long, thin, rubber-sheet rectangle. This way we may construct a surface for two reasons:

- in order to create a surface using a finite amount of yarn the yarn would have to be more than one-dimensional, and
- a surface could not be constructed with yarn of more than two-dimensions (since it itself is only two-dimensions).

Thus we will show that we can cover any surface with such rectangles, similar to the way yarn can weave a shape. We begin to see how we will form our surfaces: let us visualize the usual process of knitting. Knitting is usually done with two needles, where one row is completed, the needles are exchanged between the hands, and the next row is knit. Thus we see that a strip is formed by working up and down these rows, so that consecutive rows have touching sides. Thus we can construct our surface by connecting the sides of these long, thin, rubber-sheet rectangles. With this technique we will see that the the sides of rectangles which touch other sides are in the interior of our surface, and the sides which do not touch other sides are in the boundary. We admit that we will make a concession for the sake of mathematical coherence. In practice a stich can unravel on its own (perhaps in an idealized situation with no forces working on the knitted piece this would not happen, but this is not the case). However, in mathematics it cannot. We will therefore assume that our yarn cannot unravel on its own.

We must also minimize the geometric characteristics of our knitted objects. We wish to make topological conclusions about these knitted objects, and therefore it is important that we make the objects as abstract as possible. Thus we will ignore the usual dips and bumps in these pieces (and as they are made of knots they are full of dips and bumps) and we will imagine them to be stretched thin. We will therefore begin to imagine these knitted creations as a pieces of the plane stuck together. This is a particularly useful way for us to imagine our knitting, since it is known that every topological surface can be represented by a corresponding even-sided polygon.

We can now give a few examples of how we would use knitting to correspond to surfaces. Allow us to use the rows and columns as a sort of coordinate system. We can therefore see that we can construct a sphere using a latitude and longitude system; we can also constrcut a torus by having the rows pass through the holes so the columns of rows then cover the torus. Other surfaces (like a Klein bottle) are more difficult for us to draw neat parallels to, so we will take more time to do this.

Similarly to our description of making a moebius strip, there are a few different ways to make a Klein bottle. They also follow under two categories, one more extrinsic and one more intrinsic. The easier, extrinsic technique is to construct a cylinder with a hole in its side and then pass an end of the cylinder through the hole. The more complicated, intrinsic technique unifies the process and has the knitter basically knitting through the hole. The difference between these two techniques is not as obvious as it is for moebius strips, but it is still preferable to knit a Klein bottle intrisically. It is using this more instrinsic technique that we may construct a valid coordinate system for our knitting and therefore map a Klein bottle.

Below is a picture of a Klein Bottle Hat.

Now we have finally have the tools we need to prove our first theorem. For simplicities sake, we say that since skilled knitters can connect one ball of yarn and another ball of yarn seamlessly, we will assume that we can have a arbitrarily long strand of yarn.

## 3. Theorems

**Theorem 1.** *A sphere can be knit with a single strand of yarn.*

*Proof.* We use will use the following previously noted facts:

(1) any topological surface can be represented by an even-sided polygon, and
(2) we represent yarn as a long, thin, rubber-sheet rectangle

Thus all we need to show is that the polygonal representation of a sphere can itself be represented by a long, thin, rubber-sheet rectangle. This is not very difficult; we merely begin our rectangle at the bottom of the sphere, and then wind our way up and around it. This is also one way we could go about actually knitting a sphere. □

We work our way up in sophistication, now moving own to orientable services. Before we do that however, we need one more definition.

**Definition 1.** *Consider the space obtained from a $4n$-sided polygonal region $P$ by means of the labelling scheme*

$$(a_1 b_1 {a_1}^{-1} {b_1}^{-1})(a_2 b_2 {a_2}^{-1} {b_2}^{-1})...(a_n b_n {a_n}^{-1} {b_n}^{-1}).$$

*This space is called the* $n$**-fold connected sum of tori** *and is denoted $T\#...\#T$.*

In other words, a connected sum of two $n$-manifolds is created by the interiors of their respective $n$-balls and joining the then punctured manifolds using a homeomorphism. Thus, we see how we may create higher-dimensional manifolds from lower-dimensional manifolds, and also break down high-dimensional manifolds into lower-dimensional manifolds.

**Theorem 1.** *Every orientable surface may be knit with a single strand of yarn.*

*Proof.* We will induct on the genus of the surface.

The base case is when the genus is 1, i.e. a torus. We use the process described earlier to create a torus from a strand of yarn. We merely wind the yarn through the hole and work our way around the torus.

We use strong induction for the inductive hypothesis. Thus we assume that any orientable surface of genus less than or equal to $k$ may be knit with a single strand of yarn.

We now consider an orientable surface of genus $k + 1$. We know that it is the connected sum of an orientable surface of genus $k$ and a torus. We already know that both the surface of genus $k$ and the torus may be knit. Now all that remains to be shown is that these two surfaces can be knit together.

Both of these surfaces have no boundary, so in order to have stitches which can connect them, we must cast on more stitches onto one of the surfaces. We then cut along the final after of the other surface and use these two openings to connect the surfaces. We would use a similar process in knitting such a surface. □

And now we come to our final proof.

**Theorem 1.** *Every nonorientable surface may be knit with a single strand of yarn.*

*Proof.* We again induct on the genus.

We will use a parity argument and thus must consider two base cases: the projective plane and the Klein bottle. We have already shown that these can be knit.

We admit again that since nonorientable surfaces cannot be imbedded in $\mathbb{R}^3$ our knitted objects must have either boundaries or self-intersection (and we chose to have boundaries). Our process here will create boundaries when we cast off.

We now assume that any nonorientable surface of genus less than or equal to $k$ may be created using one piece of yarn.

Consider a nonorientable surface of genus $k+1$. If $k+1$ is odd, then it is the connected sum of a nonorientable surface of genus $k$ (which is even) with the projective plane. If $k+1$ is even, then it is the connected sum of a nonorientable surface of genus $k-1$ (which will also be even) and a Klein bottle. By the inductive hypothesis we can knit all these different shapes. Thus it remains to be shown that we can connect these two shapes (depending on the parity of $k$). We use a similar technique of grafting as we did with the previous proof. However here we must realize that if $k+1$ is odd then we will form a single boundary when cutting the nonorientable surface of genus $k$ and if $k+1$ is even, we will also form a single boundary when cutting the nonorientable surface of genus $k-1$. We therefore graft these single boundaries to their corresponding surfaces (either the projective plane or the Klein bottle) and we are done. □

## 4. Conclusion

Thus we have shown that every topological surface can be knit. As we said earlier, this should be rather intuitive. For many of the surfaces we would wish to knit, all we would have to do is wrap yarn around them. However, surfaces with larger genuses are somewhat more difficult to knit, and in these circumstances we have see ntaht we must cast on extra stitches and graft surfaces of smaller genuses together.

## 5. Extra Knitting Nerdiness

Just for the fun of it, here are the beginnings of a sierpinski shawl (obviously it can't actually be a fractal, but it looks somewhat like it):

## References

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