Change Ringing:

An Exercise in Group Theory

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"May not music be described as the mathematic[1] of sense, mathematic as the music of reason? The soul of each is the same!"

- Mathematician J.J. Sylvester

There has long been a desire to connect mathematics and music. At the base of music is a logic that is too similar to mathematics for there not to be any such connection. Indeed, there are many applications of mathematics in music. One such application is with the bell ringing technique called "change ringing."

Change ringing is a section of campanology, the art or study of bell casting and ringing. Specifically, change ringing is the ringing of a set of chimes or bells, especially in a bell tower, with every possible unrepeated variation. Change ringing is an art form separate from the rest of bell ringing in that its primary concern is the varying permutations, not a single melody.

With this topic of permutations, you may have guessed that this is where we can begin to connect the two seemingly disjoint subjects of mathematics and music. There was a British musician by the name of Fabian Stedman who contributed greatly to change ringing with his books *Tintinnalogia* (1668) (translated "the art of ringing") and *Campanologia* (1677). Stedman used group theory in his work before it was actually developed. We talk more of this later. First we will go over some basic group theory in

[1] In the quotation, Sylvester says "mathematic," not "mathematics." I am not sure why he left out the "s," but perhaps it is a product of his time period and/or culture (he lived 1814-1897 in England).

modern day change ringing. In change ringing there are five rules when designing a piece from which its structure derives. Before we list these rules we must introduce some terminology.

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Definitions:

Let us have n bells, and we shall number them 1, 2, … n, where 1 denotes the bell with the highest pitch, 2 the bell with the second highest pitch, and so on. Then bell 1 is called the *treble* and bell n is called the *tenor*.

In change ringing we consider all possible permutation of n bells, and we shall denote a single horizontal ordering of the bells to be one such permutation. Thus, in describing all our permutations for 6 bells, our writing will look something like:

123456

132546

312564

.

.

123456

with 6! = 720 such lines. Each of these permutation is called a *change*.

For mathematical purposes, we shall consider the permutation

123…n

to be the identity permutation. In change ringing, this permutation is referred to as *in rounds*.

A sequence including all possible permutations is called a *peal* or an *extent*. We shall soon see that a peal actually contains all possible permutations with one repeat; we begin and end a peal with the identity permutation for a sense of completeness.

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Now that we have the necessary terminology, we shall state the rules for ringing a peal:

1) The peal must begin and end in rounds.
2) The peal must contain all possible changes without repetition. That is, each permutation must occur once and only once.
3) A bell can move at most one place per change. That is, consider the change on 5 bells

   13245.

   This means that treble could stay in place or go to where the 3 bell is, the 3 bell could stay in place, go to where the treble or the 2 bell is, and so on. For simplicity's sake, we shall call to each such place using capital letters A, B, …, N for N bells. Thus, the treble could stay in A or go to B, and so on. This rule is more for logistic purposes; a bell can weigh as much as four tons, and would therefore have a high momentum. So changing the speed of ringing any particular bell would be difficult.
4) A bell can stay in the same place for at most two consecutive changes. (This rule is often bent somewhat; as you will soon see, change ringing can get very complicated, and it helps to have a little flexibility).

5) Each bell may move in an equally varied manner. (This rule is more vague than the others; it means that no bell should be varied more than the others. e.g. You would not want to have bell 5 move only every third permutation while every other permutation moves every permutation. Like rule 4, this rule is bent somewhat; it is treated more as a rule of thumb.)

(An outline of these axioms is provided in (5)).

In change ringing there can be anywhere from 3 to 12 bells. Each collection of bells has its own unique name:

- 3 is called singles
- 4 is called minimus
- 5 is called doubles
- 6 is called minor
- 7 is called triples
- 8 is called major
- 9 is called caters
- 10 is called royal
- 11 is called cinques
- and 12 is called maximus. (5)

However, peals using more than 7 bells are rare, as such a peal could be very long (a peal on 12 bells would contain 12! + 1 changes = 479001601 changes). For some reference, a 7 bell peal is rung at a rate of roughly five rings per second, so it takes about two hours to go through an entire peal. In contrast, a 3 bell peal can afford to be rung slowly and it is, as ringing a 3 bell peal quickly would require ringing each bell quickly (which is not an easy thing to do).

With this information, we can now make our first proposition.

Claim

When ringing three bells there are only two possible peals.

Proof

We can see with relative ease that we can only move two bells at a time; we must move the element in B, otherwise the elements in A and C would not be able to move, and then we would be keeping all of the bells in the same place. This would violate rule 2. Thus, starting with the identity, we have

$$123 \rightarrow 132 \text{ [i.e. (BC)] or } 213 \text{ [i.e. (AB)]}.$$

Thus, the two transformations that we can use for a change is (AB) or (BC). We can also see that we cannot use one of these transformations twice in a row; if we did so, we would get the following situation:

123

(BC)

132

(BC)

123.

That is, any transformation applied twice will lead to the identity. Thus, this leaves is with the two possible peals:

123

213

231

321

312

132

123

and

123

132

312

321

231

213

123

where in the first peal we apply (AB) first, and in the second peal we apply (BC) first.

QED

(An incomplete proof is provided by (5)).

From this, we will make another conjecture.

Theorem

The total number of bells and the number of bells remaining constant in a change have the same parity.

Proof

We will first prove this for odd parity.

In each change, we may apply transformations which consist of adjacent (that is, in the next spot over), disjoint (a single element may be in at most one transposition per change) elements.

Say we have n bells, where $n = 2m + 1$, for $m \in \mathbf{Z}$, and we have k transpositions, for $k \in \mathbf{Z}$ and $k \leq m$ (since each transposition uses two distinct bells). Then the number of bells staying constant is

$$n - 2k = (2m + 1) - (2|k|) = 2(m - |k|) + 1.$$

Therefore, the number of bells staying constant is odd.

Similarly, if we are ringing an even number of bells, then in a change an even number of bells must stay constant.

QED

It is interesting to note that the two previously mentioned peals involving 3 bells have names. The peal beginning with (AB) is called quick six, while the peal beginning with (BC) is called slow six (5). While there is no difference in the speed at which these two peals are peeled, slow six sounds distinctly slower than quick six. We can represent these two peals as a Cayley digraph, where the quick six is going clockwise and the slow six is going counter clockwise. See Figure 1.

In slow six and quick six, we alternate applying (AB) and (BC). Thus, we can see that

$$[(AB)(BC)]^3 = [(BC)(AB)]^3 = I, \text{ the identity.}$$

Let us explore this a little further.

$$[(AB)(BC)]^3 =$$

$$(AB)(BC)(AB)(BC)(AB)(BC) = I. \qquad (1)$$

We can see that the inverse of (AB)(BC)(AB) is (BC)(AB)(BC).

Since a transposition is its own inverse, if we multiply on the right (BC)(AB)(BC) to (1) we get

$$(AB)(BC)(AB) = (BC)(AB)(BC).$$

Thus we have

$$[(AB)(BC)(AB)]^{-1} = (AB)(BC)(AB).$$

This is a very interesting result, but we must be careful with our interpretation of what this could mean. It is not that case that any sequence of transpositions is its own inverse; we are already aware that

$$[(AB)(BC)]^2 \neq I.$$

However, if we reversed the order of the transpositions of (AB)(BC)(AB), we still get (AB)(BC)(AB). We will now show that the reversal of the ordering of transpositions is its inverse.

Theorem

Say we have a permutation ζ which is a product of transpositions. Let ζ have n transpositions, and let us denote the place of each transposition as 1, 2, …, q, where 1 is the leftmost transposition, 2 is the second leftmost transposition, and so on. Then we would generate ζ^{-1} by placing the transposition in 1 in q, the transposition in q in 1, the transposition in 2 in q-1, and so on.

This will hold whether or not ζ consists of strictly disjoint transpositions.

Proof

Let us say

$$\zeta = (AB)(CD)\ldots(JK)(NM).$$

Then

$$\zeta^{-1} = (NM)(JK)\ldots(BC)(AB).$$

Thus if we multiply them together:

$$(\zeta)(\zeta^{-1}) = (AB)(BC)\ldots(JK)(NM)\ (NM)(JK)\ldots(BC)(AB).$$

Since each transposition is its own inverse,

$$(NM)(NM) = I,\ (JK)(JK) = I,\ \ldots$$

and we will get

$$(\zeta)(\zeta^{-1}) = I.$$

QED

Note that if we are merely dealing with disjoint transpositions, then ζ^{-1} will be any order of the same transpositions (as disjoint transpositions commute).

Now that we are familiar with the singles, let us consider the minimus (ringing 4 bells). Thus we have the places

ABCD

where

$$1234$$

is the identity.

Let us consider the transpositions (BC), (CD), and (AB)(CD) [note that we have introduced a double transposition. None of the transpositions can generate an entire peal by themselves;

$$(BC)^2 = I,$$

$$(CD)^2 = I,$$

$$\text{and } [(AB)(CD)]^2 = I.$$

Thus we will need to use some sort of combination of these transpositions in order to create an entire peal. If were merely permuting the bells in positions B,C, and D, we could use (BC) and (CD), as we did in our previous example. However, neither of these transpositions does anything to permute the bell in position A.

It turns out that

$$[(CD)\ [(AB)(CD)]\ ((BC)[(AB)(CD)])^3]^3 = I. \tag{2}$$

(provided by (5)).

To make this more clear, we will let m = (BC), n = (CD), and p = (AB)(CD). Using this notation, we rewrite equation (2):

$$(np(mp)^3)^3 .$$

This still looks somewhat convoluted and is hard to decipher, but this relationship leads to cosets. Before we examine this further, we first need to make one more observation:

$$((BC)[(AB)(CD)])^4 = I, \tag{3}$$

or,

$$(mp)^4 = \mathrm{I}.$$

We observe that *m* and *p* do not generate S_4, while *m*, *p*, and *n* do. However, *m* and *p* do generate a subgroup H of S_4 of order 8, which divides S_4 into three cosets. We shall soon see that H is isomorphic to D_4 and that the cosets of H are isomorphic to the cosets of D_4 (6).

We call this peal Plain Bob Minimus. The method involved was created by Fabian Stedman. This is referred to as Plain Bob Minimus for several reasons. A *plain course* is a section of ringing that occurs with any bobs or singles (7). A bob is a device that causes an odd number of bells (usually 3) to vary, and a single is a change on three bells (7). That is, our plain course begins with the alternating applications of (AB)(CD) and (BC), and then we bob by applying (CD). Thus we have "Plain Bob" because we begin with a plain course and then add a bob, and we have "Minimus" because this we are ringing four bells. This method of naming an extent by what techniques are used and how many bells are being rung is the standard.

There is more group theory to be extracted from this method. Let us consider write our 24 changes in three column, as such

1234	4231	3142
2143	4321	1324
2413	3412	1342

3124	4123	3421
3214	1432	3241
2341	1423	2314
2431	4132	2134
4213	4312	1243

Thus, we have created an 8 x 3 matrix (as provided by (6)), and we have split our method up into leads. A lead is "a division of a method extending from when the treble leads to when it leads again." This is a common approach used to break up peals.

Let us consider the first lead of Plain Bob. By the last change (3421), we have applied $p(mp)^2$. Let this permutation equal *w*. That is,

$$w = p(mp)^2.$$

To get to the top of the next lead, we apply our bob, *n*. Going down the lead, we continue as we did before, but apply *p*, then *m*, and so on. Therefore, we can see that each row of the 8 x 3 matrix is a left coset of the subgroup of S_4 $\{e, w, w^2\}$ (6). That is, the first row is $\{e, w, w^2\}$, the second row is $\{p, pw, pw^2\}$, the third row is $\{mp, mpw, mpw^2\}$, and so forth.

It is also interesting to note that the bells all hunt (6) (though they may not begin in A). Let us consider the first lead once again. As we mentioned earlier, the treble hunts down the lead (as it does with the other two leads). Let us now turn our attention to bell 2. It begins in B, goes to A, A, B, C, D, D, C. This consecutively, stepwise movement (with the exception at the left- and right-most spots, as is standard with a hunt), shows that bell 2 is hunting. Similarly, if we examine bell 3, we see that it begins in C, goes to D, D, C, B, A, A, B, so bell 3 hunts as well. Upon observation, we see that all four bells hunt (although, with the exception of bell 1, they do not start in position A) in all three leads. This symmetry speaks to the relationship between the permutations used in each single lead, *m* and *p*. Note that this symmetry is broken by each lead, when we apply *n*. By starting with *p*, each bell is moved and since *m* and *p* only involve adjacent transpositions, our first movement (and all subsequent movements) will be piecewise. Then we apply *m*, and the bells in the middle continue their piecewise movement in the direction that they started in, while the bells on the stay where they are (thus the bells on the end stay there twice). We apply *p* once more, and again, the bells continue in the direction they began. Thus, we see that the nature of these permutations (and the fact that we only have four bells), allows all four bells to hunt.

Finally, we note that rows 1, 2, 5, and 6 generate A_4 (6). They consist of all the even permutations (or, in bell ringing terms, they consist of the all the *in-course* changes). (Note: The fact that there is a term for even permutations is telling of the intrinsic connection between bell ringing and group theory.)Let us consider row 1. Row 1 consists of e, w, and w^2. e is an even permutation, and $w = p(mp)^2$ is an even permutation (keep in mind that p is a double transposition). As w is an even permutation, so will w^2 be in an even permutation. Between rows 1 and 2 we apply p, which is an even permutation, so therefore row 2 will also be an even permutation. Between rows 1 and 3 we apply m, an odd permutation, so row 3 consist of odd permutations. We then apply p, an even permutation, so row 4 as well will consist of odd permutations. Finally, between rows 4 and 5 we apply *m* once more, so row 5 will consist of even permutations. Continuing with this process, we see that row 6 will also contain even permutations, while rows 7 and 8 will contain odd permutations. Thus, rows 1, 2, 5, and 6 generate A_4 (note that these rows contain 4*3 = 12 permutations = $|A_4|$.

Let us consider again the subgroup H of S_4 previously mentioned. This subgroup contains the elements:

1234

2143

2413

4231

4321

3412

3142

1324.

We designated this group as H for a reason; this is the *hunting group*. That is, notice that the treble (bell 1) moves one spot over to the right with each change and once it gets to the right it goes back to the left (7). When a bell "moves" like this, that is, it starts in position A, and moves one spot to the right every change until it gets to position N (for ringing on n bells), and then moves back to A in a similar manner, it is said to hunt. Thus, we use H to stand for the hunting group.

As we mentioned earlier, $H \approx D_4$. In order to make this clearer, we will graph our bells onto a square, as in Figure 3. We have the bells 1, 2, 3, 4 on the vertices of the square. We reflect over the horizontal line going through the middle of the square and over the diagonal line connected the upper left and lower right vertices. These reflections give us (AB)(CD) and (BC). This produces the same result as if we had a 90° rotation and a reflection.

We can now also make the Cayley digraph demonstrated in Figure 2. We have the subgroup H, and when we apply (CD), we get the coset H(CD). Then we can apply (CD) to any element in H(CD), we get the coset (H(CD))(CD).

As you can see, the group theory behind these peals gets more complicated as there are more bells (just as an equation becomes more complicated as more variables are added). A peal on 5 bells will have 121 changes, and there is a technique called "Stedman's Doubles" to create such a peal.

With a 5-bell peal, we have

ABCDE

for the bells

12345.

Here we have the transformations

(AB), (BC), (CD), (DE), and

(AB)(CD), (AB)(DE), (BC)(DE).

Note that we only have three double transpositions; at first glance one might think that since we have four single transpositions, we should have 4C2 = 6 double transpositions. However, we must keep in mind that our double transpositions must consist of disjoint single transpositions. Thus, we only have three double transpositions.

Stedman Doubles (named after Fabian Stedman) behaves in much the same way as in Plain Bob Minimus. It involves at first alternating between three of the permutations, and later working in the fourth (Stedman Doubles involves 5! = 125 changes, and an in-depth analysis of this peal would be a lengthy chore indeed).

We have already noted that when ringing n bells, there are n! possible permutations. We can also see that there are, theoretically, (n!)! different ways to arrange these permutations. Just to get an idea of the scope of this, let us compute this number for ringing 3 bells.

$$(3!)! = 6! = 720 \text{ possible arrangements}$$

When ringing 4 bells, we get $(4!)! = 6.2 \times 10^{23}$ possible arrangements. Thus we see that this is a very rapidly increasing number, and that we are not in danger of running out of peals. However, it is difficult to find a peal (especially if it involves a high number of bells) which satisfies the 5 axioms. Even just checking that a peal satisfies these axioms is a chore. One of the more useful contributions that Fabian Stedman made to change ringing is his creation of several methods which could be used to form viable peals.

Fabian Stedman was involved in change ringing and is known primarily for being a musician, but he acknowledged the mathematics behind change ringing and went on to create peals which contain quite a bit of group theory. He was born in Herefordshire, England in 1640, and died in 1713 (before even the time of Cauchy and Lagrange) (6). For most of his life, he was active in a bell ringing society called the Society of College Youths, and wrote the first two books on bell ringing: *Tintinnalogia* (1668) and *Campanalogia* (1677) (6).

We see evidence in Stedman's writing of a connection between bell ringing and mathematics. In his book *Campanalogia*, he writes,

> Although the practick part of *Ringing* is chiefly the subject of this Discourse, yet first I will speak something of the Art of *Changes*, its Invention being Mathematical, and produceth incredible effects, as hereafter will appear.

In addition, Stedman's most famous methods (Plain Bob on any even number of bells, Stedman on any odd number of bells) contain quite a bit of mathematical beauty. We have seen in analyzing his methods that in his work is group theory of all kinds, including permutations, transpositions, groups, subgroups, and cosets. It is telling that Fabian Stedman, the most famous man in change ringing, should involve so much mathematics in his formation and analysis of peals.

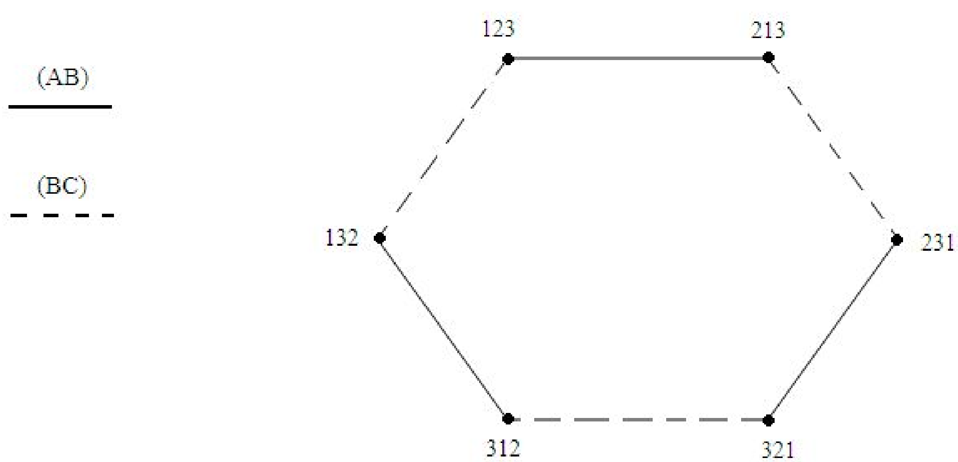
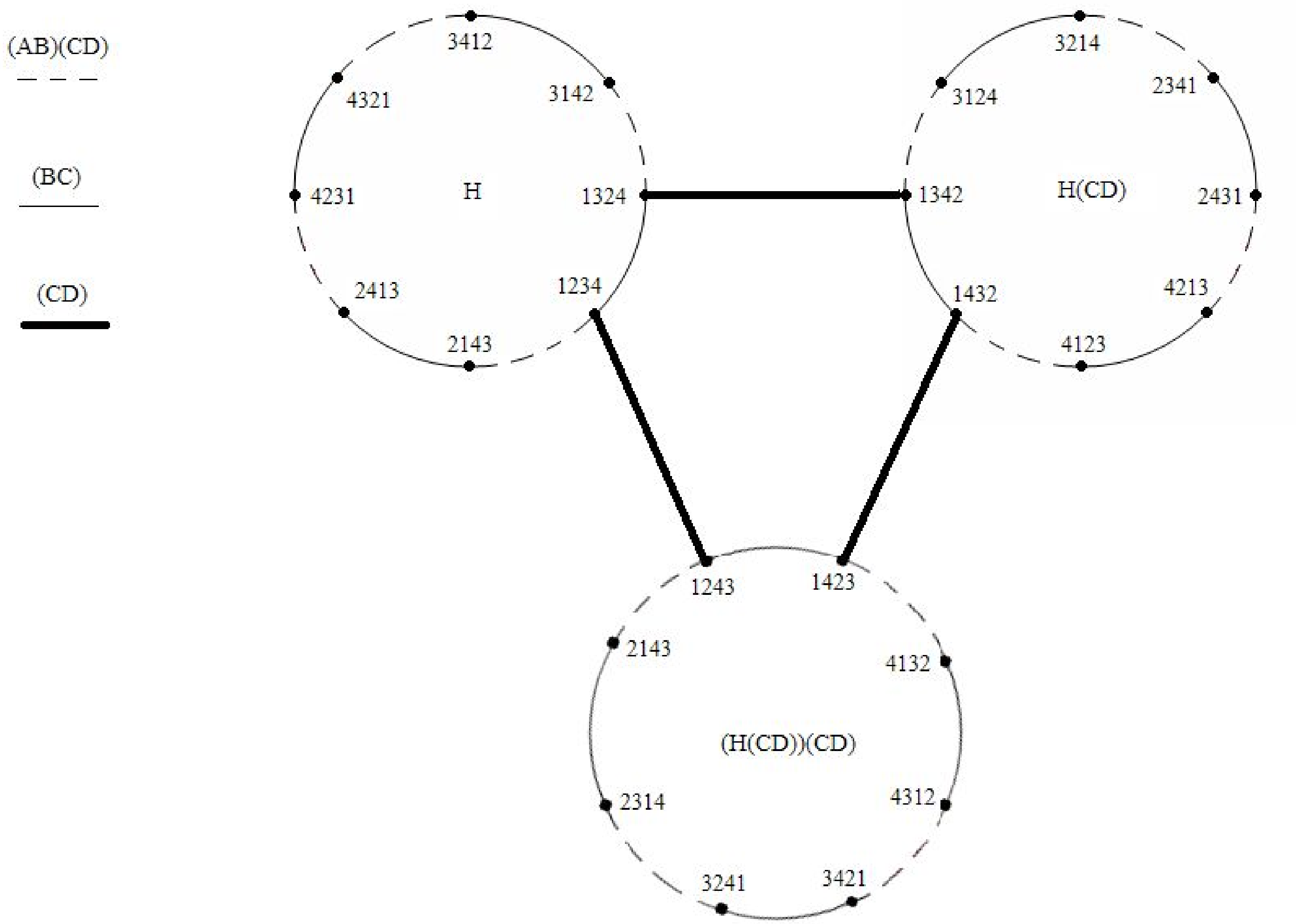

(AB)
(BC)
123
213
132
231
312
321

Figure 1: A Cayley digraph going clockwise and counterclockwise, representing the 3 bell peals.



H

Figure 2: A partial Cayley digraph of H and its cosets (this is incomplete, because for the sake of clarity, we did not draw a line for all three transpositions at each point). (5)

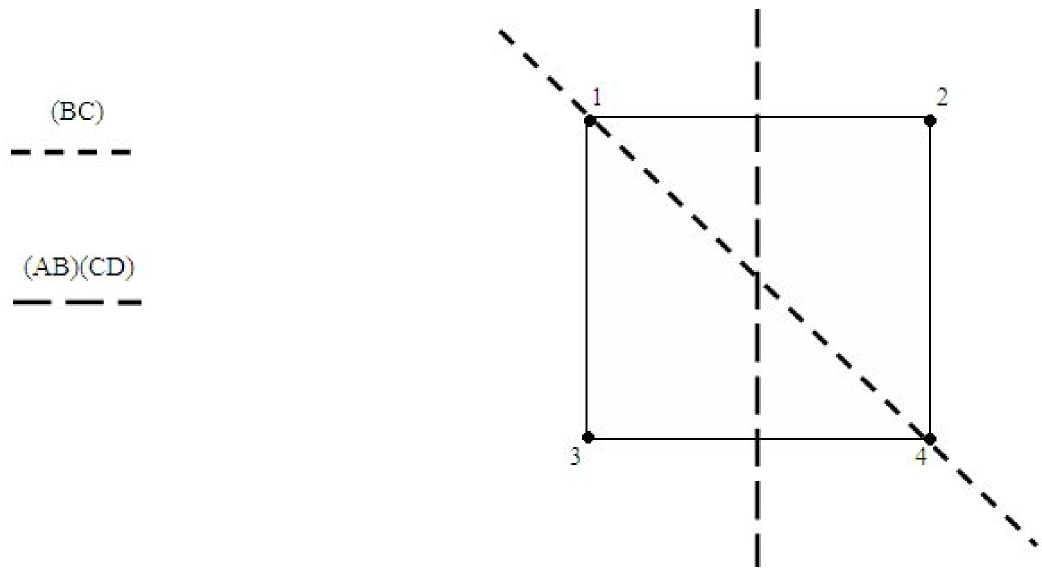


Figure 3: This is another incomplete Cayley graph, this time represent the hunting group H. (6)

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Change-ringing terms:

7) http://www.cb1.com/~john/ringing/glossary.html#Make_Places