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Name

Undergraduate Problem Solving Competition

Thursday, April 11, 2019

Welcome to the Problem Solving Competition!

You have two hours to work on these problems.

Write your name and student number on your blue book(s), on the problem sheet, and on *every* scratch paper you use. They all **must** be handed in to the proctor.

No calculators and computers are allowed. Cell phones and all other communication devices must be turned off during the exam.

Show your work and explain where your results are coming from. Every claim you make should be proved.

Give your answers in exact form (such as π or e), and not as approximations (such as 3.14159 or 2.71828).

It is better to submit fewer full solutions, rather than more incomplete solutions.

Good luck!

1. (a) Suppose x is a real number. Can you find nonzero real numbers a, b such that $x = a + b$ and $x = \dfrac{1}{a} + \dfrac{1}{b}$?

(b) Suppose x is a real number. Can you find nonzero real numbers a, b, c, d such that $x = a + b + c + d$ and $x = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} + \dfrac{1}{d}$?

2. Find all positive integers n such that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two subsets so that the product of the numbers in each subset is equal.

Turn the page

3. The quadratics $q_1(x) = x^2 + ax + b$ and $q_2(x) = x^2 + cx + d$ have real coefficients and take non-positive values on disjoint intervals. Show that there are real numbers w_1 and w_2 such that the quadratic $Q(x) = w_1 \cdot q_1(x) + w_2 \cdot q_2(x) > 0$ for all real x. †

4. Find an upper bound for the ratio

$$\frac{x_1x_2 + 2x_2x_3 + x_3x_4}{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

over all quadruples of real numbers $(x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0)$.

[Note: The smaller the bound, the better the solution.]

5. Which polygon inscribed in a given circle has the property that the sum of the squares of the lengths of its sides is maximum?

6. On an infinite checkerboard a (one-person) game is played as follows. At the start $2n$ pieces are arranged in an $2 \times n$ block of adjoining squares, one checker on each square. A move in the game is a jump in a horizontal or vertical direction (but not diagonally) over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of n for which the game can end with a single checker remaining on the board.

7. (a) Is there a set P of points in the plane such that every line contains at least one but only finitely many points of P ?

(b) Is there a set Q of points in space such that every plane contains at least one but only finitely many points of Q ?

† The original text said "negative values" instead of "non-positive values".