Algorithms Homework - Day 4

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(1) Suppose we are given an unsorted list of distinct values, and we want to find an element that is greater than exactly half of the others. Show this problem can be solved in polynomial time.

(2) The *subset sum problem* is the following:

- We are given a set S of n integers. Is there a subset of S that adds to exactly 0?

(a) Find a set for which the answer to this question is 'no' and another set for which the answer is 'yes.'

(b) Show that this decision problem is in NP.

(3) Recall the *graph isomorphism problem*:

- Given two graphs G and H with n vertices each, are G and H isomorphic?

Show that this problem is in NP.

(4) Think of at least two decision problems and show that they are in NP. (Do not pick problems that we already explicitly showed are in NP.)

(5) Consider the following two problems:

- **Problem A:** Given a graph, G, find the largest t such that K_t is contained in G.
- **Problem B:** Given a graph, G, find the size of the largest independent set of G.

Show that **problem A** has a polynomial time algorithm if and only if **problem B** has a polynomial time algorithm. [Notice! I'm not asking you to decide whether or not these problems have polynomial time algorithms. I'm asking you how you might relate these two problems.]

(6) Let M be the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(a) Find a formula for M^n in terms of Fibonacci numbers, and prove that your formula is correct. (Use the convention $F_0 = 0$, $F_1 = F_2 = 1$, et cetera)

(b) Because matrix multiplication is nice, for all n, we know $M^n M^n = M^{2n}$ (this is a general fact that you don't need to prove). Combine this fact with your formula from (a) to get a formula for F_{2n} involving F_{n-1}, F_n, and/or F_{n+1}.

(c) Try to generalize this result.

(7) Consider the following two problems

- **Problem A:** Find a proper coloring of the vertices of a graph G that uses at most three colors or state that it's impossible.

- **Problem B:** We're given a graph, G, and perhaps some of its vertices are already colored. Is it possible to color the rest of the vertices such that we end up with a proper coloring using at most 3 colors total?

You have a magic genie who can solve **problem B** for you in only 1 step (lucky you!). You can use this genie to solve **problem B** as many times as you want (costing you one step each time), and you can use his answers however you want. Using this genie as many times as you feel like, construct an algorithm to answer **problem A**. Try to make this algorithm as efficient as possible.

(8) Let's generalize Sudoku! The usual game is played with a 9×9 grid. We'll instead play on an $n^2 \times n^2$ grid, which is itself divided into an $n \times n$ grid of square regions each with area n^2. The object of the game is to fill in the grid with the numbers $1, 2, \ldots, n^2$ such that

- no number appears twice in the same row or column, and
- no number appears twice in the same $n \times n$ square region.

(a) Draw a Sudoku board for $n = 2$, and fill it in with numbers according to the rules.

(b) Draw a Sudoku board for $n = 3$ (this is the usual game) and for $n = 4$. Do not bother filling any numbers in this grid.

(c) Consider the problem "is it possible to fill in this partially filled in Sudoku board?" (we want to fill it in according to the rules of course). Show that this problem is in NP.

(d) Try to generalize or extend your ideas.