Algorithms Homework - Day 2

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(1) Prove $f(n) = \Theta(g(n))$ if and only if $f(n) = \mathcal{O}(g(n))$ and $g(n) = \mathcal{O}(f(n))$.

(2) Prove each of the following

- (a) If $f(n) = \mathcal{O}(g(n))$ then $kf(n) = \mathcal{O}(g(n))$ for all $0 < k \in \mathbb{R}$.
- (b) If $f(n) = \mathcal{O}(g(n))$ and $g(n) = \mathcal{O}(h(n))$, then $f(n) = \mathcal{O}(h(n))$.
- (c) If $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, then $f(n) = \Theta(h(n))$.

(3) Prove each of the following

- (a) If $f_1(n) = \mathcal{O}(g_1(n))$ and $f_2(n) = \mathcal{O}(g_2(n))$, then $f_1(n) + f_2(n) = \mathcal{O}(g_1(n) + g_2(n))$.
- (b) If $f_1(n) = \mathcal{O}(g_1(n))$ and $f_2(n) = \mathcal{O}(g_2(n))$, then $f_1(n) \times f_2(n) = \mathcal{O}(g_1(n) \times g_2(n))$.
- (c) If $f_1(n) = \Theta(g_1(n))$ and $f_2(n) = \Theta(g_2(n))$, then $f_1(n) + f_2(n) = \Theta(g_1(n) + g_2(n))$.
- (d) If $f_1(n) = \Theta(g_1(n))$ and $f_2(n) = \Theta(g_2(n))$, then $f_1(n) \times f_2(n) = \Theta(g_1(n) \times g_2(n))$.

(4) Prove or disprove each of the following

- (a) If $f(n) = \mathcal{O}(g(n))$, then $\lg(f(n)) = \mathcal{O}(\lg(g(n)))$.
- (b) If $f_1(n) = \Theta(g_1(n))$ and $f_2(n) = \Theta(g_2(n))$, then $|f_1(n) - f_2(n)| = \Theta(|g_1(n) - g_2(n)|)$.
- (c) If $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$.

(5) Prove that $\lg(n!) = \Theta(n \lg(n))$. (Hint: for the lower bound, use the fact that for all n, $n! \geq n \times (n-1) \times \cdots \times \lceil n/2 \rceil$.) Try to improve your result as much as possible.

(6) Suppose k is a fixed integer (like $k = 2$ or $k = 100$). Find a simple function $g(n)$ such that

$$1^k + 2^k + 3^k + \cdots + n^k = \Theta(g(n)).$$

Try to generalize and improve this result.

(7) Recall the Fibonacci numbers are given by $F_n = \begin{cases} 1, & \text{if } n = 1 \text{ or } n = 2 \\ F_{n-1} + F_{n-1}, & \text{if } n > 2. \end{cases}$

- (a) Prove by induction that $F_n \geq 2^{n/2}$ for all $n \geq 6$.
- (b) Find a constant $c < 1$ for which $F_n \leq 2^{cn}$ for all $n \geq 1$, and prove your result.
- (c) Find a function $g(n)$ so that $\lg(F_n) = \Theta(g(n))$.
- (d) Try to improve your results as much as you can.

(8) (Hint: for this problem, you can basically ignore the $\lfloor x \rfloor$ [i.e, the 'round down'] part.)

- (a) Suppose $T_1(n)$ is a function satisfying $T_1(n) = T_1(\lfloor n/2 \rfloor) + 3$ and $T_1(1) = 1$. Find a function $g(n)$ such that $T_1(n) = \mathcal{O}(g(n))$.
- (b) Suppose $T_2(n)$ is a function satisfying $T_2(n) = T_2(\lfloor n/2 \rfloor) + n$ and $T_2(1) = 1$. Find a function $g(n)$ such that $T_2(n) = \mathcal{O}(g(n))$.
- (c) Suppose $T_3(n)$ is a function satisfying $T_3(n) = 2T_3(\lfloor n/2 \rfloor) + 1$ and $T_3(1) = 1$. Find a function $g(n)$ such that $T_2(n) = \mathcal{O}(g(n))$.
- (d) Try to generalize and improve your results.