DIMACS: Probability Crash Course - Day 4

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Last time, we discussed that $1 + x \approx e^x$ is a *very* good approximation if $x \approx 0$, and we said

$$\left(1 - \frac{1}{128}\right)^{40} \approx \left(e^{-1/128}\right)^{40} = e^{-40/128} = e^{-5/16}.$$

In this situation, since $-1/128 \approx 0$, we expect this approximation to be very good, which it is.

$$\left(1 - \frac{1}{128}\right)^{40} = 0.7307184\ldots$$
$$e^{-5/16} = 0.7316156\ldots$$

This trick is convenient because e^x is literally the nicest function in existence, and e^x is actually very easy to compute via formulas such as

$$e^x = 1 + \frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which is an absolutely *amazing* fact that you'll learn in calculus. (Note the "=" above.)

Random variables

Example 1 Pat's canvas bag has toys in it. *How many* are in there?

Definition: A *random variable* (shortened as r.v.) is an unknown real number whose value depends on the outcome of the experiment. Formally speaking, a random variable is a function from the sample space to $\mathbb{R}$. A *discrete* r.v. is one that takes on only countably many values.

Example 2 Examples of random variables.

[*Story behind the name "random variable"*]

Example 3 Flip a fair coin 4 times. Let X count the number of heads that appear.

Definition: The <u>*expected value*</u> of a r.v. X is $\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$.

Example 4 Let X be the outcome of a fair die roll. Then

$\mathbb{E}[X] =$

Example 5 When John plays a game of chance, he wins one dollar with probability p and otherwise he *loses* one dollar. Let W denote the amount of money he gains (negative if he lost money). Then

$\mathbb{E}[W] =$

Example 6 On a multiple choice exam, there are five options for each question, and you get 1 point for each correct answer.

(a) What is your expected score if you randomly guess the answer?

(b) Now suppose you lose a fraction of a point for each incorrect answer. What should this fraction be in order for the expected value of random guessing to be 0?

(c) Supposing you lose the fraction calculated in (b), what is your expected score if you can eliminate two of the answer options and then you randomly choose between the remaining choices?

We can also make new random variables by considering functions of old ones or by adding together one or more random variables.

Example 7 These are all perfectly fine ways of making a new random variable out of old ones.

- $Y = 3X + 17$
- $Y = \sqrt{X^2 + 1}$
- $Y = \binom{X}{2}$
- $Y = \begin{cases} 1, & \text{if } X \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$
- $Z = X + Y$

We can then figure out expected values easily. For example, if $Y = f(X)$ then

$$\mathbb{E}[Y] = \mathbb{E}[f(X)] = \sum_x f(x) \cdot \mathbb{P}(X = x).$$

Example 8 Let Z be the number that comes up when rolling a fair die. Then

$\mathbb{E}[Z^2] =$

$(\mathbb{E}[Z])^2 =$

In general, suppose $Y = aX + b$ for real numbers a and b. Then $\mathbb{E}[Y]$ is:

If X and Y are *any* two given r.v.s, we can consider $Z = X + Y$. Then $\mathbb{E}[Z]$ is:

Theorem 9 (Linearity of expectation) *If $X_1, X_2, \ldots, X_n$ are any random variables whatsoever, and $a_1, a_2, \ldots, a_n$ are real numbers, then the following formula always holds*

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i].$$

In other words, the expected value of a sum is the sum of expected values.

Proof: Use the above calculation and induction on n.

Example 10 Suppose we roll 100 fair dice. Let Y be the sum of all the dice added together. Then

$\mathbb{E}[Y] =$

Definition: An *indicator random variable* for an event A is

$$\mathbb{1}_A = \begin{cases} 1, & \text{if the event } A \text{ happens} \\ 0, & \text{otherwise} \end{cases}$$

Then for indicator r.v.s, we have

$\mathbb{E}[\mathbb{1}_A] =$

Example 11 (Boole's inequality (*alternate proof*)) Prove the following:

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} \mathbb{P}(A_i).$$

Proof:

Example 12 (Birthday problem (*revisited*)) Suppose there are N people in a room. Let X denote the number of pairs of people who share the same birthday. (Assume all possible birthdays are equally likely.) Find $\mathbb{E}[X]$.

Example 13 (Buffon's needle) [Copy from board.]