# DIMACS: Probability Crash Course - Day 3

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## More examples

Recall the following.

**Definition:** $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.

**Definition:** Two events $A$ and $B$ are *independent* iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. In general, $n$ events $A_1, A_2, \ldots, A_n$ are independent iff (i) $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2)\cdots\mathbb{P}(A_n)$ and also (ii) every collection of $n-1$ of the events is independent.

**Law of total probability:** If $B_1, \ldots, B_n$ is a partition of the sample space, then

$$\mathbb{P}(A) = \sum_i \mathbb{P}(B_i)\mathbb{P}(A|B_i).$$

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**Example 1** Pat's canvas bag has 3 yellow stuffed animals, 2 red stuffed animals, and 4 other stuffed animals (9 total). Stuffed animals are pulled out one at a time without replacement with each equally likely. Let $Y_i$ be the event that the $i^{\text{th}}$ toy is yellow, and let $R_i$ be the event that the $i^{\text{th}}$ toy is red. Compute the following:

(a) $\mathbb{P}(Y_1)$ $\qquad$ $\mathbb{P}(Y_1^c)$

(b) $\mathbb{P}(Y_2|Y_1)$ $\qquad$ $\mathbb{P}(Y_2|Y_1^c)$

(c) $\mathbb{P}(Y_1 \cap Y_2)$ $\qquad$ $\mathbb{P}(Y_2)$

(d) $\mathbb{P}(Y_1 \cap R_2 \cap Y_3^c \cap Y_4)$

**Example 2** Alice is taking a test. She has an 80% chance of knowing how to do a question. When she is asked a question that she knows, she has a 90% chance of getting it correct. But when she is asked a question she doesn't know, then she has to guess, which only gives the correct answer 30% of the time.

(a) What percent of the time does Alice get the correct answer?

(b) Suppose we know that Alice got question 1 correct. Given this information, what's the probability that Alice had to guess on question 1?

**Theorem 3 (Multiplication rule)** *If $A_1, A_2, \ldots, A_n$ are events, then*

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2) \cdots \mathbb{P}(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

**Proof:** Use the definition of $\mathbb{P}(A|B)$ and induction on $n$.

**Example 4** Suppose 40 people each have a fair coin. Everyone starts standing up. Each round everybody independently flips their coins, and the people who got $H$ need to sit down. What is the probability that there will be at least one person in the class still standing after 7 flips?

**Example 5** You are playing a game of bridge with Armen, Burak, and Cora. Each of the four people (including you) is given 13 cards from a standard shuffled deck. You look at your cards and notice that you have exactly 5 cards that are $\spadesuit$ and you have exactly 2 Ace cards.

(a) Given the information you know about your hand, what is the conditional probability that your partner Armen has both of the other Ace cards?

(b) Given the information you know about your hand, what is the conditional probability that Burak has exactly 3 cards that are $\spadesuit$ and Cora has exactly 2 cards that are $\spadesuit$?

(c) Before you find out anything about Burak or Cora's cards, Armen puts down all of his cards face up for everyone to see, and you notice that he has exactly 3 cards that are $\spadesuit$. Given this information as well as the information you know about your hand[1], what is the conditional probability of the event in part (b)?

[1] For those who know bridge, this is exactly the sort of question you would ask yourself if you won the bidding as declarer. Although in real-life the bids [or passes] of the other players would also presumably give you valuable information to consider.

**Example 6** Google's famous *PageRank* algorithm.

**Example 7** Random walks in general. Copy from board. [Terms to look up: "random walk" or "Markov chain"]

**Example 8** Two integers $X$ and $Y$ are independently chosen from 1 to $N$ [think of $N$ as a *very large* number like $10^{30}$]. Approximate the probability that $\gcd(X, Y) = 1$.