DIMACS: Probability Crash Course - Day 2

Instructor: Pat Devlin — prd41@math.rutgers.edu

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Conditional probability

Our opinions about what probabilities should be can change over time.

Example 1 Pat has a canvas bag that you *have* seen before. What do you think is inside? How sure are you?

Definition: If A and B are events with $\mathbb{P}(B) \neq 0$, the *conditional probability of A given B* is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Example 2 A room has 30 people in it. Of this, 10 total people like Mario games and 25 total people like Zelda games. There are 9 people who like both. We pick a person uniformly at random (all outcomes equally likely). Let M be the event they like Mario and Z be the event they like Zelda.

- What is $\mathbb{P}(M|Z)$? What is $\mathbb{P}(M^c|Z)$?
- What is the conditional probability that they like Zelda given that they like Mario?

Also could think in terms of *reduced sample space*:

Example 3 Alice and Bob are playing bridge. (So they are each given thirteen cards.) Let A be the event that Alice has a void in $\diamondsuit$ [i.e., she has no $\diamondsuit$], and let B be the event that Bob has a void in $\diamondsuit$.

- Try to figure out $\mathbb{P}(A \cap B)$ in as many different ways as you can.

Example 4 A gambler has two coins in his pocket. One is a fair coin, and the other is a fake coin, on which both sides are H. The gambler picks a coin out of his pocket with each equally likely. Let R be the event that the gambler selected the real coin.

(a) Suppose the gambler flips the coin and you see that it came up H. What is probability of R given this information?

(b) Suppose the gambler flips the coin N times and you see that it came up as H every time. What is probability of R given this information?

(c) Suppose the gambler flips the coin twice and you see that it came up H the first time and T the second time. What is probability of R given this information?

Theorem 5 ("Law of total probability") *Let A be an event and $B_1, B_2, \ldots, B_n$ be events that partition the sample space. Then*

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(B_i)\mathbb{P}(A|B_i).$$

Proof:

Theorem 6 *Let A be an event where $\mathbb{P}(A) > 0$. For any event $E \subseteq S$, define $\mu(E) = \mathbb{P}(E|A)$. Then μ satisfies the axioms of a probability measure.*

[Proof as homework exercise.]

Example 7 ("Monty Hall problem") There are three doors. Behind one is a new car. Behind the others are goats. The contestant picks some door. Then the host (Monty Hall) always opens up some door the person didn't pick and reveals a goat behind it (if both of the other doors have goats, then the host opens one at random).

- Should the contestant switch doors?
- Suppose there are 100 doors with a car behind only one. The host will open up 98 other doors to show goats. Should this contestant switch doors?

Example 8 An ant is walking along the edges of a cube. Each time she gets to a corner, she picks one of the three edges at random and walks along it. Suppose she starts at corner A. What is the probability that she gets to the opposite corner of the cube before returning to A again?

Example 9 Random walks in general. Copy from board.

Example 10 Google's famous *PageRank* algorithm.

Definition: Two events A and B are *independent* iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. In general, n events $A_1, A_2, \ldots, A_n$ are independent iff (i) $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2) \cdots \mathbb{P}(A_n)$ and also (ii) every collection of $n-1$ of the events is independent.

So A, B, and C are independent iff (i) $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$ and also (ii) any two these events are independent.

Example 11 Suppose 40 people each have a fair coin. Everyone starts standing up. Each round everybody independently flips their coins, and the people who got H need to sit down. What is the probability that there will be at least one person in the class still standing after 7 flips?

Example 12 Two integers X and Y are independently chosen from 1 to N [think of N as a *very large* number like 10^{30}]. Approximate the probability that $\gcd(X, Y) = 1$.