DIMACS: Probability Crash Course

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Overview

Probability is the branch of math that deals with quantifying the unknown as best as possible. Consider for instance the following silly question.

Example 1 Pat has a canvas bag that you've never seen before. What do you think is inside? How sure are you?

Or as perhaps a more interesting example, consider the following problem, which was historically one of the *very first* probability questions mathematicians were ever able to figure out.

Example 2 ("The Problem of the Points") [*17th century France*] Two people decide to play a game of chance, where both are equally likely to win on each round. They are playing for a prize, which is $10. They keep score, and they agree beforehand that the first person to win a total of 5 rounds gets the entire prize (the other gets nothing). But, while they're in the middle of playing, the game gets interrupted, and they have to stop. How should the prize money be divided fairly?

- Without doing any computations or numbers (*yet!*), what are your thoughts on this problem? What do you think "fair" should mean in this situation? Try to explain carefully.

These examples illustrate the types of questions that probability helps us consider, and they also show the types of *answers* that probability can be expected to give us. Before we look in the bag, we cannot possibly be sure what is in there, although we might argue that saying "the bag has a pencil in it" is in some sense a *better* guess than saying "the bag has a living elephant in it." And just as we cannot *know* how the interrupted game was going to end, we can still discuss what "fair" should mean by considering the scores at the time the game stopped.

For a surprisingly long time, humans *really* struggled with how to understand probability. The first real progress in this was by seventeenth century French mathematicians such as Blaise Pascal and Pierre de Fermat. Starting with their work, people gradually began to understand simple games of chance better and better, but for hundreds of years, probability still wasn't even considered a truly mathematical subject.

No one had any idea how to view the problems in probability from any conceptual or unified viewpoint, and it wasn't until 1933 that the great Russian mathematician Andrey Kolmogorov provided the first rigorous mathematical foundation for the subject.

In this mini-course, we will follow Kolmogorov's approach and build the mathematical discipline of probability together *starting from absolutely nothing* and proving every result along the way. We will also do as many historical problems as possible. **We will learn the following:**

- Formal axioms of probability
- Probabilities with equally likely outcomes
- Conditional probability and Bayes' formula
- Random variables
- Applications of probability to other types of math
- (Various approximation techniques throughout)

Introduction to Modern Probability

Definitions: We think of each possibility as a potential *outcome* of a random experiment. The set of all outcomes is called the *sample space*. An *event* is a subset of outcomes in the sample space.

Example 3 If E and F are events of a sample space S, then we can view them as a Venn diagram. [Copy from board]

Example 4 Suppose we flip a coin and then roll a six-sided die. How many possible outcomes are there? What is the sample space?

Example 5 Four horses—A, B, C, and D—are running a race[1], and we care about the order in which they finish. Write down sets corresponding to the following events.

- Horse A comes in first.
- Horse B finishes some time before horse C, and horse D comes in third.
- The last two horses to finish are horses A and B [not necessarily in that order].
- How many total outcomes are in this sample space?

Example 6 We pick somebody from this classroom at random. Let G be the event that the person has glasses[2], let R be the event that the person is a Rutgers student, and let T be the event that the person is sitting at your table.

- Describe the event $T \cup R$ in words.
- Describe the event $G^c \cap T$ in words.
- Describe the event $(G \cup R)^c$ in words.
- Write down the event that the person is a Rutgers student not sitting at your table. Leave your answer in terms of G, R and T.
- Two events are called *mutually disjoint* if their intersection is the empty set. Let $F = R \cap T$. Are F and G disjoint? Are F and G^c disjoint?

[1] Assume no horses finish at the exact same time.

[2] This is a slight abuse of language. It is perhaps more correct to say "G is the event corresponding to the outcomes of the experiment such that the person has glasses," but that's a lot to say! In this sense mathematicians are a little bit 'sloppy' with the exact phrasing, but the meaning should always be clear.

We are now ready to define the rigorous mathematical concept of *probability*.

Definition: [Kolmogorov, 1933] If S is a sample space, then a *probability measure*, $\mathbb{P}$, on the sample space S is a way of assigning a real number to each possible event such that the following axioms hold:

(i) $0 \leq \mathbb{P}(E)$ for any event $E \subseteq S$;

(ii) $\mathbb{P}(S) = 1$; and

(iii) if E and F are any mutually disjoint events, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$.

Example 7 As an example, say we flip a coin. The sample space is $S = \{H, T\}$. If we think the coin is fair, then we would have $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = 1/2$. On the other hand, if we think the coin is biased in a way that makes H three times as likely as T, then we would have $\mathbb{P}(\{H\}) = 3/4$ and $\mathbb{P}(\{T\}) = 1/4$.

Question 8 Why do these three simple axioms of probability make sense?

Using only these three axioms, we can derive many other facts about probability that are necessarily always true.

Theorem 9 (Principle of inclusion-exclusion) *For any events $A, B \subseteq S$, we have*

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Proof:

Theorem 10 *For any event $A \subseteq S$, we have $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.*

Proof:

Corollary 11 *For any event $A \subseteq S$, we have $0 \leq \mathbb{P}(A) \leq 1$. Furthermore, $\mathbb{P}(\emptyset) = 0$.*

Proof:

Theorem 12 *Let $A, B, C, E, F \subseteq S$ be events. Then the following are true.*

(a) if $E \subseteq F$ then $\mathbb{P}(E) \leq \mathbb{P}(F)$.

(b) $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$.

(c) $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$.

(b_2) If $B_1, B_2, \ldots, B_n$ is a partition of S, then $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i)$. [this generalizes (b)]

(c_2) $\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$. [this generalizes (c)]

[Proofs left as exercises for homework.]

Equally likely outcomes

Suppose S is a finite sample space, where *the outcomes are all equally likely*. Then it must be the case that for any event $E \subseteq S$, we have

$$\mathbb{P}(E) = \frac{|E|}{|S|}.$$

Question 13 Why?

Example 14 Suppose we flip a fair coin three times in a row, where each outcome is equally likely.

(a) What is the probability the coin lands H on all three flips?

(b) What is the probability the first flip lands H and the other two both land T?

(c) What is the probability the coin lands H exactly once?

(d) What is the probability the coin lands H at least once?

Example 15 Suppose we shuffle a 52-card deck so that all possible orderings are equally likely. What is the probability that the first four cards are all the same suit? What is the probability that they are *not* all four the same suit?

Example 16 Suppose we roll two dice one at a time where all 36 possibilities are equally likely.

(a) Let A be the event that the first die lands on the number 4. What is $\mathbb{P}(A)$?

(b) Let B_n be the event that the two dice add up to the number n. What is $\mathbb{P}(B_2), \mathbb{P}(B_3), \ldots, \mathbb{P}(B_{12})$?

(c) What is $\mathbb{P}(A \cap B_{10})$? Does this equal $\mathbb{P}(A) \times \mathbb{P}(B_{10})$?

(d) What is $\mathbb{P}(A \cap B_7)$? Does this equal $\mathbb{P}(A) \times \mathbb{P}(B_7)$?

(e) What is the probability that the first die lands on a 4 *or* the dice add to 6?

Example 17 You are given thirteen cards from a shuffled 52-card deck.

(a) What is the probability that you have exactly one ace card?

(b) What is the probability that you do not have the Q of $\spadesuit$?

(c) What is the probability that you have more black cards than red cards?

(d) What is the probability that all your cards are spades?

(e) What is the probability that all your cards are black?

(f) Sort your answers to (c), (d), and (e) from smallest to largest. How can you tell which is bigger without doing anything involving numbers or computation?

Example 18 Forty-two people are lined up in a row. A bag contains forty-two white marbles *and* one black marble (43 total). The people take turns reaching in the bag and pulling out a marble. If the marble is white, then they keep the marble and sit down. But if somebody picks out the black marble, then it's all over and that person is the winner. Since there's one more marble than there are people, it's possible that nobody wins. Assume that at each stage, all remaining marbles are equally likely to be picked.

- Alice thinks that the person in the back of the line is LESS LIKELY to get the black marble than the person in the front because the only way the person in the back would get the black marble is if everybody else picked out a white marble.
- Bob thinks that the person in the back of the line is MORE LIKELY to get the black marble than the person in the front because by the time the last person picks out their marble, they would have a $1/2$ chance of it being black.

(a) Who is correct, Alice or Bob? Explain.

(b) Which person in line is most likely to draw out the black marble? Explain.

(c) What is the probability that *nobody* ends up picking the black marble?

[Please wait so that we do this next problem as a class]

Example 19 ("The Birthday Problem") How many people are in this room? Do you expect there to be two people in this room who have the same birthday?

- Are there?
- How could we do that experiment again?
- Doing the experiment several times, how many times did two people share a 'birthday'? How many total times did we try?
- What are the chances that in a room of this size *no* two people share a birthday?
- What are the chances that in a room of this size there are at least two people who share a birthday?

Example 20 You pick up a deck of cards and shuffle it a few times. You are now holding a shuffled deck of cards. You then wonder if maybe you're the very first human on earth to ever hold a deck of cards that's shuffled in that exact order. You also wonder if maybe someone else thousands of years from now might randomly shuffle their deck of cards in the exact same way as you just did. You then wonder if there have *ever* been two decks of cards shuffled in the exact same way.

(1) If we have two decks of cards, and we shuffle each deck individually, then what are the chances that they'll be in the *exact same order*?

(2) Now suppose one of the decks just came fresh from the factory, and it is in a very particular order (not at all shuffled). If we leave that deck exactly how it is but shuffle the other deck, what are the chances that they'll be in the *exact same order*?

(3) Suppose now we have N decks of cards and each deck is individually shuffled. What are the chances that these decks are *all* arranged in different orders? What are the chances that at least two decks are in the exact same order?

(4) How is this question like the birthday problem?

(5) Estimate how many decks of cards there are on earth and how many times the average deck has ever been shuffled. (This is the fun part. See hints below.)

(6) Using your answer to (5), what are the chances that there have *ever* existed two shuffled decks of cards that happen to be in the exact same order?

(7) How does changing your answer to (5) change your answer to (6)?

Hint (5): It's fun to estimate numbers like this. Sometimes it's a good idea to get an estimate that's DEFINITELY too small and one that's DEFINITELY too big. Then see how those estimates change your answer. If they don't really change your answer, then apparently it doesn't really matter how you estimate it. Another thing to do for (5) might be to ask the silly question: "if human beings were trying their hardest to shuffle as many decks of cards as possible by hand, how well could they do?"

Hint (6): If you want to know an approximation for what the number actually is, then you should somehow approximate 52!. You may also find the approximation $1+x \approx e^x$ useful, where $e = 2.71828\ldots$ is a number. And the fact $1+2+3+\cdots+(n-1) = n(n-1)/2$ might be handy.