DIMACS: Probability Crash Course — Homework

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(1) Two events are *positively correlated* iff $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$. Two events are *negatively correlated* iff $\mathbb{P}(A \cap B) \leq \mathbb{P}(A)\mathbb{P}(B)$.

  (a) Prove A and B are positively correlated iff $\mathbb{P}(A|B) \geq \mathbb{P}(A)$.

  (b) Prove that if A and B are positively correlated, then A and B^c are negatively correlated.

  (c) Find several examples of events that are positively correlated but not independent.

  (d) Find several examples of events that are negatively correlated but not independent.

  (e) Find an example of events that are both independent and mutually disjoint or prove this is not possible.

(2) Groucho, Chico, Harpo, Zeppo, and Karl decide to sit down in a line, and they do so randomly with all possible arrangements equally likely[1]. We will use notation such as $G < C$ to denote the event that Groucho ends up somewhere to the left of Chico. Similarly, $C < K < H$ denotes the event that Chico is left of Karl, who is left of Harpo.

  (a) What is $\mathbb{P}(G < C)$?

  (b) What is $\mathbb{P}(G < C | H < C)$?

  (c) What is $\mathbb{P}(C < K < H)$?

  (d) What is $\mathbb{P}(C < K < H | C < Z)$?

  (e) Between (a) and (b), which answer is larger? How about between[2] (c) and (d)?

(3) Suppose Bob is flipping a biased coin with each flip independent. It lands on H with probability x, and it lands on T with probability $1 - x$. Suppose $0 < x < 1$. Let F_n be the event that the first tails occurs on the n^{th} flip.

  (a) What is $\mathbb{P}(F_n)$?

  (b) Let $E_n = F_1 \cup F_2 \cup \cdots \cup F_n$. Describe the event E_n in words.

  (c) Find $\mathbb{P}(E_n)$ in two ways. First write this in terms of (a). Second, try to get a very simple formula by using complements.

  (d) Let $E_\infty = F_1 \cup F_2 \cup F_3 \cup \cdots$. Based on the other parts of this problem, what do you think $\mathbb{P}(E_\infty)$ is? Why does this make sense from a probability point of view?

(4) A certain factory makes car parts. 70% of the parts that the factory makes are made correctly, and the rest are defective. If a part is made correctly, then the probability that it lasts at least two years is 80%, but if a part is defective, then the probability that it lasts at least two years is only 40%.

  (a) What proportion of parts made at this factory last at least two years?

  (b) Bill bought a car part from this company two years ago. He notices that after two years, the part has not yet broken. What is the probability that Bill's car part was made correctly (given that it has lasted at least two years)?

  (c) Let q be the proportion of parts that break before two years, and let p be the proportion of parts that the factory makes correctly. If the company would like to decrease q to $2/3$ of what it currently is, what would they have to make p? (Assume p is the only parameter directly under the factory's control.)

  (d) What if the company would like to decrease q to $1/2$ of what it currently is?

[1] These guys are really great. Google "Marx brothers" later.

[2] These types of questions actually become *very* interesting in general, and some counter-intuitive things happen. Look up the "xyz conjecture" later.

(5) (**Ballot problem**) In a two-person election, candidate A receives n votes and candidate B receives m votes, where $n > m$. The votes are all placed in a hat and pulled out one at a time, with all possible orderings equally likely. As the votes are pulled out, a tally is kept counting how many votes are for each candidate. Let $P_{n,m}$ denote the probability that A is always ahead in the counting of votes. (The scores are not allowed to be tied except at the very beginning before any votes have been read.)

(a) Compute $P_{n,m}$ for all $1 \leq m < n \leq 4$.

(b) Find $P_{n,1}$ and $P_{n,2}$.

(c) Based on your results in (a) and (b), conjecture a formula for $P_{n,m}$.

(d) Derive a recurrence relation for $P_{n,m}$ in terms of $P_{n-1,m}$ and $P_{n,m-1}$ by conditioning on who received the last vote.

(e) (Optional) Use part (d) to prove your conjecture to (c) by using induction on the value of $n + m$.

(6) Everyday John eats lunch at one of two restaurants, but the quality of the cooking at these restaurants is somewhat random from day to day. On any given day, restaurant A makes food that John likes with probability a and restaurant B independently makes food he likes with probability b. Assume that these probabilities are independent from day to day and neither probability is 0 nor 1.

- On day 1, John has no idea where to eat, so he just flips a biased coin, which causes him to go to restaurant A with probability p and to B with probability $1 - p$.
- If John likes the food served to him on day n, then he'll eat there again on day $n+1$. But if doesn't like it, then he'll eat at the other restaurant on day $n + 1$.
- Let p_n denote the probability that John goes to restaurant A on day n.

(a) With this notation $p_1 = p$. Find p_2 and p_3.

(b) Argue that for all $n \geq 1$, we have

$$p_{n+1} = p_n a + (1 - p_n)(1 - b).$$

(c) Use this recurrence relation to prove the formula

$$p_n = \frac{1-b}{2-a-b} + \left(p - \frac{1-b}{2-a-b}\right) \times (a+b-1)^{n-1}.$$

(d) Let L_n be the probability that John likes his food on day n. Find a formula for L_n in terms of p_n.

(e) (Optional) Approximate p_n and L_n when n is *extremely large* (i.e., find the "limit as n goes to infinity"). **Hint:** Use the fact that $-1 < (a + b - 1) < 1$.

(7) Tom and Jerry are playing a coin flipping game. They flip three fair coins independently. Tom wins if all three land on the same side as each other, and otherwise he loses.

- Tom thinks that there is a 50% chance that *he will lose.* Tom argues that there are essentially only four outcomes: [all three coins are H], [2 coins are H], [1 coin is H], and [all three coins are T]. Of those four possibilities, Tom wins if all three are H or if all three are T. He therefore concludes that he has a $2/4 = 1/2$ chance of losing.
- Jerry thinks there is a 100% chance that *Tom will lose.* Jerry argues that the first coin flip doesn't really matter, and the only way Tom will win is if both of the other two flips are the same as the first one. The second coin has a $1/2$ chance of being different from the first, and the third coin has a $1/2$ chance of being different from the first. Therefore, the probability that the second or the third coin is different from the first is $1/2 + 1/2 = 1$.

(a) What (if anything) is wrong with Tom's argument? Be as precise as possible. If necessary, salvage his argument by altering it as little as possible.

(b) What (if anything) is wrong with Jerry's argument? Be as precise as possible. If necessary, salvage his argument by altering it as little as possible.