DIMACS: Probability Crash Course — Homework

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(1) Prove the following theorems.

  (a) Let A be an event where $\mathbb{P}(A) > 0$. For any event $E \subseteq S$, define $\mu(E) = \mathbb{P}(E|A)$. Then μ satisfies the axioms of a probability measure.

  (b) If the events A and B are independent, then the events A and B^c must be independent as well. Moreover, the events A^c and B^c are also independent.

(2) For each statement below, either prove it if it is true, or provide a counter-example.

  (a) If A and B are independent with $\mathbb{P}(B) \neq 0$, then $\mathbb{P}(A|B) = \mathbb{P}(A)$.

  (b) It is impossible for two events to be both independent and disjoint.

  (c) Suppose $0 < \mathbb{P}(B) < 1$. If $\mathbb{P}(A|B) > \mathbb{P}(A)$ then $\mathbb{P}(A|B^c) < \mathbb{P}(A)$.

  (d) Suppose $0 < \mathbb{P}(B) < 1$. If $\mathbb{P}(A|B) > \mathbb{P}(A)$ then $\mathbb{P}(A^c|B) > \mathbb{P}(A^c)$.

(3) (a) Find an example of three events—A, B, and C—such that any two of them are independent but $\mathbb{P}(A \cap B \cap C) \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$.

  (b) Generalize this.

(4) (**Gambler's ruin**) A gambler plays a game many times in a row. He starts with \$3. Each time he plays, he has an equal chance of winning and losing, and each round is independent of the others. If we wins a round, he gains \$1 and otherwise he loses \$1. The gambler will continue to play until either he is ruined (he reaches \$0) or until he has \$10.

  (a) What is the probability that starting with \$3, he will reach \$0 before he reaches \$10?

  (b) Generalize the above result.

(5) (**Euler's φ function**) In number theory, the function $\varphi(n)$ counts the number of integers x satisfying $1 \leq x \leq n$ and $\gcd(x, n) = 1$. For instance, $\varphi(6) = 2$ since the numbers 1 and 5 are the only integers meeting the requirements when $n = 6$.

  (a) Compute $\varphi(n)$ for $1 \leq n \leq 12$.

  (b) Suppose we pick Z uniformly at random from 1 to n. Let G be the event that $\gcd(Z, n) = 1$. Write $\varphi(n)$ in terms of $\mathbb{P}(G)$.

  (c) For an integer a let D_a be the event that a does *not* divide Z. If a divides n, then what is $\mathbb{P}(D_a)$?

  (d) Suppose a and b are integers such that ab divides n and $\gcd(a, b) = 1$. Show that D_a and D_b are independent.

  (e) Assume $p_1, p_2, \ldots, p_k$ is a list of all the distinct prime factors of n. Write the event G in terms of D_{p_i} and use this to find a formula for $\varphi(n)$.

(6) (**Penney's game**) Two players decide to play a coin flipping game. There are five possible *patterns* that each player is allowed to choose from. The patterns are

$$\{HHT, THH, TTH, HTT, HTH\}.$$

Each player picks one of these patterns, and they aren't allowed to pick the same one. Then the game begins, and a fair coin is flipped over and over with each flip independent of the others. The person whose pattern first appears as three consecutive flips is the winner. **For example:** Suppose player A picks the pattern HHT and player B picks the pattern THH. Suppose the coin flips are $HTHTTHTTTHH$. The game would end on the last flip since this is the first time that either player's pattern showed up as three flips in a row. This pattern is THH, so player B would be the winner.

(a) For each possible pair of patterns, P_1 and P_2, determine the probability that pattern P_1 appears before pattern P_2. (Hint: condition on the first flip.)

(b) If you were playing this game, would you rather select your pattern first or second? Describe your strategy.

(7) A bug is hopping on a number line. He starts at 0. Each hop, he rolls a fair six-sided die. He then hops to the right the number shown on the die, *skipping over* any numbers in between. Assume the die rolls are independent.

(a) Suppose the bug starts at 0 and he hops many times in a row. What is the probability that at some point, he will have exactly landed on the number 10? (Exact answer.)

(b) Suppose the bug starts at 0 and he hops many times in a row. What is the *approximate* probability that at some point, he will have exactly landed on the number one million? (*Approximate* this answer somehow and explain your approximation. If you can come up with multiple ways to approximate this, that's even better.)