DIMACS: Probability Crash Course — Homework

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(1) Prove the following.

(a) If $E \subseteq F$ then $\mathbb{P}(E) \leq \mathbb{P}(F)$.

(b) If $\mathbb{P}(A) = 0.9$ and $\mathbb{P}(B) = 0.8$ then show $\mathbb{P}(A \cap B) \geq 0.7$. In general, prove *Bonferroni's inequality*, which is

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

(c) Generalize Bonferroni's inequality to show that

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots A_n) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n) - (n-1).$$

(2) Prove the following.

(a) If A and B are events, then $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$.

(b) If $B_1, B_2, \ldots, B_n$ is a partition of S, then $\mathbb{P}(A) = \displaystyle\sum_{i=1}^{n} \mathbb{P}(A \cap B_i)$.

(c) If A and B are events, then $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$.

(d) If $A_1, A_2, \ldots, A_n$ are events, then $\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \displaystyle\sum_{i=1}^{n} \mathbb{P}(A_i)$. (This is called *Boole's inequality* or the *union bound*.)

(3) Suppose we pick an integer X between 1 and 20, where each of these 20 options is equally likely.

(a) What is the probability that X is a prime number? (Note: 1 is not a prime number)

(b) What is the probability that X can be written as the average of two primes? (the primes need not be different)

(c) Now suppose Y is an integer picked between 1 and one million with each option equally likely. What is the probability that Y is divisible by 2?

(d) What is the probability that Y is divisible by 3?

(e) What is the probability that Y is a perfect square *or* a perfect cube? (or both)

(4) Alice loves coins, but she *hates* it when all the coins are heads up. It's her birthday, so you want to mail her a box full of coins. But when the coins ship in the mail, they are jostled around inside the box, and when Alice opens the box, each coin is randomly laying either heads or tails up with all the outcomes equally likely.

(a) If you mail Alice 6 coins, then what's the probability that Alice will hate it? [i.e., what's the probability they are all heads up]

(b) How many coins do you need to mail her so that you can be 99.9% sure that Alice *won't* hate it?

(c) How many coins do you need to mail so that the probability that Alice hates it is less than one in 6 million? (Hint: use the convenient fact that $2^{10} \approx 10^3$)

(d) How many coins do you need to mail so that the probability that Alice hates it is less than 10^{-30}? (Hint: use $2^{10} \approx 10^3$ multiple times)

(e) Bob is even more demanding than Alice. Bob hates it when the coins are either all heads *or* all tails. Repeat questions (a) through (d) for Bob instead of Alice.

(5) A bug is hopping around on a grid. He starts at the point $(0,0)$. Each time he hops, he flips a fair coin. If the coin lands H then he hops *to the right* one step [increasing his x-coordinate by 1], and if the coin lands T then he hops *up* one step [increasing his y-coordinate by 1]. Assume all sequences of flips are equally likely.

(a) Suppose he takes 8 steps and the coin flips in order were H, T, T, T, H, T, H, T. Draw the path corresponding to these flips.

(b) If the bug starts at $(0,0)$, then what are all the possible locations of where the bug could be after exactly N hops? For each of these possible locations, what is the probability that the bug ends up there after exactly N hops[1]?

(c) If the bug takes exactly 10 hops, what is the probability that he *never* landed on the point $(1,2)$? How would this probability change if instead he takes exactly 23 hops?

(6) The inclusion-exclusion principle can be generalized to more than two events.

(a) Prove the following inclusion-exclusion principle for three events:

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).$$

(b) State and prove an inclusion-exclusion principle for four events.

(c) How could this be generalized to even more events?

(7) There are three 6-sided dice—A, B, and C—with the following numbers on their faces:

$$\begin{aligned} A &: \; 2,2,4,4,9,9, \\ B &: \; 1,1,6,6,8,8, \qquad \textit{and} \\ C &: \; 3,3,5,5,7,7. \end{aligned}$$

Two people are going to play a game. The first player chooses any one of the dice. Then the second player chooses one of the two remaining ones. The players then roll their chosen dice, and the person who rolls the higher number is the winner.

(a) In order to maximize your chances of winning, would you rather pick your die first or second? Explain your strategy.

(b) Now suppose the rules of the game are slightly changed. The first player still gets to pick which die he wants. But then the second player has to flip a coin to decide which of the other two dice he gets. With this rule change, which die should the first player pick to maximize his chances of winning?

[1] This same idea actually gives a probabilistic proof of the binomial theorem.