Math 477
Summer 2015
Midterm 2
June 25, 2015
Time Limit: 110 Minutes

Name (Print): ____________________

Instructor: Pat Devlin

This exam has 11 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam. However, you may use your formula sheet.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the place without a clear ordering is a pain in the butt to read.
- **Mysterious or unsupported answers rarely receive credit**. Some answers require no work. However, the majority require at least some work or explanation or both.
- **When in doubt, show work and explanations**. But no question requires all that much explanation—don't waste your time writing a book!
- **If you get stuck** then move to another problem. Don't erase failed attempts, just put a line through them (that way you can get partial credit for attempts).
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	5	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	15	
Total:	100	

1. (10 points) What is the difference between an event and a random variable?

   Provide a real-life example of two random variables that are independent.

   Provide a real-life example of two random variables that are not independent.

2. (5 points) Suppose X is a random variable and you know $\mathbb{E}[X] = -2$ and $\mathrm{Var}(X) = 3$. Compute the following:

   - $\mathbb{E}[3X - 2]$
   - $\mathbb{E}[X^2]$
   - $\mathrm{Var}(3X - 2)$

3. (10 points) Suppose X and Y are two discrete random variables and suppose their joint pmf $p(a,b) = \mathbb{P}(X = a, Y = b)$ is given by

$$p(-1,1) = 0.3, \qquad p(2,-5) = 0.1, \qquad p(4,1) = 0.1, \qquad p(2,1) = 0.3, \qquad p(-1,3) = 0.2.$$

   (a) Find $\mathbb{P}(Y \geq 1)$.

   (b) Find $\mathbb{E}[X]$.

   (c) Find the conditional probability $\mathbb{P}(Y = -5 \mid X > 0)$.

   (d) Find the conditional expectation $\mathbb{E}[X \mid Y = 1]$.

4. (10 points) Tommy and Gina like thinking about random numbers. Tommy picks a number X uniformly at random from the interval $(0, 2)$ and Gina independently picks a number Y uniformly at random from $(1, 2)$. What is the probability that the numbers they pick multiply to something less than 1? [Hint: Draw a careful picture for the region of interest.]

5. (10 points) One day Pat writes a computer program that simulates rolling a fair 20-sided die [the die is labelled with numbers 1,2,3,. . . , 20, and each outcome is equally likely]. He turns on the program but then forgets about it. When he comes back, he finds that the computer has simulated independently rolling this 20-sided die 10000 times. Let X be the number of times that the die came up as a multiple of 5 (i.e., either 5, 10, 15 or 20).

   (a) What is $\mathbb{E}[X]$?

   (b) What is $\operatorname{Var}(X)$? What is $\sigma = \sqrt{\operatorname{Var}(X)}$?

   (c) Use a normal approximation to estimate $\mathbb{P}(X \leq 1950)$. Write your answer in the form

   $$\Phi(t) = \mathbb{P}(Z \leq t),$$

   where $Z \sim N(0, 1)$ is the "standard normal" (i.e., Z is normally distributed with mean 0 and variance 1). [Values of Φ for different numbers t is that table in every probability book ever.]

   (d) Estimate $\mathbb{P}(2040 \leq X \leq 2100)$. Leave your answer in terms of Φ.

6. (10 points) A pot has 3 black marbles, 2 white, and 1 red marble. Antonio reaches into the pot and pulls out marbles one at a time without replacement (so after he pulls out the sixth marble, the pot will be empty). He stops as soon as he gets the red marble. Let B denote the number of black marbles pulled out in this manner, and let W denote the number of white marbles pulled out.

   (a) Find $\mathbb{P}(B = 1, W = 1)$.

   (b) Find $\mathbb{P}(B + W = 0)$.

   (c) Find $\mathbb{P}(B + W \leq 2)$.

7. (10 points) Daniel wants to buy a car. If X is the random variable for how long (in years) a **new car** will last before breaking down, then Daniel knows from TV that X is an exponentially distributed continuous random variable with parameter $\lambda = 1/8$.

   (a) If Daniel buys a new car, what is the probability that he will be able to drive it for at least 5 years without it breaking down?

   (b) If Daniel buys a new car, what is the probability that he will be able to drive it for at least 12 years without it breaking down?

   (c) Now suppose Daniel is considering buying a used car, which is already 5 years old. Given that it hasn't broken down yet, what's the probability that Daniel would be able to drive that car for at least 12 ***additional years*** without it breaking down? (i.e., what's the probability the car will last 17 total years given it already lasted 5)

8. (10 points) Pat started erasing with his hand again, so a student throws an Angry Bird at the chalkboard. However, the student's aim isn't always perfect, and let's say that the coordinates of where the bird lands are given by X and Y. Suppose that X and Y are continuous random variables with the following joint probability density function (pdf):

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(x+xy), & \text{if } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) The place the student was aiming for could be computed by expected values. What is $\mathbb{E}[X]$? (Answers with absolutely no justification or work receive no credit.)

(b) What is $\text{Cov}(X,Y)$? (Answers with absolutely no justification or work receive no credit.)

9. (10 points) *It was the best of ideas; it was the worst of ideas...* There is the following story[1] about the famous English writer Charles Dickens (1812–1870).

> Charles Dickens purchased a train ticket, which he planned to use on the last day of March. However, when the day arrived for him to get on the train, he noticed that there hadn't yet been any train accidents at all in that month. He knew that there were on average two accidents each month, so he thought it would therefore be much more likely for an accident to happen on the day he wanted to travel. For that reason, he cancelled his ticket and bought another one for a day in the next month.

(a) Do you think his reasoning makes sense? Briefly explain why or why not using ideas and terms learned in this class.

(b) Let X be the number of train accidents in a month, and suppose X has a Poisson distribution with $\mathbb{E}[X] = 2$. What are the chances that a particular month has at least 4 or more train accidents? (Don't simplify.)

(c) Continuing from (b), what are the chances that a particular month has no train accidents? Use the fact that e is between 2 and 3 to get a rough approximation of this number.

[1] As with most "good stories," this is perhaps more apocryphal than anything else.

10. (15 points) The following special random variables have been discussed in class: Bernoulli, binomial, exponential, geometric, normal, Poisson, and uniform.

Write your own short "story" that incorporates **at least four** of these random variables in a reasonable way. When you do this, write what type of random variable it is. Your random variables should be different types, and they should be different from any examples on this test. You may include more than four random variables, in which case I will grade the best. [You will be graded on the appropriateness of your examples and not your ability to write.]

Example: One day Mohammed Li decided to get a pet kitten. At the pet store, there were 12 cats. Mohammed knows that each cat is male independently with probability 0.5 (so the number of male cats in the store is [—r.v.—]). Walking by the snake section, Mohammed figures that the store probably only rarely sells those snakes, and he thinks that the number of snakes sold in a week is [—r.v.—]. Mohammed finds a cute kitten and picks it up. He notices it is quite light, and he suspects that the weight of a kitten is probably [—r.v.—] . . .

Topic ideas: Sports, school, parties, games, action/adventure, biology, finance, et cetera . . .

Extra credit 1: Suppose we flip two coins where all outcomes are equally likely. Let X be 1 if the first coin is H and 0 otherwise. Similarly, let Y be 1 if the *second* coin is H and 0 otherwise. Let $Z = (-1)^{X+Y}$. Show that any *two* of these random variables are independent but that collectively all three random variables together are *not* independent [since knowing any two of them uniquely determines the third].

Extra credit 2: An ant is crawling along the edges of a cube. Every time he gets to a corner, he stops and then picks which edge to use next randomly with each option being equally likely [when he gets to a corner, he has *three* options of where to go next including going back the way he came]. After picking where to go next, he walks along that edge until he gets to the next corner. If the ant starts at corner A, then what are the chances that after walking along exactly 11 edges he will be at corner A again? (Unsupported answers receive no credit.)

Extra credit 3: Continuing the above, if the ant starts at corner A, then what are the chances that after walking along exactly n edges he will be at corner A again? (Unsupported answers receive no credit, however answers for small n may receive some credit.)