Math 152 - Section C2
Summer 2013
Midterm 2
July 2, 2013
Time Limit: 80 Minutes

Name (Print): ____________________

Instructor: Pat Devlin

This exam has 11 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam, but you may use your 3 by 5 index card. This card is to be turned in with your test.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the place without a clear ordering is a pain the butt to read.
- **Mysterious or unsupported answers rarely receive credit**. Some answers require no work. However, the majority require at least some work or explanation or both.
- **When in doubt, show work and explanations**. But no question requires all that much explanation—don't waste your time writing a book!
- **If you get stuck** then move to another problem. Don't erase failed attempts, just put a line through them (that way you can get partial credit for attempts).
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	15	
4	15	
5	15	
6	15	
7	10	
8	10	
9	0	
10	0	
Total:	100	

1. (10 points) Compute the following (using any method you want).

   (a) Suppose $a_0 = 1$ and for all $n \geq 1$ we have $a_n = 2a_{n-1} + 1$. Compute a_1, a_2, and a_3.

   (b) Suppose $f(n) = \dfrac{3^n}{n!} + \dfrac{3n^2 - n}{2n^4 - 5} + \left(\dfrac{1}{2}\right)^{n^2}$. Compute $\lim_{n\to\infty} f(n)$ or state it does not exist.

   (c) Find $\lim_{n\to\infty} \left(1 - \dfrac{1}{n}\right)^n$ or state it does not exist.

2. (10 points) The following series converge to the following limits:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{9}{n\pi + \sqrt{n^2\pi^2 - 9}}\right) &= -\frac{\pi^2}{12e^3} \approx -0.0409482\ldots \\
\sum_{k=1}^{\infty} \frac{1}{1 + k^2\pi^2} &= \frac{1}{e^2 - 1} \\
\sum_{n=0}^{\infty} \frac{2n+1}{1 + e^{\pi(2n+1)}} &= \frac{1}{24}
\end{aligned}$$

Pick **one** of the series above, and show that it converges [you do *not* have to show what it converges to!].

3. (15 points) Choose five of the following six series and determine if they converge or not.

- $\displaystyle\sum_{n=6}^{\infty}(-1)^n \cos\left(\frac{1}{n}\right)$
- $\displaystyle\sum_{n=19}^{\infty} n^{-0.8}$
- $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{\pi}\right)^n$
- $\displaystyle\sum_{n=0}^{\infty} ne^{-n^2}$
- $\displaystyle\sum_{n=5}^{\infty}(-1)^n \sin\left(\frac{1}{n}\right)$
- $\displaystyle\sum_{n=9}^{\infty}\frac{1}{n\ln(n)}$

4. (15 points) Consider the power series $F(x) = \sum_{n=4}^{\infty} \left(\frac{5n^2+1}{n^3-1} \right) \frac{(x-3)^n}{2^n}$. For which x does it converge?

5. (a) (10 points) Find a power series representation for $f(x) = \arctan(x^2)$ centered at $x = 0$. Write out the first few non-zero terms, and also write your answer using summation notation.

(b) (5 points) Determine the limit $\lim_{x \to 0} \dfrac{\arctan(x^2) - x^2}{x^6}$. [Hint: instead of using l'Hôpital's rule six times in a row, you could expand the numerator as a power series and simplify.]

6. (a) (8 points) Find a simple numerical expression that approximates $\sin(1/2)$. **You should NOT waste time simplifying your approximation *whatsoever*.** You <u>do not</u> have to argue how good the approximation is[1].

(b) (7 points) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges. What is the exact value of this summation?

[1]Your expression should be simple enough for a gifted middle school student to simplify [factorials are fine], and your error should be at most 10^{-4} [if you have three or more terms, that's sufficiently accurate].

7. (10 points) Alice and Bob are arguing about the series $\sum_{n=0}^{\infty}(-2)^n = 1-2+4-8+16-32+\cdots$.

- Alice says that since the individual terms do not go to zero, the series must diverge.
- Bob says that since $1+x+x^2+\cdots = \dfrac{1}{1-x}$, the series must converge to $\dfrac{1}{1-(-2)} = 1/3$.

(a) Who's right: Alice or Bob? Verify that this person is correct by doing another test on the series that confirms their conclusion.

(b) What's wrong with the other person's reasoning?

8. (10 points) Answer any ***one*** of the following three questions. If you do more than one, you must circle which question you would like to have graded. For this question, I am just looking for how you would informally explain this to a friend. The idea is for you to demonstrate that you have at least *some* internalization of the concepts.

   (a) State the integral test, and explain (with a careful picture or two) why it is true.

   (b) How can you get the Maclaurin series of $\sin(x)$ if you've never ever seen it before? How can you remember which is the series for $\sin(x)$ and which is for $\cos(x)$?

   (c) Explain (as if to a friend) why $e^{ix} = \cos(x) + i\sin(x)$ and also explain what things like e^{2i} even mean in the first place. [Here, i is the imaginary number $i = \sqrt{-1}$. Recall $i^2 = -1$.]

9. (7 points) **Extra credit 1:** In 1995, Bailey, Borwein, and Plouffe discovered the following *very remarkable*[2] formula for π

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right).$$

Their formula is based on the fact that

$$\pi = \int_0^1 \frac{16y-16}{y^4 - 2y^3 + 4y - 4}\, dy.$$

(a) How do you think they showed that this integral equals π?

(b) How do you think they used this integral to show that their series converges to π?

[Do NOT actually do any math at all for this problem!!! Just talk about an idea of what you *could* do. A basic example would be "First, you could try such and such. Then you could probably do this/that/the other thing. And then after this, you're probably done."]

[2]This formula is so amazing because it can be used to find the millionth digit of π without finding any of the digits that come before that! ('Digit' here is actually base 16 [hexidecimal].)

10. (8 points) **Extra credit 2:** The following is a series developed in 1987 by the Chudnovsky brothers for their ground-breaking computer algorithm to compute[3] π. This series is based on the work of the *brilliant* Indian mathematician Srinivasa Ramanujan (1887-1920).

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!(13591409 + 545140134n)}{(3n)!(n!)^3 640320^{3n+3/2}}$$

This series converges to π faster than any other series we've discovered (so you don't need to add up that many terms to get very good approximations to π). What makes a series converge "faster" or "slower"? Give examples of a few convergent series (you do not need to know what they converge to), and discuss which ones converge faster.

[3]This formula is used for the world records for most digits of π computed. The current world record is 10 trillion digits of π computed by Japanese businessman Shigeru Kondo in his spare time [October 2011].