Math 103 - Honors - Assignment 11
The power of contradiction

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Due: 4-November-2014

This assignment is in two parts. You are to turn it in electronically (through sakai or via email), and you are also to hand in a physical copy in class on Tuesday.

Reminders and Background

In class, we discussed several very important facts about numbers discovered by the ancient Greeks. First, we discussed that there are infinitely many primes, and we gave Euclid's highly celebrated proof of this.[1] A second classical result in number theory that we discussed is that the square root of 2 is not a rational number.

The arguments we gave for the irrationality of $\sqrt{2}$ were ***proofs by contradiction***. And although we approached Euclid's proof of the infinitude of primes in a more "direct" or "constructive" way, it is very common for this too to be presented as a proof by contradiction in the following way:

- Pretend there are only *finitely many* primes, and write them all down in some list.
- Using the same argument presented in class (multiply all the numbers in this list together, et cetera), we can definitely find some new prime number that isn't on our list.
- But this is a contradiction since our list was supposed to be a list of "all the prime numbers in the universe."
- Therefore, the only way for this to make sense is if there are actually *infinitely many* primes. □

And even though I didn't mention the phrase '*proof by contradiction*' by name before, it's actually a concept we've already encountered earlier in the course. For instance, when discussing Ramsey theory, we essentially gave a proof by contradiction to show that in any group of six people, you can always find either a group of three friends or a group of three perfect strangers. In fact, even our very first proof about tiling a "mutilated chessboard" could be phrased in terms of a proof by contradiction.[2]

Remark

A brief list of useful online resources for your reference is given at the end.

Part 1

This notion of 'proof by contradiction' is one of the most powerful tools available when crafting any type of argument, and yet this type of reasoning often initially strikes people as strange, unsettling, and even invalid (this is especially true for those who have little or no experience creating mathematical arguments).

Please write a reflection on the following [at least 1 page]: In general, how do *you* feel about this type of argument? Is it something you think you already do, or does it feel new to you? How did you feel about the proofs given in class that $\sqrt{2}$ is irrational? Identify and discuss some aspects about a 'proof by contradiction' approach that people might find confusing or foreign.

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[1] Namely, if we multiply together all the primes we know about and then add one, then the number that pops out must have some factor that is a new prime we didn't know about before.

[2] Pretend you *can* tile it. Then each domino would. . . but that would mean. . . but that would make no sense. So you *can't* tile it.

Part 2

When you tell someone you are a mathematician, they usually either say something about how they felt in grade-school math or they make some joke about how good you must be at taxes. But occasionally they ask, "so what does a mathematician *do*?" At that point, you as the mathematician are usually very excited that somebody asked, and you quickly try to think of the simplest and most elegant proof you know to try to give them a feeling for what math is all about. For most mathematicians, this is either that $\sqrt{2}$ is irrational or that there are infinitely many primes, and both of these proofs are by contradiction.

However, unbeknownst to many mathematicians, a proof by contradiction—unless it is carefully prepared and rehearsed—is often very confusing for most people to hear, and this has therefore led to many mutually frustrating elevator conversations, airplane rides, and even first dates.

Please read the following link: This is one such story involving the Hungarian mathematician Paul Erdős (1913-1996) [pronounced like AIR-dish]. He was a prolific researcher, and he was the most famous and influential discrete mathematician of all time.[3] `http://musingsonmath.com/2013/03/24/erdos-and-%E2%88%9A2/`

Please write the following [as long as needed]: Please write a conversation in the style of a Socratic dialogue wherein a mathematician character (e.g., you!) convinces (or tries to convince) another character that the square root of 2 is irrational by using a proof by contradiction.[4] Give both characters names, and make sure that a complete proof of the theorem is eventually given. Try to make the conversation similar to a realistic dialogue whereby the characters interact and question each other. (In fact, if you like you could just have this conversation with some friends and essentially just record what happens!)

Resources

- Some brief general discussion about proofs by contradiction and another example of it can be found here. `http://educ.jmu.edu/~taalmala/235_2000post/235contradiction.pdf`
- This page is a collection of many different proofs that $\sqrt{2}$ is irrational (almost all of which are by contradiction). The top of the page is an actual Socratic dialogue about this fact, but this dialogue (unlike yours) does not discuss any proofs. `http://www.cut-the-knot.org/proofs/sq_root.shtml`
- For another example of a conversation reminiscent of a Socratic dialogue, see that starting at the bottom of page 6 of Paul Lockhart's *A Mathematician's Lament*. `https://www.maa.org/external_archive/devlin/LockhartsLament.pdf`

[3] If you like interesting people, look him up.

[4] Or if you prefer, instead the subject could be that there are infinitely many primes or that say $\sqrt{3}$ is irrational. You can present any proof of these facts that you like, but just make sure that it is in fact a valid proof! If you're unsure, ask me.