Math 152 - Section C2 - Summer 2013
Workshop 6

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Worked on Thursday June 20, 2013 — Due Wednesday June 26, 2013

Problem 1: Under the hypotheses of the integral test, if $a_n = f(n)$ then for any positive integer N, we have

$$\sum_{N+1}^{\infty} a_n \leq \int_N^{\infty} f(x)\,dx.$$

(a) Using this bound, how large does N have to be to ensure[1] that $\sum_{n=1}^{N} \frac{1}{n^5}$ is within 10^{-6} of $\sum_{n=1}^{\infty} \frac{1}{n^5}$? How about to ensure that $\sum_{n=1}^{N} ne^{-n^2}$ is within 10^{-6} of $\sum_{n=1}^{\infty} ne^{-n^2}$?

(b) Get a decimal approximation for the sum of one of the series with error less than 10^{-6}.

Problem 2: A 1×1 square is "dissected" by three equally spaced horizontal lines and by three equally spaced vertical lines. The central square is shaded. Then the bordering Northeast, Northwest, Southeast, and Southwest squares are similarly dissected, with the central square shaded. Each of *those* dissected squares has a similar process done to their borders, etc. The diagram to the right shows this only for the first three steps but it is supposed to continue indefinitely.

(a) How many new shaded squares are introduced at the n^{th} step? (There is one shaded square at the first step.) What is the side length of the squares which are introduced at the n^{th} step?

(b) What is the sum, as n goes from 1 to ∞, of the shaded area (all the shaded squares)? What is the sum, as n goes from 1 to ∞, of the perimeters of all the shaded squares?

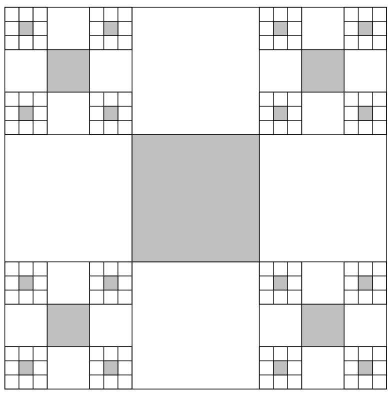

Figure 1: Dissection of square.

[1] Notice that we can estimate the error of a partial sum without actually doing any summations! This is computationally *very* useful, and it leads to a branch of math called "numerical analysis".

Problem 3: "Book stacking problem" follow-up! Recall that if each book is 1 unit long, then with a stack of n books, we can slide the top book a total of

$$\frac{H_n}{2} = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} = \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right)$$

book-lengths off the edge of the table, where H_n is the sequence of partial sums of the Harmonic series.

(a) Using this formula, how many books would you need until the top book is a total of 1.5 book-lengths over the edge of the table? [Hint: the answer happens to be less than 13, so just take some partial sums.]

(b) Under the hypotheses of the integral test, if $a_n = f(n)$, then for each positive integer n we have

$$\int_1^n f(x)\,dx \le \sum_{j=1}^{n} a_j \le a_1 + \int_1^n f(x)\,dx.$$

Use this fact to show that $\ln n \le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \le 1 + \ln n$ for all positive integers n.

(c) Part (b) shows that $1/2 + \ln(n) \approx 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is a good approximation for the partial sums of the harmonic series. Use this approximation to estimate how many books you would need to use to get the top book 5 book-lengths over the edge. [Remember to divide by two in our formula for how far the top book can reach.]

(d) Using this same approximation, how many books would you need to get the top book 6 book-lengths over the edge? How about 8? How about 10? (Your calculator may have trouble with such large numbers!)

(e) If you had $n = 10^{87}$ books available (which is more books than the number of atoms estimated in the visible universe), then about how many book-lengths over the edge of the table could you get the top one[2]? [Hint: I'm guessing that your calculator can't do computations with numbers that big, so use some properties of logs to first simplify your answer by hand a little bit.]

[2] Realistically, the problem would change by a lot if our "book tower" got high enough. We would have to consider how gravitational forces change based on distance and even how the book tower itself would create its own gravitational pull!