Math 152 - Section C2 - Summer 2013
Workshop 4

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Worked on Wednesday June 12, 2013
Due Monday June 17, 2013

Problem 1: Let's find a formula for e, everybody's favorite mathematical constant[1]! For this problem, recall that $n!$ is the "factorial" function, which is equal to $n! = 1 \times 2 \times 3 \times 4 \times \cdots \times (n-1) \times n$ (also recall that $0! = 1$ by convention).

(a) First show that $e = \frac{1}{0!} + e \int_0^1 e^{-t}\, dt$ simply by integrating (again, recall that $0! = 1$).

(b) Now! Use integration by parts to show that $\int e^{-t}\, dt = te^{-t} + \int te^{-t}\, dt$. Then use part (a) to show the formula $e = \frac{1}{0!} + \frac{1}{1!} + e \int_0^1 \frac{t}{1!} e^{-t}\, dt$.

(c) Use integration by parts again to show $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + e \int_0^1 \frac{t^2}{2!} e^{-t}\, dt$.

(d) Integrate by parts yet again to show $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + e \int_0^1 \frac{t^3}{3!} e^{-t}\, dt$.

(e) In general, if we just keep integrating by parts, then this suggests the following (true!) formula[2] for e

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Using a calculator[3], compute $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{10!}$. How does this compare to the value of e?

[1] You'll have to wait a week until we get a formula for for π.
[2] We'll mathematically prove this formula is in fact true with 'Taylor series', but this problem is already very close to a proof.
[3] Standard calculators cannot go much further than this since they will run out of digits.

Problem 2 Fun fact: $\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx = \ln(2)$. Show that this is correct.

Problem 3: Suppose A is a positive real number and m_A is the average value of $\sin^3(Ax)$ on the interval $[0, 2]$.

(a) Compute m_A. (Hint: this answer will have several terms and will *not* be simple.)

(b) What is $\lim_{A\to\infty} m_A$? (Hint: this answer *should* be simple.) Explain briefly why it is correct. You may refer to graphs of functions like $\sin^3(x), \sin^3(10x), \sin^3(100x)$ [et cetera] if that is helpful.