Special Homework 1 – Best Problems

Pat Devlin – Calculus 151 – Section C1

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The first special homework assignment was to come up with a bunch of (ideally creative) calculus questions in preparation for the second midterm. Everyone did very well, and the responses were quite creative. Some did themes like Christmas, puppies, the circus, and Toy Story, while others tried to make each problem different. There were a lot of shapes filling up with water, objects with known velocity but unknown position, and things flying through the air (including of course Angry Birds). Here are just some of the best ones from each category, although for some categories there were so many good ones that it was very hard to pick only a few. Note, some problems have been slightly modified. Within each section, the problems are presented in no particular order.

1 Implicit Differentiation

1.1 Implicit by Sarina

The general equation for an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are constants each greater than 0. Show that the tangent lines at $(0, b)$ and $(0, -b)$ are horizontal.

1.2 Implicit by Edel – World Cup Theme

On the day that USA plays in the World Cup, the Jones family is trying to find an equation for the tangent line to a soccer ball. The ball sort of looks like part of the graph $y^5 - 3x^2 + xy = 3x^4 - 2x - 12$, and they want to know the tangent line at $(3, 3)$. They're an interesting family to say the least.

2 Related Rates

2.1 Related rates by Kristy – Puppy Theme

My dinner table stands 3 feet tall and has a tablecloth draping over it. My little troublemaker dog Tina starts tugging at the tablecloth, and she's moving away at a rate of 0.25 feet per minute. When Tina is four feet away from the table, how fast is the tablecloth being pulled off?

2.2 "Monty Pythagoras and the Holy Grail of Related Rates Problems" by Sten

A flock of African swallows migrating to Camelot is carrying coconuts. When they reach King Arthur's Court, they drop the coconuts into a pile at the rate of 2 cubic feet of coconuts per minute. The pile of coconuts is shaped like a cone whose height and base diameter are always equal. At what rate is the height of the pile growing when the pile is 5 feet high?

2.3 Related rates by Anu – Christmas Theme

Tom forgot to pay his electricity bill, and the ornaments decorating his Christmas tree are beginning to shut off one by one. The tree is a cone 15 feet high with radius 10 feet, and the lights are turning off from the bottom up at a rate of 2 feet per second. How fast is the tree "virtually disappearing" (i.e., rate of change of the volume of the part of the tree that is still illuminated) when the ornaments are still on at 5 feet above the bottom of the tree? [**Remark:** The lights are going out at a rate of 2 feet per second means that the *height* of where the lights are still on is changing at this rate.]

3 Optimization

3.1 Optimization by Ashley

Your mother told you that you can get a puppy for your birthday—but only if you build a fence for it in your backyard to keep it safe. The fence needs to be a rectangle that encloses an area of 120 feet, and you're going to make it out of two different types of wood. The longer sides will be made out of cheaper wood that only costs \$5 per foot, and the two shorter sides will be made out of a more expensive wood that costs \$6 per foot. How should you design the fence so as to minimize its cost?

3.2 Optimization by Kishan – Toy Story Theme

Woody is tied up, and Buzz needs to go over and rescue him. Buzz is on the ground, and there is a long shelf 20 inches above him that extends the length of the room. Woody is on this shelf 30 inches horizontally away. Buzz needs to use his flying ability to reach the shelf, and then he needs to walk for the rest of the way. But he needs to conserve his energy so that he can untie Woody.

Flying drains Buzz's battery at a rate of 6 units for every inch he flies, and walking only drains his battery at a rate of 3 units for each inch walked. How should Buzz plan his route so as to minimize the amount of battery he uses?

3.3 Optimization by Xavier – Space Theme

You are a researcher on an incredibly boring and unproductive deep space expedition. To pass the time, you decide to find the minimum value of the function $f(x) = (x^2 - 4x + 4)^{2/5} + 3$, which the idiot navigator who dumped you here simply could not figure out.

4 Linear Approximation

4.1 "(You Make Me Feel Like) A Natural Logarithm" by Sten

You and a group of "bohemian" friends are living off the grid in the mountains of Northern California. While you've eschewed electricity and technology, you still love a good old-fashioned organic, cage-free, non-GMO natural log problem every now and then. "But duuuuude," one of your semiconscious confederates opines, "it's impossible to know what $\ln(0.942)$ is without a calculator." Is this true?

4.2 Linear approximation by Omar

You and a friend are sharing a watermelon. You tried to split it evenly, but you notice that your friend's piece weighs 4 ounces but your piece only weighs $\sqrt[3]{63.2}$ ounces. Approximate how much more watermelon your friend got than you did. Is your approximation larger or smaller than the truth?

5 Newton's Method

5.1 Newton's method by Murtala

Pat loves to throw Angry Birds at people (for fun). Unfortunately, he's not very good at throwing things. When he threw one bird, he was aiming for someone located at $y = 8$, but he accidentally hit $y = \sqrt{65}$. Approximate his error using Newton's method.

5.2 "Fig Newton's Method" by Sten

A local baker plans on creating the world's largest Fig Newton. He has purchased enough fig filling to cover an area of 23 square feet, but he is not sure what the dimensions the large square cookie should be given that amount of filling. Use Newton's method to help the chef out.

5.3 Newton's method by Christian B

King Frederick III has once again managed to upset the neighboring kingdom of Asymptotia. Asymptotia's king, Monsieur de L'Hôpital, has created a cannon that is able to fire projectiles into the air. If the projectile follows the path $f(x) = -x^2 - 7x + 6$, then use Newton's method to estimate the positive x-value where the projectile will hit the ground (at $y = 0$).

6 Curve Sketching

6.1 Curve sketch by Kiki – Pokémon Theme

Bayleef is learning calculus and wants to see the function $y = -x^3 + 3x^2 + 9x + 2$. Please sketch.

6.2 Curve sketch by Murtala

A computer engineer created a program that detects mood change by checking for heart-rate and pulse activity. This program compares a certain person's mood to an already collected database from 10,000 people using an electrocardiogram. It then translates the information collected into an equation. Mandy's mood is collected from time $t = 0$ (at noon) until time $t = 2\pi$ (at 2π pm), and the equation produced by the machine is

$$f(t) = 2t^3 - 15t^2 + 24t + 7.$$

State when Mandy was the happiest and saddest (if ever) during that time interval. Also determine when her mood started to change (if there was a change in mood). Make a sketch of her mood over time, and explain what's happening.

6.3 Curve sketch by Kristy

On my way to class, I am driving along the path $f(x) = -x^3 - 6x^2 + x + 5$. At what point do I change which direction I am turning? [Hint: this is asking inflection points]

6.4 Curve sketch by Jason

A company has been in business for the past two and a half years. Their revenue during this time is given by the equation $R(t) = 2t^3 - 4t^2 + 8$, where $t = 0$ represents the time they started, and $t = 2.5$ represents today. Make a careful graph so that the company can see its highs and lows in revenue over the period $0 \le t \le 2.5$ and whether or not their projected future revenue looks promising.

7 L'Hôpital's Rule

7.1 L'Hôpital by Maya – Circus Theme

During intermission, Bobo the clown said that the limit $\lim_{x\to 0} \dfrac{3\sin(x) - \sin(3x)}{x - \sin(x)}$ is equal to $0/0$. He called you up from the audience to help because he got stuck. Show Bobo how to solve the problem.

7.2 "L'Hôpital Is No Place to Be Sick" by Sten

Guillaume François Atoine, Marquis de L'Hôpital était un mathématicien français du dix-septième siècle, lorsque les limites étaient très difficiles à calculer. "J'en ai ras le bol!" il s'est exlamé. Mais tout d'un coup, une solution lui arrive: un règle pour faire des limites. Décrivez le "régle d'Hôpital" et résoudrez le limite suivant, quand x s'approche de 1:

$$\lim_{x\to 1} \frac{4x^2 - 5x + 1}{x^2 - 1}.$$

[*Find the limit of the above equation as x approaches 1.*]

7.3 L'Hôpital by Anehita

Jennifer was home sick from school the other day. When Bobby gave her the homework about limits, she didn't know what to do. Show Jennifer how to solve the following problem, and state why it is an indeterminate form:

$$\lim_{x\to 0^+}(1-2x)^{1/x} \qquad \text{``}=\text{''}(1-0)^{\infty}.$$

8 Antiderivatives

8.1 Antiderivatives by Christian S

Suppose $f(x)$ and $g(x)$ are continuous functions. Is it necessarily true that

$$\left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right)=\int f(x)g(x)\,dx?$$

If so, explain why this must be the case. If not, provide some specific counter-examples[1].

8.2 "That Seventies Antiderivative" by Sten

Alice is in the Brady's kitchen preparing tater tots when she hears Marsha scream hysterically: "You've ruined it, Greg! You've destroyed it forever!" Alice runs upstairs to see what all the "ruckus" is about. She finds Marsha crying while Greg sheepishly extends his hand. He's holding what looks like a derivative.

"He ruined it! He ruined my favorite polynomial formula by taking the derivative, and now I'll never get it back!" Marsha laments. "Let me see, Greg," asks Alice. He hands the derivative over to Alice:

$$x^2+10x+1.$$

"I didn't mean to, Alice, I promise – cross my heart!" Greg pleads. "Greg, why don't you go out and play some ball with the boys out back. Marsha and I have some '*girls' business*' to attend to." [*audience laughs*]

With Greg gone, Alice looks briefly at the derivative and reassures Marsha: "Now, now, Marsha, no need to cry. Mind you, we can't undo the derivative *entirely*; but applying the antiderivative and a little hairspray certainly does work wonders!" [*applause*]

8.3 Antiderivatives by Maya – Circus Theme

The contortionist is on a motorcycle travelling at 50 m/s. She then starts slowing down with constant acceleration -38 m/s^2. Determine the distance travelled before the motorcycle comes to a stop.

9 Conceptual

9.1 Conceptual by Ryan

What is the limit definition of the derivative? How does this connect with the mean value theorem?

9.2 Conceptual by Chris

A hiker is climbing a mountain. Initially, her elevation was 0 feet, and at the top her elevation was 4200 feet. Show that at some time, her elevation must have been exactly 3000 feet.

9.3 Conceptual by Amber

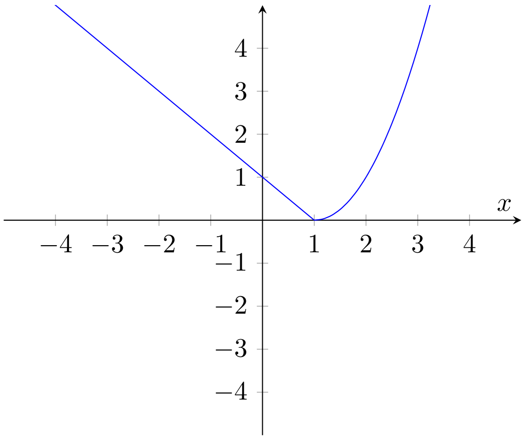
What does the mean value theorem say about the curve $1/x$ with points $a=-1$ and $b=1$? If it does not say anything, then how could you alter the question so that the mean value theorem becomes applicable?

[1] It's actually more interesting to try to come up with examples where this *does* in fact work (there are *not* that many).

9.4 "Bad, Bad, MVT. Meanest Value Theorem in the Whole [Darn] Town" by Sten

Actually, MVT isn't really so mean — he's just *average*. That's not to say MVT can't be a bit "fussy" at times. Consider the piecewise function below and its corresponding graph.

$$\text{Graph of } f(x) = \begin{cases} -x+1, & \text{if } x < 1 \\ (x-1)^2, & \text{if } x \geq 1 \end{cases}$$



Knowing what you know about MVT's preferences, write down a few intervals $[a, b]$ that would satisfy MVT's requirements. Then come up with a and b values that MVT would "dislike". Explain why you chose these particular pairs of values.

9.5 Conceptual by Min

How does the mean value theorem relate to linear approximations?

9.6 Conceptual by Ravi

Two traffic lights with cameras are located 6 miles apart from each other. You pass through the first intersection at 53 miles per hour. Five minutes later, you pass the second intersection driving 48 miles per hour. These cameras are connected to a centralized computer that mails speeding tickets to drivers who at some point exceed the 65 mile per hour speed limit. Does the computer have a reason to give you a speeding ticket?