Math 151 – Thanksgiving Break Assignment!

TA - Pat Devlin

Due in Recitation on Tuesday 27-Nov-2012

Instructions: This assignment is a (Thanksgiving-themed) review for the second midterm. You must do at least *one from each section* below and a *total of eight* problems [you pick which ones you want to do, but it would certainly benefit you to work on as many problems as you can]. Do not write out these solutions as you would a workshop problem. For each problem you pick, just write up all your work and solution as you would for a homework assignment.

Remember: *In addition to* this special assignment, you also have a workshop due on Tuesday 27-Nov-2012.

Section A

Problem A.1

A child is tracing their hand on some graph paper in order to draw a 'hand turkey'. You glance at the picture and notice that the part for the thumb looks like it is described by the curve $x^3 + y^3 = 3xy + e^{xy-x^2}$. What is the slope of this curve at the point $(0, 1)$?

Section B

Problem B.1

For the Macy's Day parade, a crew of people is inflating a spherical pumpkin balloon. The crew is using a hose that pumps in air at a rate of 12 cubic feet per second. How fast is the radius of the balloon changing when it is filled with $2,000$ cubic feet of air?

Problem B.2

During the Macy's Day parade, the same balloon as in problem B.1 is being carried around. At one time, the parade is halted, and the people guiding the balloon stop walking. At this point, an employee is holding a rope attached to the balloon. The balloon is twenty feet in the air and fifteen feet in front of the employee. A wind then starts up that causes the balloon to drift horizontally away from the employee at a rate of 3 feet per second. At what speed does the rope that the employee is holding begin to slide through his hands?

Section C

Problem C.1

You're sitting in the kitchen and the chef asks you to taste the pumpkin soup. You think it could use a little curry powder, but you're not quite sure how much. You do know that if you added x teaspoons of curry powder then the amount that you'd like the soup would be given by

$$L(x) = -x^4 - 4x^3 + 36x^2 + 6.$$

How much curry should you add to maximize the function $L(x)$?

Problem C.2

Your friend wants to build a traditional 'cornucopia', which is a cone made out of a basket-like material (with no material covering the circular opening). Your friend needs the cone to be 1 cubic foot in volume, and she wants to minimize the amount of material needed to construct it. How should she design the radius and height of the cone so that its volume is 1 cubic foot and its surface area is minimized?

Section D

Problem D.1

You stare longingly at the delicious green bean casserole that just came out of the oven, but (unfortunately) you know it's still much too hot to eat any. Let $\theta(t)$ represent the temperature of the casserole t minutes after it comes out of the oven. Then from physics class, you know that θ satisfies the equation:

$$\frac{d\theta}{dt} = -0.3(\theta - 75),$$

where 75 is the temperature of the room, and 0.3 is a constant that depends on the properties of the casserole[1].

(a) Suppose that when the casserole just comes out of the oven, its temperature is $\theta(0) = 350$. What would $\frac{d\theta}{dt}$ be at this time?

(b) If $\theta(0) = 350$, then what would be an equation for the line tangent to $\theta(t)$ at $t = 0$?

(c) Use the equation of this tangent line to estimate $\theta(0.1)$ and $\theta(0.5)$ assuming again that $\theta(0) = 350$.

Problem D.2

At the dinner table, the stuffing is being consumed at an alarming rate. In fact, t minutes after the dinner starts, there are only $S(t) = -t^3 - 5t^2 - 6t + 10$ cups of stuffing left. Use Newton's method (with initial guess $t = 1$ and three iterations) to estimate the time t at which the stuffing runs out.

Section E

Problem E.1

Carter, that high school student who lives next door to your folk's place, got stuck on his Thankgiving break math homework. He knows you're now a super-smart calculus wiz, so he facebooked you one of his problems. Unfortunately, you forgot everything you knew about "Descartes' rule of signs" (which his teacher expects him to use to solve the problems). But lucky for Carter, you can still solve his homework problems using calculus!

(a) Prove that the function $f(x) = x^5 + 4x^3 + 3x - 2$ has *at least* one root in the interval $[0, 1]$. (Hint: think about things like the mean value theorem or the intermediate value theorem)

(b) Prove that the function $f(x) = x^5 + 4x^3 + 3x - 2$ has *exactly* one root. (Hint: first find the function $f'(x)$ and argue that it is never negative. Then use this to conclude the claim.)

Problem E.2

(a) Suppose $f(x)$ is a differentiable function such that $f'(x)$ is never equal to 0. Prove that $f(x)$ is *injective* (i.e., that it passes the 'horizontal line test', which means that $f(a) = f(b)$ if and only if $a = b$). [Hint: notice that $f(a) = f(b)$ if and only if the slope of the secant line through these two points is 0, and try to use something like intermediate value theorem or mean value theorem]

(b) On the other hand, prove (by finding an example) that there is some injective differentiable function, $f(x)$, such that $f'(a) = 0$ for some number a.

[1]This equation is called 'Newton's law of heating and cooling'. It has many applications including estimating the time of death at a crime scene.

Problem E.3

So far in calculus, we have learned theorems such as the intermediate value theorem, the mean value theorem, and the theorem that every continuous function on a closed interval has an absolute maximum and minimum. Although they may seem ‘elementary’ or ‘obvious’, they are actually quite useful (even in real life!). However, you need to make sure that you apply the theorems correctly (the intermediate value theorem only works on continuous functions, et cetera). Keep these theorems (and the conditions under which they hold) in mind for the following questions.

(a) “The local football team was losing at half-time, but now they’re winning! So, by the intermediate value theorem, there must have been some point in the game after half-time where the teams were tied.” Justify this reasoning or explain why it is incorrect[2].

(b) Uncle Bill drove 110 miles to get to grandma’s house for the family dinner, and it took him an hour and a half to get there. Grandma’s suspicious that uncle Bill might have gone above the speed limit (55 miles per hour), but he insists that he never went above 60. Is it possible that uncle Bill is telling the truth?

(c) Uncle Bob (also in a hurry to get to grandma’s house) drove 80 miles in an hour and a half. To comfort the ever worrying grandma, he then said, “my average speed was less than 55 miles per hour, so (by the mean value theorem) there’s no way I went faster than the 55 mile per hour speed limit”. Justify his reasoning or explain why it is incorrect.

(d) Cousin Claire is 4 years old, and she’s 3 and a half feet tall (42 inches). She wants to grow up to be five and a half feet tall (66 inches), but [since four is her least favorite number] she never ever wants to be exactly four feet tall (48 inches). Is it possible for Claire to reach a height of 5 and a half feet without ever being exactly four feet tall?

Section F

Problem F.1

Over Thanksgiving dinner, you accidentally end up sitting next to your know-it-all cousin who (for whatever reason!) asks you to evaluate the following limits[3]:

(a) $\lim_{x\to 3} \frac{2^{x-3}-1}{x}$

(b) $\lim_{x\to 0^+} x\ln(x)$

(c) $\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$

(d) $\lim_{x\to 0^+} x^x$ (Hint: use natural logs as shown in recitation)

Problem F.2

Charlie Brown is minding his own business when Lucy invites him to run up and kick the football she is holding. Charlie Brown would love to kick the football, but he knows from experience that if he tries, Lucy will probably just pull it away from him at the last moment (making him miss it and feel like an idiot). But to entice Charlie Brown, Lucy promises that she won’t pull it away until one and a half seconds after he starts running.

Now Charlie Brown knows (from his classes in cartoon physics) that if he starts running as fast as he can, then his velocity after t seconds would be $v(t) = 6t^2 + 8t$ (measured in feet per second). If Lucy is 15 feet away, can Charlie Brown make it in time, or should he not even try? [Hint: we are given *velocity*, so find the anti-derivative to get to *displacement*]

[2] If you are unfamiliar with American football, you just need to know that every time that a team gets points, their score increases by 3 or by 7 [the actual scoring is a bit more complicated, but that’s not important].

[3] I couldn’t think of any Thanksgiving-themed examples of L’Hôpital’s rule!