MATH 555 Problem List 1 Due Sep. 15, 2020

Let A be a (finite dimensional, as always) $\mathbb{F}$-algebra.

1.1. Let U be an A-module. Show that the following statements are equivalent.
(a) U is completely reducible.
(b) every submodule of U is completely reducible.
(c) every homomorphic image of U is completely reducible.

1.2. Let V be an A-module. Prove that V is completely reducible iff the intersection of all maximal submodules of V is 0. (Isaacs' book, problem 1.1).

1.3. A is semisimple iff every A-module is completely reducible. (Isaacs' book, problem 1.3)

1.4. Let G be a finite group and p a prime dividing $|G|$. Let $\mathbb{F}$ be a field of characteristic p. Prove that $\mathbb{F}G$ is not semisimple. (Isaacs' book, problem 1.9).
(*Hint*: Apply Problem **1.2** to the regular module $\mathbb{F}G$ and consider the element $\sum_{g\in G} g$.)

MATH 555 List 2 Due Sep. 29, 2020

2.1. (Isaacs' book, problem 2.1) a) Let Φ be an irreducible $\mathbb{F}$-representation of a finite group G over an arbitrary field $\mathbb{F}$. Show that $\sum_{g\in G}\Phi(g)=0$, unless Φ is the principal representation.

b) Let H be a subgroup of G and $g\in G$ be such that all elements of the coset Hg are G-conjugate. Let $\chi\in \mathrm{Irr}(G)$ be such that $(\chi|_H, 1_H)_H=0$. Show that $\chi(g)=0$. (**Hint** : Apply a) to H and compute $tr(\sum_{h\in H}\Phi(hg))$, Φ a representation affording χ.)

From now on, all characters are $\mathbb{C}$-characters.

2.2. (Isaacs' book, problem 2.3) Let χ be a character of G. Choose a representation Φ affording χ and define $\det\chi : G\to\mathbb{C}$ as follows: $(\det\chi)(g)=\det(\Phi(g))$. Show that $\det\chi$ does not depend on the choice of Φ, and it is a linear character of G.

2.3. (Isaacs' book, problem 2.4) a) Let G be a non-abelian group of order 8. Show that G has exactly four linear characters and one more irreducible character, say χ, which is of degree 2. Show that $\chi(g)=-2$ if $g\in Z(G)$ and $g\neq 1$, and $\chi(g)=0$ if $g\in G\setminus Z(G)$. (**Hint** : Use the fact that $Z(G)=G'\simeq\mathbb{Z}_2$, the cyclic group of order 2, and then compute $(\chi,\chi)_G$.)

b) If G is the dihedral group D_8 of order 8:

$$G=<a,b\mid a^4=b^2=1, bab=a^{-1}>,$$

then $\det\chi\neq 1_G$. If G is the quaternion group Q_8 of order 8:

$$G=<a,b\mid a^2=b^2, a^4=1, bab^{-1}=a^{-1}>,$$

then $\det\chi=1_G$. On the other hand, observe that D_8 and Q_8 have the same character table. (**Hint** : Find $\Phi(b)$, Φ a representation affording χ.)

2.4. (Isaacs' book, problem 2.13) Let G be a finite group such that $G'\leq Z(G)$ and $|G'|=p$ a prime. Let $\chi\in\mathrm{Irr}(G)$ and $\chi(1)>1$. Show that χ vanishes on $G\setminus Z(G)$ and $\chi(1)^2=(G:Z(G))$.

2.5. (Isaacs' book, problem 2.9) a) Let χ be a character of an abelian group A. Show that $\sum_{a\in A}|\chi(a)|^2\geq |A|\cdot\chi(1)$. (**Hint** : Decompose $\chi|_A$ into irreducible characters of A.)

b) Let G be a finite group with an abelian subgroup A. Show that $\chi(1)\leq (G:A)$ for any $\chi\in\mathrm{Irr}(G)$.

MATH 555 List 3 Due Oct. 13, 2020

3.1. (Isaacs's book, problem 3.3) Show that no simple group can have an irreducible character of degree 2. (**Hint** : Suppose G has such a character of degree 2, afforded by a representation Φ. Then $|G|$ is divisible by 2 and so G has an involution j. Use Problem 2.2 to show that $\Phi(j) = -I$, hence $j \in Z(G)$.)

3.2. (Isaacs's book, problems 3.16 and 3.17) Let G be a finite group of odd order.

a) Let $\chi \in \mathrm{Irr}(G)$ and suppose χ is not the principal character. Show that χ and its complex conjugate $\bar{\chi}$ are different. (**Hint** : Using the oddness of $|G|$, show that

$$G = \{1\} \cup \{a_1, a_1^{-1}\} \cup \{a_2, a_2^{-1}\} \cup \cdots \cup \{a_m, a_m^{-1}\}.$$

Suppose $\chi = \bar{\chi}$. Computing $(\chi, 1_G)_G$, show that $\chi(1)/2$ is an algebraic integer.)

b) Suppose G has exactly r conjugacy classes. Show that $|G| \equiv r (mod\, 16)$. (**Hint** : Use part a) and the formula $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$.)

3.3. Let G be a non-abelian group of order p^3, p a prime. Show that G has exactly $p^2 + p - 1$ irreducible characters: p^2 of degree 1 and $p-1$ of degree p. (**Hint** : First show that $Z(G)$ has order p. Then show $[G, G] = Z(G)$ and use Corollary 2.23 and Theorem 3.12.)

This problem shows that Theorem 3.13 is false if one removes the assumption that $P \in Syl_p(G)$ is abelian.

3.4. (Isaacs's book, problems 3.4) Let G be a simple group having an irreducible character χ of prime degree p. Show that p, but not p^2, divides $|G|$. (**Hint** : Let $P \in Syl_p(G)$. If P is abelian, apply Theorem 3.13. If not, show that $\chi|_P$ is irreducible, hence $Z(P) \leq Z(\chi)$ by Schur's Lemma.)

3.5. Show that any finite non-abelian simple group has order at least 60. (**Hint** : You can use Burnside's $p^a q^b$-theorem.)

3.6. Let g be an element of a finite group G and let k be any integer coprime to $|g|$. Show that g is a commutator in G if and only if g^k is a commutator in G. (**Hint** : Use the character formula proved in class.)

MATH 555 List 4 Due Oct. 27, 2020

4.1. Let G be a finite group and let G have a subgroup H of prime index p. Suppose p is the smallest prime divisor of $|G|$. Show that $H \lhd G$. (**Hint** : Consider the action of G via conjugation on the set of G-conjugates of H.)

4.2. (Isaacs's book, problems 4.1) Let χ, ψ be characters of a finite group G. Prove that $\det(\chi\psi) = (\det\chi)^{\psi(1)} \cdot (\det\psi)^{\chi(1)}$.

4.3. (Isaacs's book, problems 4.3) Let $G = H \times K$ be the direct product of two finite groups H, K. Let $\alpha \in \mathrm{Irr}(H)$ and $\beta \in \mathrm{Irr}(K)$ be faithful. Show that $\alpha \otimes \beta$ is faithful if and only if $(|Z(H)|, |Z(K)|) = 1$.

4.4. Let G be a finite group generated by two involutions x, y. Show that, if $z = xy$ has order $n > 2$ then

$$G = \langle x, z | x^2 = z^n = 1, xzx^{-1} = z^{-1} \rangle$$

is the dihedral group of order $2n$. Show that

a) If n is odd, then any two involutions in G are conjugate.

b) If n is even, then $Z(G)$ has order 2; namely, it is generated by $z^{n/2}$.

c) Determine the character table of G.

4.5. Let g be an element of order n in a finite group G. Suppose g is a product of two squares in G and k is coprime to n. Show that g^k is also a product of two squares in G. (**Hint** : Use the relevant character formula proved in class.)

4.6. Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The symplectic group $S = Sp_{2n}(\mathbf{R})$ is defined to be the set of all $2n \times 2n$-matrices A over $\mathbf{R}$ such that ${}^tAJA = J$. Suppose the finite group G has a representation $\Phi : G \to S$, with character χ. Prove that the complex character χ cannot be irreducible. (**Hint** : Use the Frobenius-Schur indicator.)

4.7. (Isaacs's book, problems 4.11) Let G be a simple group and let $S \in Syl_2(G)$ be elementary abelian of order $q = 2^f$. Suppose $S = C_G(x)$ for any x, $1 \neq x \in S$. Prove the following statements.

a) Any 2-element of G is conjugate to an element of S. If $g \in G$, then either g is an involution, or g has odd order.

b) Let x be an involution in S. Then G has at least one involution $y \notin S$. Moreover, x and y are conjugate in G. (**Hint** : Suppose x, y are not conjugate. Using Problem **4.4**, show that x, y are centralized by some involution z. Then show that x, y commute with each other and hence belong to S.)

c) Using b), show that G has exactly $|G|/q$ involutions, and they are all conjugate.

d) $\nu_2(\chi) = 1$ for all $\chi \in \mathrm{Irr}(G)$. If $\chi \in \mathrm{Irr}(G)$ and $\chi \neq 1_G$, and x any involution, then $\chi(x) = \chi(1) - q$. (**Hint** : Suppose $\chi \neq 1_G$. First compute the integer $(\chi|_S, 1_S)_S$ to show that $\chi(x) \leq \chi(1) - q$. Then estimate $\sum_{\chi \in \mathrm{Irr}(G)} \chi(x)\chi(1)$ and use Corollary 4.6.)

MATH 555 List 5 Due Nov. 10, 2020

5.1. For any finite group X, set $b(X) = \max\{\chi(1) \mid \chi \in \mathrm{Irr}(X)\}$. If H is a subgroup of G, show that $b(H) \leq b(G) \leq (G : H)b(H)$. (**Hint** : Use Ind_H^G.)

5.2. Let χ be a monomial character of a finite group G and suppose K is a subgroup of G such that $\chi|_K \in \mathrm{Irr}(K)$. Show that $\chi|_K$ is a monomial character of K. (**Hint** : Use Mackey's formula.)

5.3. Consider the character table C of a finite group G. Show that the sum of all entries of any row is a non-negative rational integer. (**Hint** : Consider the action of G on G via conjugation, and the corresponding permutation character.) Also show that $|\det(C)|^2$ is a positive integer if we view C as a matrix. (**Hint** : Use the proof of Isaacs' Theorem 2.18.)

5.4. Let G act doubly transitively on a finite set Ω. Let $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$. Denote $H = Stab_G(\alpha)$, $K = Stab_G(\alpha) \cap Stab_G(\beta)$. Suppose $\phi \in \mathrm{Irr}(H)$ and $\phi|_K \in \mathrm{Irr}(K)$. Show that $(\phi^G, \phi^G)_G \leq 2$. (**Hint** : Use Mackey's formula.)

5.5. For any $\chi \in \mathcal{C}(G)$, set $\chi^{(n)}(g) = \chi(g^n)$. Suppose $G = H \times A$ with A abelian and $(|H|, n) = 1$. Show that $\chi^{(n)} \in \mathrm{Irr}(G)$ for all $\chi \in \mathrm{Irr}(G)$. (**Hint** : You may use – or may try to prove – Problem 4.7 in Isaacs' book.)

5.6. Consider the permutation actions of the symmetric group $G := Sym_n$, $n \geq 5$, on $\{1, 2, 3, \ldots, n\}$, respectively on the set of **unordered pairs** $\{i, j\}$, $1 \leq i \neq j \leq n$, with permutation character α, respectively, β. Show that there are some irreducible characters χ_1 of degree $n-1$ and χ_2 of degree $n(n-3)/2$ of G such that $\alpha = 1_G + \chi_1$ and $\beta = 1_G + \chi_1 + \chi_2$. (**Hint** : Mimick the analysis done in class for rank 2 and 3 permutation actions of $GL_n(q)$.)

5.7. (Continuation of problem **4.7**) Let G be a simple group and let $S \in Syl_2(G)$ be elementary abelian of order $q = 2^f$. Suppose $S = C_G(x)$ for any x, $1 \neq x \in S$. Prove that $|G| = q(q+1)(q-1)$ following the steps below.

a) Let $N = N_G(S)$. Show that N acts transitively on $q-1$ involutions of S, hence $|N| = q(q-1)$. (**Hint** : By **4.7** (c), any two involutions

$s, s' \in G$ are G-conjugate. Assuming $s, s' \in S$ and $s' = gsg^{-1}$, show that $g \in N$.)

b) Fix a $\lambda \in \mathrm{Irr}(N)$ with $\lambda \neq 1_N$ and $Ker(\lambda) \geq S$. Show that $\lambda^G(s) = \lambda(1)$ for all $s \in S$, $s \neq 1$. (**Hint** : Any involution of S is inside N, so $s^G \cap N = s^N = S \setminus \{1\}$.)

c) If $\chi \in \mathrm{Irr}(G)$ is a constituent of λ^G, then $\chi(1) \geq q+1$. (**Hint** : Since $|N/S|$ is odd, $\lambda \neq \overline{\lambda}$. Due to **4.7** (d), χ is real, hence $\lambda + \overline{\lambda}$ enters $\chi|_N$ and $(\chi|_S, 1_S)_S \geq 2$. Using **4.7** (d), compute $(\chi|_S, 1_S)_S$.)

d) Decompose $\lambda^G = \sum_\chi a_\chi \chi$ with $\chi \in \mathrm{Irr}(G)$, $a_\chi \geq 1$. Let $s \in S \setminus \{1\}$. By c) and **4.7** (d),

$$\lambda(1) = \lambda^G(s) = \sum_\chi a_\chi(\chi(1) - q) = (G : N)\lambda(1) - q \sum_\chi a_\chi,$$

hence $\sum_\chi a_\chi \leq \lambda(1)$. Using this inequality and the fact that $\lambda(1)$ divides $q - 1 = |N/S|$, show that $(G : N) = q + 1$. Consequently, $|G| = q(q^2 - 1)$.

MATH 555 List 6 Due Nov. 24, 2020

6.1. Let $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$. Show that $\theta^G \in \mathrm{Irr}(G)$ if and only if $I_G(\theta) = N$.

6.2. Let G be solvable and suppose that every $\chi \in \mathrm{Irr}(G)$ is quasiprimitive. Show that G is abelian. (**Hint** : First show that every normal abelian subgroup of G is central. Then apply induction hypothesis about quasiprimitivity to $G/Z(G)$.)

6.3a (not assigned, only for reference) Let $N \lhd G$ and suppose G/N is solvable. Let $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(N)$ with $(\chi|_N, \theta)_N > 0$. Show that $\chi(1)/\theta(1)$ divides $(G : N)$. (**Hint** : Use Corollary 6.19.)

6.3. Suppose G has exactly one nonlinear irreducible character. Show that G' is an elementary abelian p-group (for a prime p). However, G is not always nilpotent. (**Hint** : Look at λ^G, λ any nonprincipal irreducible character of G'.)

6.4. Let G be an M-group and suppose $N \lhd G$ with $(|N|, |G/N|) = 1$. Show that N is an M-group. (**Hint** : Let $\theta \in \mathrm{Irr}(N)$. Using Mackey's Theorem, find an $H \leq G$ and $\lambda \in \mathrm{Irr}(H)$ such that $\lambda(1) = 1$, $\lambda^G \in \mathrm{Irr}(G)$ and $((\lambda^M)|_N, \theta)_N > 0$ for $M = NH$ (you may need to replace θ by a conjugate of it). Use Problem **6.3a** to show that $\lambda^M(1) = \theta(1)$, then use Problem **5.2**.)

6.5. Let $N \lhd G$ with G/N cyclic. Let $\theta \in \mathrm{Irr}(N)$ be invariant in G and suppose that $(\theta(1), |G/N|) = 1$. Show that θ is extendible to G. (**Hint** : Using the fact G/N is cyclic, show that $G/Ker\,(det\,\theta)$ is abelian. Then use Theorem 6.25.)

6.6. Let $1 = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G$. Suppose H_i/H_{i-1} is nonabelian for all i. Show that G has an irreducible character χ of degree at least 2^n. (**Hint** : Proceed by induction on n and use Corollary 6.17.)

6.7 Let $\mathbb{F}$ be a field of characteristic 5, and let $G = \mathrm{Sym}(\{1, \ldots, 5\}) \cong S_5$ act on $V = \langle e_1, \ldots, e_5 \rangle_{\mathbb{F}}$ via $\sigma(e_i) = e_{\sigma(i)}$. Consider the submodules $I \subset U \subset V$, where $I = \langle \sum_i e_i \rangle$ and $U = \{\sum_i a_i e_i \mid \sum_i a_i = 0\}$.

(a) Using the bilinear form $(\sum_i x_i e_i, \sum_i y_i e_i) = \sum_i x_i y_i$, show that $V/I \cong U^*$ as G-modules. (**Hint** : Assign to $v \in V$ an element in U^*.)

(b) Show that $\mathrm{Hom}_{\mathbb{F}G}(\mathbb{F}, U) \neq 0$, but $\mathrm{Hom}_{\mathbb{F}G}(U, \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}G}(\mathbb{F}, V/I) = 0$.

MATH 555 Problem List 7 Due Dec. 8, 2020

7.1. Problem (7.1) from Isaacs' book.

7.2. Problem (7.2) from Isaacs' book. (**Hint :** Note that $H \setminus M$ is a T.I. set. Mimic the proof of Frobenius' theorem for $\mathrm{Irr}(H/M)$.)

7.3. Problem (7.3) from Isaacs' book. (**Hint :** Mimic the proof of Theorem (7.2) from the book to construct $\chi \in \mathrm{Irr}(G)$ such that $\vartheta = \chi - \xi(1)1_G$. Also use Frobenius' reciprocity to get $\chi|_H = \xi$.)

7.4. Problem (7.4) from Isaacs' book. (**Hint :** Again follow the proof of Theorem (7.2) from the book and use Frobenius' reciprocity as in the previous exercise. Use Lemma (7.3) from the book, and estimate $|(G \setminus \cup_{g \in G} H^g) \cup \{1\}|$.)

7.5. Let A be any nontrivial subgroup of the multiplicative group of the finite field $\mathbf{F}_q$, and consider the group G of all maps $\varphi : x \mapsto ax+b$ on $\mathbf{F}_q$, where $a \in A$ and $b \in \mathbf{F}_q$, under the usual composition. Show that G is a Frobenius group.

7.6. Let V be a 2-dimensional vector space over the field $\mathbb{F}_q$ of $q \geq 4$ elements. Consider the group $G = SL(V)$ of all linear transformations of V with determinant 1 and its action on the set Ω of 1-dimensional subspaces of V. Show that the induced action of $G/Z(G)$ on Ω yields a Zassenhaus group.