MATH 553 Problem List 1 Spring 2022

Let k be an algebraically closed field.

1.1. Let G be a linear algebraic group.
(a) Show that $C_G(x)$ is a closed subgroup, for any $x \in G$.
(b) Show that $Z(G)$ is a closed subgroup of G.
(c) If H is a subgroup of G, then so is its *closure* $\bar{H}$ (the smallest closed subset of the variety G which contains H). (**Hint**: Use continuity of the inversion and the left multiplication by $x \in H$ to get $\bar{H}^{-1} = \bar{H}$ and $x\bar{H} = \bar{H}$.)

1.2. Show that $CSp_{2n} = Sp_{2n}.Z(CSp_{2n})$.

1.3. Show that the set $\{(x, y) \in k^2 | xy = 0\}$ is not irreducible but is connected in the Zariski topology.

1.4. Prove that GO_{2n} is not connected when $char(k) \neq 2$.

1.5. Prove that each of the groups T_n, U_n, and D_n is connected.

1.6. Show that $\dim T_n = n(n + 1)/2$, $\dim U_n = n(n - 1)/2$, and $\dim D_n = n$.

MATH 553 Problem List 2

2.1. Show that the set G_u of unipotent elements in any linear algebraic group G is closed. (**Hint** : Embed G in some GL_n and look at characteristic polynomials of unipotent elements in GL_n.)

2.2. Consider the *projective general linear group* $PGL_n := GL_n/Z$, where $Z = \{cI_n | c \in k^\times\}$, as abstract group. Let V be the n-dimensional vector space over k on which GL_n acts naturally, and let $V^* := Hom(V, k)$ be the dual space. The action of GL_n on $V \otimes V^*$ defines a group homomorphism $\varphi : GL_n \to GL_{n^2}$.

(a) Show that φ is a morphism of algebraic groups. Conclude that its image is a closed subgroup of GL_{n^2}.

(b) Show that $ker(\varphi) = Z$. Conclude that PGL_n is a linear algebraic group. (**Hint** : Using the formula for tensor product of two matrices, first show that any $g \in ker(\varphi)$ is diagonal.)

2.3. Show that the group of all automorphisms of the **algebraic** group $\mathbf{G}_a$ is isomorphic to $\mathbf{G}_m$. (**Hint** : Recall that $k[G] = k[T]$ in this case. If φ is such an automorphism, find $\varphi^*(1)$ and $\varphi^*(T)$.)

2.4. Work out the details of Example 3.13 given in class.

2.5. Suppose G is a nontrivial connected nilpotent linear algebraic group. Show that $\dim Z(G) \geq 1$.

MATH 553 Problem List 3

3.1. (i) Show that the rank of Sp_{2n}, resp. SO_{2n+1}, is n.

(ii) Show that $Sp_{2n} \cap T_{2n}$, resp. $SO_{2n+1} \cap T_{2n+1}$, is a Borel subgroup of Sp_{2n}, resp. of SO_{2n+1}.

3.2. Let G be a connected linear algebraic group.

(i) Suppose that a Borel subgroup of G is nilpotent. Prove that $G = B$ and so G is nilpotent. (**Hint**: Consider a counterexample of minimal dimension and use Exercise 2.5 and Proposition 6.8.)

(ii) Show that G is solvable if $\dim G \leq 2$.

3.3. (i) Let G be a connected linear algebraic group. Then $G = G_s$ if and only if G is a torus. (**Hint**: Apply Theorem 4.4 to a Borel subgroup of G and then use Exercise 3.2.)

(ii) Give an example of a closed subgroup of GL_n which consists of only semisimple elements but which is not conjugate to any subgroup of D_n. (**Hint**: Think about finite groups!)

3.4. (i) Let G be a linear algebraic group. Show that $R(G) = (\cap_{B \text{ Borel}} B)^\circ$.

(ii) Show that Sp_{2n} is semisimple. (**Hint**: Follow Example 6.17 in class.)

3.5. Let $V = k^3$ be a 3-dimensional vector space with a fixed basis (e_1, e_2, e_3), and let $P = Stab_{SL(V)}(\langle e_1, e_2 \rangle_k)$. Consider

$$S = \{g \in P | g(e_i) = \lambda_{g,i} e_i \text{ for some } \lambda_{g,i} \in k^\times, 1 \leq i \leq 3\},$$

a torus of P. Find $N_P(S)$, $N_P(S)^\circ$, $C_P(S)$, and $C_P(S)^\circ$, and verify that $N_P(S)/C_P(S)$ is finite.

MATH 553 Problem List 4

4.1. Let G be a linear algebraic group and $i : g \mapsto g^{-1}$ be the inversion morphism. Show that the differential of i is the map $X \mapsto -X$ for $X \in Lie(G)$. (**Hint**: First compute the differentials of the morphisms $\mathrm{Id} : g \mapsto g$ and $g \mapsto 1$. Then apply Proposition 7.7 and Example 7.8 to the morphism $\mu \circ (i, \mathrm{Id}) : G \to G \times G \to G$.)

4.2. Show that the Lie algebra of T_n, respectively of U_n, D_n, can be identified with the Lie subalgebra of all upper triangular, strictly upper triangular, diagonal $n \times n$-matrices, respectively. (**Hint**: Use Theorem 7.9.)

4.3. Let k be a field of characteristic $p > 0$, $G = SL_3$, and let

$$\varphi : \mathbf{G}_a \to G, \; t \to \begin{pmatrix} 1 & t & t^p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Show that φ defines an isomorphism of algebraic groups between $\mathbf{G}_a$ and $H = Im(\varphi)$.

(ii) Determine $Lie(H)$, as a subalgebra of gl_3.

(iii) Show that $C_G(H)$ is **not** equal to

$$C_G(Lie(H)) := \{g \in G \mid Ad(g)X = X, \; \forall X \in Lie(H)\}.$$

(iv) Suppose $p = 3$. Show that $Lie(C_G(H))$ is **not** equal to

$$C_{Lie(G)}(H) := \{X \in Lie(G) \mid Ad(h)X = X, \; \forall h \in H\}.$$

4.4. (i) Let G be a linear algebraic group. Show that $Z(G) \leq Ker(Ad)$.

(ii) Determine $Ker(Ad)$ for each of the following groups: GL_n, U_n, T_n.

(iii) Let k be a field of characteristic $p > 0$, and let

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^p & b \\ 0 & 0 & 1 \end{pmatrix} \mid a \in k^\times, b \in k \right\}.$$

Show that $Z(G) < Ker(Ad) < G$ (proper containments!).

4.5. Show that the Lie algebras of PGL_n and SL_n are isomorphic if and only if $char(k)$ does not divide n. (**Hint**: For the non-isomorphism, think about the center $Z(\mathcal{L}) := \{x \in \mathcal{L} \mid [x, y] = 0, \; \forall y \in \mathcal{L}\}$ of the Lie algebras $\mathcal{L}$ in question.)

5.1. Let $G = Sp_{2n}$. It is known that

$$Lie(G) = \{X \in gl_{2n} \mid X^T J_{2n} = -J_{2n} X\},$$

where J_{2n} is as in the definition of symplectic groups. Let $T = D_{2n} \cap G$ be a maximal torus of G (see Exercise 3.1).

(i) Assume $n = 2$. Find the roots and root subspaces of $Lie(G)$. For each root α, exhibit a one-dimensional closed subgroup $U_\alpha \leq G$ whose Lie algebra is the corresponding root subspace.

(ii) Generalize to the case of arbitrary n: show that the root system of Sp_{2n} is of type C_n. Conclude that $\dim Sp_{2n} = 2n^2 + n$.

5.2. Show that

$$[GL_n, GL_n] = SL_n,\ [CSp_{2n}, CSp_{2n}] = Sp_{2n},\ [CO_n^\circ, CO_n^\circ] = SO_n.$$

(**Hint**: Compare the root systems of the two groups in each case and apply Proposition 6.20(c) and List of Isogeny types for simple algebraic groups).

5.3. (*A general construction of abstract root systems.*) Let $E = \mathbb{R}^n$ be a Euclidean space with scalar product $(\cdot,\cdot)$. Let $(e_1, \dots, e_n)$ and $\Lambda = \langle e_1, \dots, e_n \rangle_{\mathbb{Z}}$, the free $\mathbb{Z}$-module generated by $e_1, \dots, e_n$. Suppose that $(u, v) \in \mathbb{Z}$ for all $u, v \in \Lambda$. Let

$$\Lambda_1 := \{v \in \Lambda \mid (v, v) = 2\},\ \Lambda_2 := \{v \in \Lambda \mid (v, v) \in \{1, 2\}\}.$$

Show that, for each $i = 1, 2$, if Λ_i is non-empty, then it is an abstract root system in $E_i := \Lambda_i \otimes_{\mathbb{Z}} \mathbb{R}$.

5.4. Let $E = \mathbb{R}^9 = \langle e_1, \dots, e_9 \rangle_{\mathbb{R}}$ be a Euclidean space with standard scalar product: $(e_i, e_j) = \delta_{i,j}$. Consider the free $\mathbb{Z}$-submodule

$$\Lambda := \left\{ \sum_{i=1}^{9} x_i e_i \in E \mid \sum_{i=1}^{9} x_i = 0,\ x_i + x_j + x_k \in \mathbb{Z},\ 1 \leq i, j, k \leq 9 \right\}.$$

In the notation of Exercise 5.4, show that Λ_1 is an abstract root system of type E_8. (This construction actually arises from a so-called *orthogonal decomposition* of the complex simple Lie algebra of type A_2.)

5.5. Let $V = \mathbb{F}_2^3$ and let $E = \mathbb{R}^8 = \langle e_v \mid v \in V \rangle_{\mathbb{R}}$ be a Euclidean space with scalar product: $(e_u, e_v) = \delta_{u,v}/2$. A subset X of V is called an *affine plane* in V, if $|X| = 4$ and the four vectors in X add up to 0. Let $\mathcal{A}$ be the set of all affine planes in V. Show that

$$\Phi := \{\pm 2e_v \mid v \in V\} \cup \left\{ \sum_{x \in X} \epsilon_x e_x \mid \epsilon_x = \pm 1,\ X \in \mathcal{A} \right\}$$

is an abstract root system in E. What is the type of Φ, and why?