Introduction to Mathematical Reasoning
Homework 7
Due Tuesday December 10, 2019

(1) Prove that the function $h : \mathbb{Z} \longrightarrow \mathbb{O}$ defined by $h(x) = 2x - 3$ is bijective.

(2) Prove that the function $f : [1, \infty) \to [2, \infty)$ defined by $f(x) = x + 1/x$ is bijective.

(3) Define $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ by $(m, n) \mapsto 2^{m-1}3^{n-1}$. Prove that f is injective, but not surjective. Does this fact imply that $\mathbb{N} \times \mathbb{N}$ is not equivalent to $\mathbb{N}$, and why?

(4) Let $\mathbb{Z}[\sqrt{2}] = \{x \in \mathbb{R} \mid x = a + b\sqrt{2},\ a, b \in \mathbb{Z}\}$. Is $\mathbb{Z}[\sqrt{2}]$ is countable? Why? (*Hint*: You may need the irrationality of $\sqrt{2}$ and some operations with countable sets.)

(5) Let $\mathbb{D}$ denote the set of integers $n > 1$ that are powers of 2 or of 3. Construct a bijection $\mathbb{N} \to \mathbb{D}$ and hence conclude that $\mathbb{D}$ is equivalent to $\mathbb{N}$.

(6) Let $X = \{3n + 2 \mid n \in \mathbb{N}\}$ and Y be the set of integers which are perfect squares (like 1, 4, 9, etc.) Find a bijective map $f : X \longrightarrow Y$.

(7) Determine all pairs of real numbers m, b such that the function $f(x) = mx + b$ yields a bijection between $\mathbb{R}$ and $\mathbb{R}$.

(8) (a) For any $n \in \mathbb{Z}_{\geqslant 0}$, construct a bijection between $\mathbb{N}$ and

$$\{0, -1, -2, \ldots, -n + 1\} \cup \mathbb{N} = \{-n + 1, -n + 2, \ldots, 0, 1, 2, 3, \ldots\}.$$

(b) Using (a), but without using the Theorem proved in class about unions of countable sets, prove that if A is a finite set and B is an infinite countable set disjoint from A, then $A \cup B$ is countable.

(9) Prove that the set $\mathbb{X}$ of all finite subsets of $\mathbb{N}$ is countable. (*Hint:* You can either use operations with countable sets, or perhaps use Prime Factorization Theorem and the set $\mathbb{P}$ to construct an injection between $\mathbb{X}$ and $\mathbb{N}$.)

(10) Let S be the set of all open intervals in $(-\infty, \infty)$ of the form

$$S = \{(0, \frac{1}{2^n}) \mid n \in \mathbb{N}\}.$$

Prove that S is countable.

(11) Prove that $\mathbb{R} \setminus \mathbb{N}$ is uncountable. (*Hint*: Use the fact that $\mathbb{R}$ is uncountable.)