Math 300, Homework 6
Due Wednesday, November 27, 2019

(1) Let $\mathcal{X}$ denote the set of all possible finite subsets of $\mathbb{N}$. Define a relation $\sim_1$ on $\mathcal{X}$ by $S_1 \sim_1 S_2$ if S_1 and S_2 have the same cardinality, and a relation $\sim_2$ by $S_1 \sim_2 S_2$ if $S_1 \cap S_2 \neq \emptyset$. Which ones among $\sim_1$ and $\sim_2$ are an equivalence relation? Prove your answer.

(2) Let S be the set of all quadratic polynomials with real roots. That is,

$$S = \{\ f(x) \ = \ ax^2 \ + \ bx \ + \ c \mid a, b, c \in \mathbb{R},\ b^2 - 4ac \geq 0\ \}.$$

Define a relation $\sim$ on S by

$$f \sim g \iff \text{the two equations } f(x) = 0,\ g(x) = 0 \text{ have the same sets of solutions.}$$

Show that $\sim$ is an equivalence relation. Name three elements in the equivalence class of $x^2 - 3x + 2$.

(3) Define a relation on $\mathbb{R}$ given by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Show that $\sim$ is an equivalence relation. Find the equivalence classes of 0, 26/11, and $\sqrt{2}$ under this equivalence relation. Name three elements in each of these equivalence classes.

(4) Define a relation on $\mathbb{N}_{>1}$ given by $x \sim y$ if and only if the prime factorizations of x and y have the same number of 2's. (For instance, $8 \sim 24$ but $4 \not\sim 24$.) Show that $\sim$ is an equivalence relation and find the equivalence classes of 2 and 5 under this equivalence relation. Name three elements in each of these equivalence classes.

(5) Let $x, y \in \mathbb{R}$ and define a relation $\sim$ on $\mathbb{R}$ by $x \ \sim \ y$ if $|x| = |y|$. Prove that $\sim$ is an equivalence relation. Find the equivalence classes of 0, 5 and $\sqrt{2}$ under the relation $\sim$.

(6) Find the remainder when (a) $238 + 494 - 44$ is divided by 9; (b) $182 \cdot 144$ is divided by 13; (c) 11^{27} is divided by 3.

(7) Prove by induction on n that if $x \equiv y (\text{mod } m)$ then $x^n \equiv y^n (\text{mod } m)$ for all $n \in \mathbb{N}$.

(8) Use congruences modulo 7 and exhaustion of cases to prove that, for any integer k, $5(k^6 - k^3) \equiv 0$ or 3 modulo 7.