1. (a) What does it mean to say that a set $S = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of vectors in $\mathbb{R}^n$ is *linearly independent*? Give the precise definition in one or more full sentences. Then describe what this means in terms of the matrix A with columns $\mathbf{u}_1, \dots, \mathbf{u}_k$.

(b) Do the three vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ make up a linearly independent set? Justify your answer using the matrix A as in (a).

2. (a) What is meant by the *span* of a set $S = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of vectors in $\mathbb{R}^n$? Give the precise definition in one or more full sentences. Then describe what this means in terms of the matrix A with columns $\mathbf{u}_1, \dots, \mathbf{u}_k$.

(b) Suppose that $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Is the span of the set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ all $\mathbb{R}^3$? Justify your answer using the matrix A as in (a).

3. Answer questions 1 and 2 when $\mathbf{u}_3$ is changed to $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

4. In (a)-(c) below we suppose that we have a system of equations $A\mathbf{x} = \mathbf{b}$ and that we have used row operations to transform the augmented matrix $[A \;\; \mathbf{b}]$ to the reduced row-echelon form $[R \;\; \mathbf{c}]$ given below. In each case, determine (i) whether the original equations have a solution; (ii) if they do have a solution, whether or not it is unique; and (iii) if it is not unique, how many free parameters there are in the solution. Then write the solution explicitly as a fixed vector plus a linear combination of vectors $\mathbf{y}$ that satisfy $A\mathbf{y} = 0$ with the free variables as coefficients.

(a) $[R \;\; \mathbf{c}] = \begin{bmatrix} 1 & 5 & 0 & 2 & 8 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $[R \;\; \mathbf{c}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

(c) $[R \;\; \mathbf{c}] = \begin{bmatrix} 0 & 1 & 2 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

5. In each part below, give a $m \times n$ matrix R *in reduced row-echelon form* satisfying the given condition, or explain briefly why it is impossible to do so.

(a) $m = 3$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{c}$ has a solution for all $\mathbf{c}$.

(b) $m = 3$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{0}$ has a unique solution.

(c) $m = 4$, $n = 3$, and the equation $R\mathbf{x} = \mathbf{c}$ has a solution for all $\mathbf{c}$.

(d) $m = 4$, $n = 3$, and the equation $R\mathbf{x} = \mathbf{0}$ has a unique solution.

(e) $m = 4$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{0}$ has no solution.

(f) $m = 4$, $n = 4$, and the equation $R\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

(g) $m = 4$, $n = 4$, and for every $\mathbf{c}$ the equations $R\mathbf{x} = \mathbf{c}$ have a solution containing a free parameter.

6. (a) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are solutions of the system of equations $A\mathbf{x} = \mathbf{0}$. Show that $c\mathbf{u} + d\mathbf{v}$ is also a solution, for any scalars c and d.

(b) Why does the above conclusion not hold (in general) if the system of equations is $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b}$ a nonzero vector?

7. Suppose that

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 \\ 3 & -3 \\ 4 & 1 \end{bmatrix}.$$

Which of the following quantities are defined? Calculate those that are defined.

(a) BA (b) AB (c) $3C - 2B^T$ (d) BC (e) CAB (f) $C + 2A$ (g) C^TC.

8. Let A be an $m \times n$ matrix of rank r. What can you conclude about m, n, and r (other than $r \le m$ and $r \le n$ which is always true) if the equation $A\mathbf{x} = \mathbf{b}$ has

(a) exactly one solution for some $\mathbf{b}$ and no solution for other $\mathbf{b}$?

(b) infinitely many solutions for all $\mathbf{b}$?

(c) exactly one solution for every $\mathbf{b}$?

(d) infinitely many solutions for some $\mathbf{b}$ and no solutions for other $\mathbf{b}$?

(e) exactly one solution when $\mathbf{b} = \mathbf{0}$?

9. (a) Suppose that A and B are 4×5 matrices and that B is obtained from A by the row operation given below. In each case give an elementary matrix E such that $B = EA$.

(i) $\mathbf{r}_1 \leftrightarrow \mathbf{r}_4$, (ii) $\mathbf{r}_3 + 3\mathbf{r}_2 \to \mathbf{r}_3$, (iii) $7\mathbf{r}_2 \to \mathbf{r}_2$.

(b) Give the inverses of the elementary matrices found in (i), (ii), and (iii) above. (You can do this without calculation; think about reversing the corresponding row operations.)

10. A certain 3×3 matrix $A = [\,\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\,]$ has reduced row echelon form $R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) Find a nontrivial linear relation on the columns of A, that is, a relation $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = \mathbf{0}$ with c_1, c_2, and c_3 not all zero.

(b) Suppose that $\mathbf{a}_1 = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\mathbf{a}_3$.

11. (a) Suppose that A is a square matrix. What does it mean to say that A is *invertible*? (Give the definition, not one of the many equivalent conditions in Theorem 2.6 of the text.)

(b) Suppose that A and B are invertible $n \times n$ matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

(c) Suppose that A is an invertible $n \times n$ matrix. Show that $(A^T)^{-1} = (A^{-1})^T$.

12. Use row reduction to show that the matrix $\begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ is invertible and to find its inverse.

13. Do the True-False questions from Sections 1.1–1.4, 1.6, 1.7, 2.1, 2.3, and 2.4 that are listed in the homework assignments.