Sub-level sets of the Bergman kernel on tube domains

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January 8, 2016

Setup

$D \subset \mathbb{R}^n$, a bounded convex domain.

$\Omega := \mathbb{R}^n + iD$, a pseudoconvex tube domain in $\mathbb{C}^n$.

$K_\Omega : \Omega \times \Omega \to \mathbb{C}$, the Bergman kernel of Ω.

$x \in D$,

$$K_D(x) := K_\Omega(ix, ix).$$

$M > 0$,

$$D^M := \{x \in D : K_D(x) < M\}.$$

- equi-affine invariant;
- strongly convex;
- exhaust D as $M \to \infty$.

Fact: For strongly convex D,

$$\lim_{M\to\infty} a_n \frac{\mathrm{vol}(D \setminus D^M)}{M^{\frac{-1}{n+1}}} = \int_{\partial D} \kappa(x)^{\frac{1}{n+1}}\, dS_{\mathrm{Euc}}(x) = \sigma_{\mathrm{aff}}(\partial D).$$

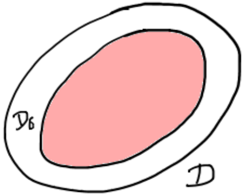
σ_{aff}: Blaschke affine surface area measure.

Q. Does the LHS limit exist for all convex domains?

The Convex Floating Body

$D \subset \mathbb{R}^n$ is a convex body.

$D_\delta :=$ intersection of all the halfspaces whose defining hyperplanes cut off a set of volume δ from D.



- equi-affine invariant.
- strictly convex (but not necessarily $\mathcal{C}^1$-smooth);
- exhaust D as $\delta \to 0$.

The Convex Floating Body

Extended affine surface area:

Theorem (Schütt-Werner)

Let $D \subset \mathbb{R}^n$ be a convex body. If D is strongly convex, then

$$\lim_{\delta \to 0} b_n \frac{\mathrm{vol}(D \setminus D_\delta)}{\delta^{\frac{2}{n+1}}} = \int_{\partial D} \kappa(x)^{\frac{1}{n+1}} \, dS_{\mathrm{Euc}}(x) = \sigma_{\mathrm{aff}}(\partial D)$$

for some $b_n > 0$. The limit on the LHS always exists.

'Lower'-dimensional affine surface area:

Theorem (Schütt)

Let $P \subset \mathbb{R}^n$ be a convex polytope with non-empty interior. Then,

$$\lim_{\delta \to 0} c_n \frac{\mathrm{vol}(P \setminus P_\delta)}{\delta |\ln \delta|^{n-1}} = \sigma(\partial P)$$

for some $c_n > 0$, where σ is a measure supported on the vertices of P.

Main Result

For a strongly convex domain $D \Subset \mathbb{R}^n$, $\{D^M\}_{M>0}$ and $\{D_\delta\}_{\delta>0}$ yield the same boundary measure.

Theorem (G.)

Let $D \subset \mathbb{R}^n$ be a bounded convex domain. There are dimensional constants $\ell_n > 0$ and $u_n > 0$ such that

$$D^{\frac{\ell_n}{\delta^2}} \subseteq D_\delta \subseteq D^{\frac{u_n}{\delta^2}}.$$

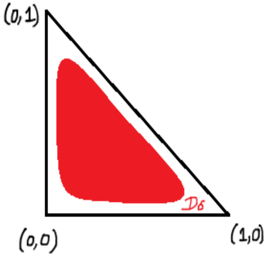
for small enough $\delta > 0$. Or,

$$D_{\sqrt{\frac{u_n}{M}}} \subseteq D^M \subseteq D_{\sqrt{\frac{\ell_n}{M}}}.$$

for large enough $M > 0$.

$\ell_n = \dfrac{1}{4^{n+1}}$ and $u_n = \dfrac{n!n^{2n}(\omega_n)^2}{\pi^n}$, $\omega_n =$ volume of the unit ball in $\mathbb{R}^n$.

An Example



$$D = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, x + y < 1\}.$$

$$K_D\big((x, y)\big) = \frac{1}{\pi^2} \int_{\mathbb{R}^2} \frac{st(s - t)}{s(1 - e^{-2t}) - t(1 - e^{-2s})} e^{-2xs-2yt} \, ds \, dt.$$

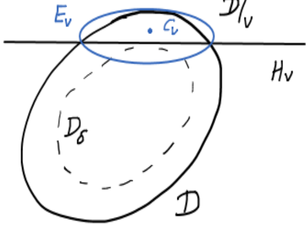
$$D_\delta = \Big\{(x, y) \in \mathbb{R}^2 : \min\Big(xy, (1 - x - y)y, (1 - x - y)x\Big) > \delta/2\Big\}.$$

As $M \to \infty$, $D_{\frac{1}{4\sqrt{M}} + o(\frac{1}{\sqrt{M}})} \subseteq D^M \subseteq D_{\frac{1}{2\pi\sqrt{M}} + o(\frac{1}{\sqrt{M}})}$.

Proof of the Theorem - Step 1

For $v \in S^{n-1}$,

$$H_v := \text{hyperplane perpendicular to } v \text{ that cuts off volume } \delta \text{ from } D.$$
$$D|_v := \text{slice of } D \text{ of volume } \delta \text{ cut off by } H_v.$$



- $E \subset_{\text{cvx.}} \mathbb{R}^n$ origin-symmetric. Nazarov and Blaschke-Santaló:

$$K_E(0) \leq \frac{n!\omega_n^2}{\pi^n} \frac{1}{\text{vol}(E)^2}.$$

- $E_v :=$ min. vol. ellipsoid $\supseteq D|_v$.
  $c_v :=$ center of E_v.
  John: $\widetilde{E}_v := c_v + \frac{1}{n}(E_v - c_v) \subseteq D|_v$.

$$K_D(c_v) \leq K_{\widetilde{E}_v}(c_v) \leq \frac{n!\omega_n^2}{\pi^n} \frac{1}{\text{vol}(\widetilde{E}_v)^2} = \frac{n!n^{2n}\omega_n^2}{\pi^n} \frac{1}{\text{vol}(E_v)^2} \leq \frac{n!n^{2n}\omega_n^2}{\pi^n} \frac{1}{\delta^2}.$$

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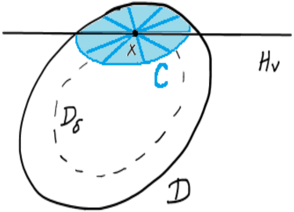
The image of $v \mapsto c_v$ 'surrounds' D_δ. So, $D_\delta \subseteq D^{u_n \delta^{-2}}$.

Proof of the Theorem - Step 2

Fix a $v \in S^{n-1}$ and $x \in H_v \cap D$. WLOG, let x be the origin and $H_v = \{z_n = 0\}$.

- Lempert, Błocki: $K_D(x) \geq \dfrac{1}{\mathrm{vol}_{\mathbb{C}^n}(\mathrm{I}_\Omega(x))}$,

  $$\mathrm{I}_\Omega(x) := \{\phi'(0) : \phi \in \mathcal{O}(\mathbb{D};\Omega), \phi(0) = ix\}.$$



- $\frac{1}{2}\mathrm{I}_\Omega(x) \subseteq C + iC$,

$C =$ the largest origin-symmetric convex body contained in D.

- $\mathrm{vol}(C) \leq 2\delta \Rightarrow \mathrm{vol}_{\mathbb{C}^n}(\mathrm{I}_\Omega(x)) \leq (2)^{2n}\, 4\delta^2$.

Every $x \in \partial D_\delta$ is in some $H_v \cap D$. So, $D_\delta \supseteq D^{\ell_n \delta^{-2}}$.

On the constants

- ℓ_n and u_n are unlikely to be optimal.
- For the triangle, better constants than $\ell_2 = \frac{1}{64}$ and $u_2 = 16$.
- Affine invariant constants:

$$\begin{aligned} \ell_D &:= \liminf_{\delta \to 0} \left(\sup\{\ell > 0 : D^{\ell/\delta^2} \subseteq D_\delta\} \right); \\ u_D &:= \limsup_{\delta \to 0} \left(\inf\{u > 0 : D_\delta \subseteq D^{u/\delta^2}\} \right). \end{aligned}$$

  - D strongly convex $\Rightarrow \ell_D = u_D (= \frac{4}{9\pi^2}$, when $n = 2)$.
    $\ell_D = u_D \Rightarrow D$ is strongly convex?
  - D triangle or parallelogram $\Rightarrow \ell_D = \frac{1}{4\pi^2}$ and $u_D = \frac{1}{16}$.
    Same value for all planar polygons?

THANK YOU.