# Revised Proof for the Uniqueness Theorem for "No Collapse" Interpretations of Quantum Mechanics

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**Abstract**

We show that the uniqueness theorem for 'no collapse' interpretations of Bub and Clifton [2] can be proved without the 'weak separability' assumption.

In [2], Bub and Clifton proved a uniqueness theorem for 'no collapse' interpretations of quantum mechanics. The proof was repeated, with minor modifications, in [1]. The original proof involved a 'weak separability' assumption (introduced to avoid a dimensionality constraint) that required several preliminary definitions and considerably complicated the formulation of the theorem. Sheldon Goldstein has pointed out that the proof goes through without this assumption.

The question at issue is this: Consider an arbitrary 'pure' quantum state represented by a ray $e$ on a Hilbert space $\mathcal{H}$ and the Boolean algebra or lattice, $\mathcal{B}(R)$, generated by the eigenspaces of a single observable $R$ on $\mathcal{H}$.

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The probabilities defined by $e$ for the ranges of values of $R$ can be represented by a probability measure over the 2-valued homomorphisms on $\mathcal{B}(R)$. What is the maximal lattice extension $\mathcal{D}(e,R)$ of $\mathcal{B}(R)$, generated by eigenspaces of observables other than $R$, on which there exist sufficiently many 2-valued homomorphisms so that we can represent the probabilities defined by $e$, for ranges of values of $R$ and these additional observables, in terms of a measure over the 2-valued homomorphisms on $\mathcal{D}(e,R)$? The theorem provides an answer to this question on the assumption that $\mathcal{D}(e,R)$ is invariant under automorphisms of the lattice $\mathcal{L}$ of all subspaces of Hilbert space that preserve the state $e$ and the 'preferred observable' $R$, without the further assumption of 'weak separability.'

We sketch a revised proof of the theorem here.

*Theorem:* Consider a quantum system $S$ in a (pure) quantum state represented by a ray $e$ in an $n$-dimensional Hilbert space $\mathcal{H}$, and an observable $R$ with $m \leq n$ distinct eigenspaces $r_i$ of $\mathcal{H}$. Let $e_{r_i} = (e \vee r_i^{\perp}) \wedge r_i$, $i = 1, 2, \ldots, k \leq m$, denote the nonzero projections of the ray $e$ onto the eigenspaces $r_i$. Then $\mathcal{D}(e,R) = \mathcal{L}_{e_{r_1} e_{r_2} \ldots e_{r_k}} = \{p : e_{r_i} \leq p \text{ or } e_{r_i} \leq p^{\perp}, i = 1, \ldots, k\}$ is the unique maximal sublattice of $\mathcal{L}(\mathcal{H})$ satisfying the following three conditions:

(1) *Truth and probability (TP):* $\mathcal{D}(e,R)$ is an ortholattice admitting sufficiently many 2-valued homomorphisms, $h : \mathcal{D}(e,R) \to \{0,1\}$, to recover all the (single and joint) probabilities assigned by the state $e$ to mutually compatible sets of elements $\{p_i\}_{i\in I}, p_i \in \mathcal{D}(e,R)$, as measures on a Kolmogorow probability space $(X, \mathcal{F}, \mu)$, where $X$ is the set of 2-valued homomorphisms on $\mathcal{D}(e,R)$, $\mathcal{F}$ is a field of subsets of $X$, and

$$\mu(\{h : h(p_i = 1, \text{ for all } i \in I\}) = tr(e \prod_{i\in I} p_i)$$

(2) *R-preferred (R-PREF):* the eigenspaces $r_i$ of $R$ belong to $\mathcal{D}(e,R)$.

(3) *$e, R$-definability (DEF):* for any $e \in \mathcal{H}$ and observable $R$ of $S$ defined on $\mathcal{H}$, $\mathcal{D}(e,R)$ is invariant under lattice homorphisms that preserve $e$ and $R$.

*Proof:* The strategy of the proof is to show, firstly, that if $p \in \mathcal{D}(e,R)$, then for any $e_{r_i}, i = 1, \ldots, k$, either $e_{r_i} \leq p$ or $e_{r_i} \leq p^{\perp}$, and secondly, that the sublattice $\mathcal{L}_{e_{r_1} e_{r_2} \ldots e_{r_k}} = \{p : e_{r_i} \leq p \text{ or } e_{r_i} \leq p^{\perp}, i = 1, \ldots, k\}$ satisfies the conditions of the theorem. Maximality then requires that $\mathcal{D}(e,R) = \mathcal{L}_{e_{r_1} e_{r_2} \ldots e_{r_k}}$.

Consider an arbitrary subspace $p$ of $\mathcal{H}$, and suppose $p \in \mathcal{D}(e, R)$. Clearly, if $r_i$ is included in $p$ or in $p^\perp$, for any $i$, then $e_{r_i} \leq p$ or $e_{r_i} \leq p^\perp$.

Suppose this is not the case. That is, suppose some $r_i$ is neither included in $p$ nor in $p^\perp$. Then either the intersection of $r_i$ with $p$ or the intersection of $r_i$ with $p^\perp$ is nonzero. (For, by the condition TP, there must be a 2-valued homomorphism mapping $r_i$ onto 1, and this homomorphism would have to map both $p$ and $p^\perp$ onto 0 if $r_i \wedge p = r_i \wedge p^\perp = 0$, which contradicts TP.) Assume that the intersection of $r_i$ with $p$ is nonzero (a similar argument applies if we assume that the intersection of $r_i$ with $p^\perp$ is nonzero). Call this intersection $b$.

If $p$ belongs to $\mathcal{D}(e, R)$, then $b = r_i \wedge p$ also belongs to $\mathcal{D}(e, R)$ (by the lattice closure assumption of TP, since $r_i$ belongs to $\mathcal{D}(e, R)$). Now, either $b$ is orthogonal to $e_{r_i}$ in $r_i$, or $b$ is skew to $e_{r_i}$ in $r_i$. We shall show that in either case $e_{r_i}$ must belong to $\mathcal{D}(e, R)$. It follows that $e_{r_i} \leq p$ or $e_{r_i} \leq p^\perp$, because if $e_{r_i} \not\leq p$ and $e_{r_i} \not\leq p^\perp$, then—since $e_{r_i}$ is a ray—$e_{r_i}$ must be skew to both $p$ and $p^\perp$; that is, $e_{r_i} \wedge p = 0$ and $e_{r_i} \wedge p^\perp = 0$. But this contradicts TP, because there must be a 2-valued homomorphism mapping $e_{r_i}$ onto 1, and this homomorphism would have to map both $p$ and $p^\perp$ onto 0.

So if $r_i$ is included in $p$ or in $p^\perp$ then $e_{r_i} \leq p$ or $e_{r_i} \leq p^\perp$, and if $r_i$ is not included in $p$ or in $p^\perp$, then $e_{r_i}$ must belong to $\mathcal{D}(e, R)$, from which it follows that $e_{r_i} \leq p$ or $e_{r_i} \leq p^\perp$.

To see that $e_{r_i}$ must belong to $\mathcal{D}(e, R)$ if (i) $b = r_i \wedge p$ is orthogonal to $e_{r_i}$ in $r_i$, or (ii) $b$ is skew to $e_{r_i}$ in $r_i$, consider each of these cases in turn.

(i) Suppose b is orthogonal to $e_{r_i}$ in $r_i$; that is, $b \leq e'_{r_i}$ (where the $'$ here denotes the orthogonal complement in $r_i$). If $r_i$ is 1-dimensional, $e_{r_i} = r_i$ and it follows immediately from R-PREF that $e_{r_i} \in \mathcal{D}(e, R)$. If $r_i$ is 2-dimensional, $b = e'_{r_i}$ and $e_{r_i} = r \wedge b^\perp$, so $e_{r_i} \in \mathcal{D}(e, R)$ by lattice closure (that is, by TP), because both $r_i$ and $b^\perp$ belong to $\mathcal{D}(e, R)$. If $r_i$ is more than 2-dimensional, consider all lattice automorphisms $U$ that are rotations about $e_{r_i}$ in $r_i$ and the identity in $r_i^\perp$. Such rotations preserve $e$ and $R$, because they preserve the eigenspaces of $R$ and the projections of $e$ onto the eigenspaces of $R$. It follows that $U(b) \in \mathcal{D}(e, R)$, by DEF. There are clearly sufficiently many rotations to generate a set of elements $U(b)$ whose span is $e'_{r_i}$. So $e_{r_i} \in \mathcal{D}(e, R)$, and hence $e_{r_i} \in \mathcal{D}(e, R)$ by lattice closure.

(ii) Suppose $b$ is skew to $e_{r_i}$ in $r_i$. Consider first the case that the eigenspace $r_i$ is more than 2-dimensional. In that case, we may suppose that $b$ is not a ray (or else we apply the following argument to $b'$). The subspace $b$ can therefore be represented as the span of a subspace $c$ orthogonal to $e_{r_i}$ in $r_i$, and the ray $d$, the projection of the ray $e_{r_i}$ onto $b$. Consider a lattice automorphism $U$ that is a reflection through the hyperplane $e_{r_i} \vee c$ in $r_i$ and the identity in $r_i^{\perp}$. (So $U$ preserves $e_{r_i}$ and $c$, but not $d$.) As in (i) above, $U(b) \in \mathcal{D}(e, R)$ because $U$ preserves $e$ and $R$, and $b \wedge U(b) = c$, so $c \in \mathcal{D}(e, R)$. We can now consider rotations of $c$ about $e_{r_i}$ in $r_i$ as in (i) to show that ${e_{r_i}}' \in \mathcal{D}(e, R)$, and hence that $e_{r_i} \in \mathcal{D}(e, R)$.

The argument fails if $r_i$ is 2-dimensional. If $b$ is skew to $e_{r_i}$ in $r_i$ and $r_i$ is 2-dimensional, then $b$ must be a ray in $r_i$; that is, $c$ is the null subspace and $d = b$. The automorphism $U$ reduces to a reflection through $e_{r_i}$ in $r_i$. Suppose $b$ is not at a 45° angle to $e_{r_i}$. The four rays $b, b', U(b), U(b)'$ would all have to belong to $\mathcal{D}(e, R)$ if $b$ belongs to $\mathcal{D}(e, R)$, but this contradicts TP. A 2-valued homomorphism would have to map one of the rays $b$ or $b'$ onto 1, and one of the rays $U(b)$ or $U(b)'$ onto 1, and hence the intersection of these two rays—the null subspace—would also have to be mapped onto 1. So $b$ cannot be skew to $e_{r_i}$ in $r_i$, unless $b$ is at a 45° angle to $e_{r_i}$. In this case, reflecting $b$ through $e_{r_i}$ yields $b'$, and there is no contradiction with TP.

So, if $r_i$ is 2-dimensional (and $r_i$ is not included in $p$ or $p^{\perp}$), we could conclude from this argument only that $e_{r_i} \leq p$ or $e_{r_i} \leq p^{\perp}$, or that $b \leq p$ or $b \leq p^{\perp}$, where b is a ray at a 45° angle to $e_{r_i}$ in $r_i$. In the original formulation of the proof, the 'weak separability' condition was introduced to exclude this one anomalous possibility, that the determinate sublattice $\mathcal{D}(e, R)$ might contain propositions $p$ such that $b \leq p$ or $b \leq p^{\perp}$, where $b$ is at a 45° angle to $e_{r_i}$ in $r_i$.

To see that such a condition is not required, suppose $r_i$ is 2-dimensional. Consider a lattice automorphism $U$ that is the identity in $r_i^{\perp}$, preserves $e_{r_i}$ in $r_i$, and maps b onto $U(b) \neq b$. For example, suppose we represent $e_{r_i}$ by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $r_i$ and take the action of $U$ on $r_i$ as represented by the matrix $\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$. If $b$ is represented by a vector $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ in $r_i$, then:

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix} = i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} ib_1 \\ b_2 \end{pmatrix} \neq k \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

for any constant $k$. So $U$ maps $e_{r_i}$ onto the same ray but maps $b$ onto a different ray, whether or not $b$ is at a 45° angle to $e_{r_i}$. The four rays $b$, $b'$, $U(b)$, $U(b)'$ would all have to belong to $\mathcal{D}(e, R)$ if $b$ belongs to $\mathcal{D}(e, R)$, which contradicts TP, by the argument above.

We have now established that if $p \in \mathcal{D}(e, R)$ then either $e_{r_i} \leq p$ or $e_{r_i} \leq p^{\perp}$, for all $i = 1, \ldots, k$. The final stage of the proof involves showing that the set of *all* such elements $\{p : e_{r_i} \leq p \text{ or } e_{r_i} \leq p^{\perp}, i = 1, \ldots, k\} = \mathcal{L}_{e_{r_1} e_{r_2} \ldots e_{r_k}}$ satisfies the three conditions TP, R-PREF, and DEF. Maximality then requires that $\mathcal{D}(e, R) = \mathcal{L}_{e_{r_1} e_{r_2} \ldots e_{r_k}}$. (For details, see [1].)

# References


[1] J. Bub, *Interpreting the Quantum World* (Cambridge: Cambridge University Press, 1997).

[2] Bub and Clifton, ‘A Uniqueness Theorem for “No Collapse” Interpretations of Quantum Mechanics,’ *Studies in the History and Philosophy of Modern Physics* **27**, 181-219 (1996).