Quantum Mechanics in Multiply-Connected Spaces

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Abstract

We explain why, in a configuration space that is multiply connected, i.e., whose fundamental group is nontrivial, there are *several* quantum theories, corresponding to different choices of topological factors. We do this in the context of Bohmian mechanics, a quantum theory without observers from which the quantum formalism can be derived. What we do can be regarded as generalizing the Bohmian dynamics on $\mathbb{R}^{3N}$ to arbitrary Riemannian manifolds, and classifying the possible dynamics that arise. This approach provides a new understanding of the topological features of quantum theory, such as the symmetrization postulate for identical particles. For our analysis we employ wave functions on the universal covering space of the configuration space.

Key words: topological phases, multiply-connected configuration spaces, Bohmian mechanics, universal covering space
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Contents

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1 Introduction

We shall be concerned here with topological effects in quantum mechanics, and shall elaborate on some results first described in [13]. The kind of statement on which we shall focus asserts that if the configuration space $\mathcal{Q}$ is a multiply-connected[1] Rieman-

[1] Recall that a manifold $\mathcal{Q}$ is *simply connected* if all closed curves in $\mathcal{Q}$ are contractible. Otherwise, it is *multiply connected*. (Note that "multiply connected" is different from the notion "n-connected" for $n \geq 0$, which is sometimes used in the literature on algebraic topology [38, p. 51] and means that the first n homotopy groups $\pi_n(\mathcal{Q})$ are all trivial.) Examples of simply-connected spaces are $\mathbb{R}^d$ for any $d \geq 0$, the spheres S^d for $d \geq 2$, or the punctured spaces $\mathbb{R}^d \setminus \{0\}$ for $d \geq 3$; examples of multiply-connected spaces are the circle S^1, the torus $S^1 \times S^1$, the punctured plane $\mathbb{R}^2 \setminus \{0\}$, or

nian manifold then there exist *several* quantum theories in $\mathcal{Q}$. More precisely, the dynamics is not completely determined by specifying $\mathcal{Q}$ (whose metric we regard as incorporating the "masses of the particles") together with the potential and the value space of the wave function; in addition, one can choose *topological factors*, which form a representation (or twisted representation) of the fundamental group $\pi_1(\mathcal{Q})$ of $\mathcal{Q}$. In each of the theories, the Hamiltonian is locally equivalent to $-\frac{\hbar^2}{2}\Delta + V$, though not globally. The investigation in this paper is continued with other methods in three follow-up papers [14, 15, 16].

Our interest lies in explaining why there is more than one quantum theory and how the several possibilities arise, and in classifying the possibilities. The formulation of quantum mechanics we use for this purpose is Bohmian mechanics [8, 4, 18, 6, 19, 25], a quantum theory without observers; it describes a world in which particles have trajectories, guided by a wave function ψ_t; observers in this world would find that the results of their experiments obey the quantum formalism [8, 4, 18, 20]. We will give a brief review of Bohmian mechanics in Section 4. Most of our mathematical considerations and methods are equally valid, relevant, and useful in orthodox quantum mechanics, or any other version of quantum mechanics. Bohmian mechanics, however, provides a sharp mathematical justification of these considerations that is absent in the orthodox framework.

The motion of the configuration in a Bohmian N-particle system can be regarded as corresponding to a dynamical system in the configuration space $\mathcal{Q} = \mathbb{R}^{3N}$, defined by a time-dependent vector field v^{ψ_t} on $\mathcal{Q}$ which in turn is defined, by the Bohmian law of motion, in terms of ψ_t. We are concerned here with the analogues of the Bohmian law of motion for the case that $\mathcal{Q}$ is, instead of $\mathbb{R}^{3N}$, an arbitrary Riemannian manifold.[2] The main result is that, if $\mathcal{Q}$ is multiply connected, there are several such analogues: several Bohmian dynamics, which we will describe in detail, corresponding to different choices of the topological factors.

To define a Bohmian theory in a manifold $\mathcal{Q}$ ultimately amounts to defining trajectories in $\mathcal{Q}$ and their probabilities. This leads to clear mathematical classification questions, while from the orthodox point of view the ground rules with respect to the issue of the existence of several quantum theories with different topological factors are less clear. We will review the differences between the two viewpoints in Section 2.

The topological factors consists of, in the simplest case, phase factors associated with non-contractible loops in $\mathcal{Q}$, forming a character[3] of the fundamental group $\pi_1(\mathcal{Q})$. All characters can be physically relevant; we emphasize this because it is easy to overlook the multitude of dynamics by focussing too much on just one, the simplest one, which we will define in Section 5: the *immediate generalization* of the

generally $\mathbb{R}^d \setminus U$ where U is a subspace of dimension $d-2$, $d \geq 2$.

[2]Manifolds will throughout be assumed to be Hausdorff, paracompact, connected, and C^∞. They need not be orientable.

[3]By a *character* of a group we refer to what is sometimes called a unitary multiplicative character, i.e., a one-dimensional unitary representation of the group.

Bohmian dynamics from $\mathbb{R}^{3N}$ to a Riemannian manifold, or, as we shall briefly call it, the *immediate Bohmian dynamics.*

Apart from the mathematical exercise, what do we gain from studying the possible Bohmian dynamics on manifolds?

- A new understanding of how topological factors in quantum mechanics can be regarded as arising.
- A presumably complete *classification* of the topological factors in quantum mechanics, including some, corresponding to what we call *twisted representations* of $\pi_1(\mathcal{Q})$, that have not, to our knowledge, been considered so far in the literature.
- An explanation of the fact that the wave function of a system of identical particles is either symmetric or anti-symmetric, a fact that (at least insofar as nonrelativistic quantum mechanics is concerned) is usually, instead of being derived, introduced as a *symmetrization postulate.* This application is discussed in detail in a sister paper to this one [17], and will only be touched upon briefly here.

Our main motivation for studying the question of Bohmian dynamics on manifolds was in fact the investigation of the symmetrization postulate for identical particles.

As we have already mentioned, one of the different Bohmian dynamics on a manifold $\mathcal{Q}$ is special, as it is the immediate Bohmian dynamics on $\mathcal{Q}$.[4] The other kinds of Bohmian dynamics come in a hierarchy of increasing complexity. There are three natural classes $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ of Bohmian dynamics, related according to

$$\mathscr{C}_0 \subseteq \mathscr{C}_1 \subseteq \mathscr{C}_2 \subseteq \mathscr{C}_3 \,, \tag{1}$$

where $\mathscr{C}_0$ contains only the immediate Bohmian dynamics. The dynamics of class $\mathscr{C}_1$, defined in Section 6.4, involve topological phase factors forming a character of the fundamental group $\pi_1(\mathcal{Q})$. Those of class $\mathscr{C}_2$, defined in Section 8.1 and in a more general setting in Section 8.4, involve topological factors that are given by matrices, forming a unitary representation of $\pi_1(\mathcal{Q})$ or, in the case of a vector bundle, a twisted representation (see the end of Section 8.4 for the definition). Those of class $\mathscr{C}_3$ will not be discussed here but in [14]; they involve changes in connections and potentials that are not based on multiple connectivity. As we shall explain, the dynamics of bosons belongs to $\mathscr{C}_0$ while that of fermions belongs to $\mathscr{C}_1$. More precisely, fermions can be regarded as belonging either to $\mathscr{C}_0$, for a certain nontrivial vector bundle defined in [17] (for which bosons are of class $\mathscr{C}_1$), or to $\mathscr{C}_1$, for the trivial bundle $\mathcal{Q} \times \mathbb{C}$. In Section 7 we derive a dynamics of class $\mathscr{C}_1$ for the Aharonov–Bohm effect. We

[4]That is, if one considers the value space of the wave function as given. In [16], in contrast, we obtain several bundles of spin spaces from the Riemannian geometry of the configuration space. We could take any of these bundles as the starting point for defining the dynamics, and then which one of the dynamics is immediate will depend on this choice.

will define dynamics here in a non-rigorous way; a rigorous definition of the classes $\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ is given in [14].

It is not obvious what "all possible kinds of Bohmian dynamics" should mean. We will investigate one approach here, while others, as already mentioned, are studied in [14, 15, 16]. The present approach is based on considering wave functions ψ that are defined not on the configuration space $\mathcal{Q}$ but on its universal covering space $\widehat{\mathcal{Q}}$. We then study which kinds of periodicity conditions, relating the values on different levels of the covering fiber by a topological factor, will ensure that the Bohmian velocity vector field associated with ψ is projectable from $\widehat{\mathcal{Q}}$ to $\mathcal{Q}$. This is carried out in Section 6 for scalar wave functions and in Section 8 for wave functions with values in a complex vector space (such as a spin-space) or a complex vector bundle. In the case of vector bundles, we derive a novel kind of topological factor, given by a twisted representation of $\pi_1(\mathcal{Q})$.

Let us mention the other approaches to defining the dynamics of classes $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_3$. Since wave functions can be regarded as sections of Hermitian bundles, i.e., complex vector bundles with a connection and parallel Hermitian inner products, one approach [14] considers all Hermitian bundles that are locally (but not globally) isomorphic to a given one. Another approach [14] expresses the dynamics in terms of the Hamiltonian and considers all Hamiltonians H that are locally (though not necessarily globally) equivalent to $-\frac{\hbar^2}{2}\Delta + V$. Another approach [16] regards the value space of the wave function as a representation space (such as a spin-space) of a suitable group (such as the rotation group) and classifies the Hermitian bundles consisting of representation spaces. A last approach [15] removes a surface κ from configuration space $\mathcal{Q}$, such that $\mathcal{Q} \setminus \kappa$ is simply connected, and imposes a periodic boundary condition relating the wave function on both sides of the new, "virtual", boundary κ by a topological factor.

In some cases, some of the classes coincide: When $\mathcal{Q}$ is simply connected, then $\mathscr{C}_0 = \mathscr{C}_1 = \mathscr{C}_2$. When the wave function is a scalar (as for spinless particles), then $\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{C}_3$. For generic potentials, $\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{C}_3$.

We encounter examples of multiply-connected configuration spaces in two ways: either, as in the Aharonov–Bohm effect, by ignoring an existing part of physical (or configuration) space, or, as for identical particles or multiply-connected cosmologies, from the very nature of the configuration space. In the former case the topological factors of the *effective* dynamics on the available configuration space depend on external fields, while in the latter case the topological factors of the *fundamental* dynamics should be compatible with any choice of external fields. This compatibility is a strong restriction, which allows only the dynamics of class $\mathscr{C}_1$. Hence, as we shall argue more fully later, the several fundamental quantum theories in $\mathcal{Q}$ are those given by $\mathscr{C}_1$. This conclusion we call the *Character Quantization Principle*, since the dynamics of $\mathscr{C}_1$ are defined using the characters of the fundamental group of $\mathcal{Q}$. It is formulated and discussed in Section 9. We conclude in Section 10.

The notion that multiply-connected spaces give rise to different quantum theories is not new. Here is a sampling of the literature. A covering space was used at least as early as 1950 by Bopp and Haag [9] for the configuration space of the spinning top; it was more fully exploited by Dowker [11], and was used by Leinaas and Myrheim [30] for the configuration space of identical particles. Vector potentials on multiply connected spaces were used by Aharonov and Bohm in [1]. Path integrals on multiply connected spaces began largely with the work of Schulman [35, 36] and that of Laidlaw and DeWitt in [29]; see [37] for details. There is also the current algebra approach of Goldin, Menikoff, and Sharp [24]. Most of these works are dedicated to scalar wave functions. The study of arbitrary manifolds began with Laidlaw and DeWitt [29], which deals with path integration on the universal covering space for scalar wave functions. Nelson [32] derives the topological phase factors for scalar wave functions from stochastic mechanics. Gamboa and Rivelles [22] consider relativistic Hamiltonians using a path-integral approach. Ho and Morgan [28] provide a study of quantum mechanics on $\mathbb{R}^d \times S^1$ for scalar wave functions.

2 Perspective on Orthodox Quantum Mechanics

When one considers a Bohmian dynamics and removes the Bohmian trajectories, there still remains the wave function ψ, and a number of nontrivial things can be said about it, such as, which space ψ can be taken from and how its evolution is defined. As a consequence, much of the mathematical discussion in this paper would be equally valid, applicable and relevant for any other formulation of quantum mechanics. However, our analysis of the emergence of the further kinds of wave functions, whose main role in Bohmian mechanics is to define trajectories, would not work in the same way if one were to dispense with the trajectories.

One would meet, when trying to carry out the program of this paper in orthodox quantum mechanics, some difficulties that are absent in the Bohmian approach. This is mainly because of two traits of Bohmian mechanics: first, it is clear in the Bohmian framework at which point the specification of a theory is complete; and second, it is clear whether two variants of a theory are physically equivalent or not. Let us explain.

In the Bohmian framework, once the possible trajectories of all particles have been defined (together with the appropriate equivariant probability distribution, see Section 4) then the theory has been completely specified, and there is neither need nor room for further axioms. In orthodox quantum mechanics, in contrast, it is not obvious what it is that needs to be specified in order to have a variant of the theory. The Hilbert space $\mathscr{H}$ and the Hamiltonian H? While they certainly must be specified, they are certainly not enough.

One could think, for example, of different possible position observables in the same Hilbert space, and these would lead to different predictions for position measurements. Thus, one should specify, it would seem, $\mathscr{H}$, H, and the operator, or commuting set of operators, for the position observable. But would that be enough? Need we not also be told what operator represents the momentum observable? Need we not be told what operators represent *all* the observables? This should be contrasted with the fact that in Bohmian mechanics, once the dynamics of the particles is specified, also the outcomes of all experiments are specified.[5]

And what are, by the way, "all" observables? It seems clear that the list of all observables should begin with position, momentum, and energy, but where it should end is rather obscure. In addition, the notion of observable becomes somewhat problematic when the configuration space $\mathcal{Q}$ is a manifold. The problem is not so much that the position observable can no longer be represented by a set of commuting position operators, as the manifold may not permit global coordinates (e.g., on the circle); one should conclude that the appropriate notion of position observable is then a PVM (projection-valued measure) on $\mathcal{Q}$ acting on $\mathscr{H}$, associating with every subset of $\mathcal{Q}$ a projection in $\mathscr{H}$. The more serious problem concerns the momentum observable: already on the half-line, the operator $p = -i\hbar\, d/dq$ does not have a self-adjoint extension. More generally, on a Riemannian manifold $\mathcal{Q}$ the notion of the momentum observable becomes obscure, as it is based on a translation symmetry that may not exist in $\mathcal{Q}$. Thus, a momentum observable may not exist. This brings us back to the point that it is not clear which observables need to be specified in order to specify an orthodox quantum theory.

The contrast between the clarity of Bohmian mechanics and the vagueness of orthodox quantum theory is perhaps even more striking when we consider the issue of the physical equivalence of theories. In this paper we shall always treat Bohmian theories, when they are mathematically different but lead to the same trajectories (and probabilities), as physically equivalent. For example, the dynamics we shall define using wave functions on the covering space with the trivial character is physically equivalent to the immediate Bohmian dynamics, using wave functions on the configuration space.

In orthodox quantum mechanics, when should we regard two variants of the theory as physically equivalent? The answer in the spirit of orthodox quantum mechanics is, when they predict the same statistics for outcomes for all experiments; that is, when they are empirically equivalent.[6] This answer leads again to the question, what are "all" observables? In addition, it leads us to the possibly separate problem of identifying the observables of one theory with the observables of another. Within the Bohmian framework, based on a clear ontology and a correspondingly sharp

[5]One could argue that for exactly the same reason, to specify the position observable in orthodox quantum mechanics would be sufficient, as it would fix the statistics of the outcomes of every measurement. This is true, and we think that this is a healthy attitude. However, it is also quite against the spirit of orthodox quantum mechanics which sets a high value on the "democracy" for all observables.

[6]This answer can be criticized on the grounds that there are known examples of theories that are empirically equivalent though physically inequivalent, such as Bohmian mechanics and stochastic mechanics [32], or the variants of Bohmian mechanics in which some of the particles do not possess actual positions while their coordinates get integrated over in the law of motion [26].

specification of the relevant physical structures and their behavior, no such questions and problems can arise.

3 Perspective on Spontaneous Collapse Theories

Another approach besides Bohmian mechanics leading to quantum theories without observers is that of spontaneous wave function collapse [33, 23, 5, 3]; the simplest and best known model of this kind is due to Ghirardi, Rimini, and Weber (GRW) [23]. Its situation with respect to topological factors is very different from that of Bohmian mechanics. For example, the situation of identical particles in the GRW theory is different from that in Bohmian mechanics because the latter is (in a suitable sense) automatically compatible with bosons and fermions, whereas the equations of the GRW model require modification for identical particles as follows [10, 40].

In the original GRW model (corresponding to N distinguishable particles), collapses are associated with points in 3-space and labels $i \in \{1, \dots, N\}$. Given the wave function $\psi : \mathbb{R}^{3N} \to \mathbb{C}$, a collapse with label i and location $\boldsymbol{x} \in \mathbb{R}^3$ occurs with rate

$$r_i(\boldsymbol{x}|\psi) = \langle \psi | \Lambda_i(\boldsymbol{x})\, \psi \rangle \,, \tag{2}$$

where the collapse rate operator $\Lambda_i(\boldsymbol{x})$ is a multiplication operator defined by

$$\Lambda_i(\boldsymbol{x})\, \psi(\boldsymbol{q}_1, \dots, \boldsymbol{q}_N) = \lambda \, \exp\Big(-\frac{(\boldsymbol{x} - \boldsymbol{q}_i)^2}{2a^2}\Big)\, \psi(\boldsymbol{q}_1, \dots, \boldsymbol{q}_N) \,. \tag{3}$$

The constants λ and a are parameters of the model. A collapse at time t and location $\boldsymbol{x}$ with label i changes the wave function according to

$$\psi_{t-} \mapsto \psi_{t+} = \frac{\Lambda_i(\boldsymbol{x})^{1/2} \psi_{t-}}{\|\Lambda_i(\boldsymbol{x})^{1/2} \psi_{t-}\|} \,. \tag{4}$$

In the version for identical particles, collapses are associated with locations $\boldsymbol{x}$ only, without labels. Letting ψ be either a symmetric or an anti-symmetric function on $\mathbb{R}^{3N}$, a collapse occurs at location $\boldsymbol{x} \in \mathbb{R}^3$ with rate

$$r(\boldsymbol{x}|\psi) = \langle \psi | \Lambda(\boldsymbol{x})\, \psi \rangle \,, \tag{5}$$

where

$$\Lambda(\boldsymbol{x}) = \sum_{i=1}^{N} \Lambda_i(\boldsymbol{x}) \,, \tag{6}$$

and changes ψ according to

$$\psi \mapsto \frac{\Lambda(\boldsymbol{x})^{1/2} \psi}{\|\Lambda(\boldsymbol{x})^{1/2} \psi\|} \,. \tag{7}$$

Thus, the collapsed wave function is again symmetric respectively anti-symmetric.

The arguments used in the present paper for deriving the topological factors cannot be repeated in the context of the GRW theory because they rely on particle configurations, which do not exist in the GRW theory. As a consequence, indeed, configuration space does not play, in the GRW theory, the same central role as in Bohmian mechanics, but merely that of a convenient tool for representing the state vector as a function (similar to the role, in Bohmian mechanics, of momentum space or of the set of spin eigenvalues). Thus, it is hard to see how the GRW theory could provide any reasons for the existence of several possibilities, corresponding to different topological factors, in situations in which the configuration space of the Bohmian theory is multiply connected. Moreover, for the GRW theory, for which there are no particles to begin with, it is hard to see why the multiply-connected natural configuration space ${}^N\mathbb{R}^3$ for N identical particles, see Section 5, should be considered at all. Nonetheless, topological factors can always be introduced into GRW theories, as in the example above, as we shall explain later in Section 6.5, Remark 6.

4 Bohmian Mechanics in $\mathbb{R}^{3N}$

Bohmian mechanics is a theory about particles with definite locations. The theory specifies the trajectories in physical space of these particles. The object which determines the trajectories is the wave function, familiar from quantum mechanics. More precisely, the state of the system in Bohmian mechanics is given by the pair (Q, ψ); $Q = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) \in \mathbb{R}^{3N}$ is the configuration of the N particles in our system and ψ is a (standard quantum mechanical) wave function on the configuration space $\mathbb{R}^{3N}$, taking values in some *Hermitian vector space* W, i.e., a finite-dimensional complex vector space endowed with a positive-definite Hermitian (i.e., conjugate-symmetric and sesqui-linear) inner product $(\,\cdot\,,\,\cdot\,)$. The state of the system changes according to the guiding equation and Schrödinger's equation:

$$\frac{d\boldsymbol{Q}_k}{dt} = \frac{\hbar}{m_k}\mathrm{Im}\frac{(\psi, \nabla_k \psi)}{(\psi, \psi)}(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) =: v_k^{\psi}(Q), \quad k = 1, \ldots, N \tag{8}$$

$$i\hbar\frac{\partial \psi}{\partial t} = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k}\Delta_k \psi + V\psi \tag{9}$$

where V is the potential function with values given by Hermitian matrices (endomorphisms of W). We call $(\phi(q), \psi(q))$, the inner product on the value space W, the *local inner product*, in distinction from the inner product $\langle \phi, \psi \rangle$ on the Hilbert space of wave functions. For complex-valued wave functions, the potential is a real-valued function on configuration space and the local inner product is $\overline{\phi(q)}\psi(q)$, where the bar denotes complex conjugation.

The empirical agreement between Bohmian mechanics and standard quantum mechanics is grounded in equivariance [18, 20]. In Bohmian mechanics, if the configuration is initially random and distributed according to $|\psi_0|^2$, then the evolution is such that the configuration at time t will be distributed according to $|\psi_t|^2$. This property is called the equivariance of the $|\psi|^2$ distribution. It follows from comparing the transport equation

$$\frac{\partial \rho_t}{\partial t} = -\nabla \cdot (\rho_t v^{\psi_t}) \tag{10}$$

for the distribution ρ_t of the configuration Q_t, where $v^\psi = (v_1^\psi, \ldots, v_N^\psi)$, to the quantum continuity equation

$$\frac{\partial |\psi_t|^2}{\partial t} = -\nabla \cdot (|\psi_t|^2 v^{\psi_t}), \tag{11}$$

which is a consequence of Schrödinger's equation (9). A rigorous proof of equivariance requires showing that almost all (with respect to the $|\psi|^2$ distribution) solutions of (8) exist for all times. This was done in [7, 39]. A more comprehensive introduction to Bohmian mechanics may be found in [25, 6, 19].

Spin is already incorporated in (8) and (9) if one chooses for W a suitable spin space [4]. By assumption, for one particle moving in $\mathbb{R}^3$, W is a complex, irreducible representation space of $SU(2)$, the universal covering group[7] of the rotation group $SO(3)$. If it is the spin-s representation then $W = \mathbb{C}^{2s+1}$.

5 The Immediate Generalization to Riemannian Manifolds

We now consider, in the role of the configuration space, a Riemannian manifold $\mathcal{Q}$ instead of $\mathbb{R}^{3N}$. The primary physical motivation is the study of identical particles, for which the natural configuration space is the set ${}^N\mathbb{R}^3$ of all N-element subsets of $\mathbb{R}^3$,

$${}^N\mathbb{R}^3 := \{S | S \subseteq \mathbb{R}^3, |S| = N\}, \tag{12}$$

which naturally carries the structure of a Riemannian manifold, in fact a multiply-connected one. This configuration space was first suggested in [29] and [30]; for further discussion see [17].

But the generalization to manifolds is also very natural mathematically. In addition, there are further cases of physical relevance: One could consider, instead of $\mathbb{R}^3$, a curved physical space. And in cases like the Aharonov–Bohm effect, the phase shift that occurs can be attributed to the topology of the effectively available configuration space, a subset of the entire configuration space that can be viewed as a multiply-connected manifold.

5.1 Euclidean Vector Spaces

It is in fact easy to find a generalization of the Bohmian dynamics to a Riemannian manifold $\mathcal{Q}$, which we call the immediate Bohmian dynamics on $\mathcal{Q}$. One reason why

[7]The universal covering space of a Lie group is again a Lie group, the *universal covering group*. It should be distinguished from another group also called the *covering group*: the group $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ of the covering (or deck) transformations of the universal covering space $\widehat{\mathcal{Q}}$ of a manifold $\mathcal{Q}$, which will play an important role later.

it is so easy is that the law of motion for the point $Q_t = Q(t) = (\mathbf{Q}_1(t), \ldots, \mathbf{Q}_N(t))$ in the configuration space $\mathbb{R}^{3N}$ representing the positions of all particles at time t is almost independent of the way in which $\mathbb{R}^{3N}$ is composed of N copies of $\mathbb{R}^3$. In fact, (8) can be written as

$$\frac{dQ_t}{dt} = \hbar\, \mathfrak{m}^{-1}\, \mathrm{Im} \frac{(\psi, \nabla\psi)}{(\psi, \psi)}(Q_t) \tag{13}$$

where $\mathfrak{m}$ is the diagonal matrix with the masses as entries, each mass m_k appearing 3 times. That is, as soon as $\mathfrak{m}$ is given, the information about which directions in $\mathbb{R}^{3N}$ correspond to the single factors $\mathbb{R}^3$ becomes irrelevant for defining the dynamics of Q_t. Eq. (13) would as well define a dynamics on any Euclidean vector space $\mathscr{E}$ of finite dimension, given a wave function ψ on $\mathscr{E}$ and a positive-definite symmetric endomorphism $\mathfrak{m} : \mathscr{E} \to \mathscr{E}$.

It will be convenient to include the mass matrix $\mathfrak{m}$ in the metric g_{ab} of $\mathscr{E}$,

$$g_{ab} = \sum_{c=1}^{\dim \mathscr{E}} g'_{ac} \mathfrak{m}^c_b\,, \tag{14}$$

where g'_{ab} is the metric of $\mathscr{E}$ before the inclusion of masses, and indices a, b, c run through the dimensions of $\mathscr{E}$. In the standard example of $\mathbb{R}^{3N}$, this amounts to introducing the metric

$$g_{i\alpha, j\beta} = m_i \delta_{ij} \delta_{\alpha\beta}\,, \tag{15}$$

where $i, j = 1, \ldots, N$ and $\alpha, \beta = 1, 2, 3$ (and the index i occurring twice on the right is not summed over). With ∇ then defined using g instead of g', (13) becomes

$$\frac{dQ_t}{dt} = \hbar\, \mathrm{Im} \frac{(\psi, \nabla\psi)}{(\psi, \psi)}(Q_t)\,. \tag{16}$$

(Note that in order to turn the covector given by the differential of ψ into a vector, one uses g^{ab}.)

Similarly, the Schrödinger equation (9) can then be written

$$i\hbar \frac{\partial \psi}{\partial t} = -\tfrac{\hbar^2}{2} \Delta\psi + V\psi\,, \tag{17}$$

where Δ, the Laplacian on $\mathscr{E}$, is to be understood as the metric trace of the second derivatives,

$$\Delta = g^{ab} \partial_a \partial_b \tag{18}$$

(in abstract-index notation with sum convention). Thus, also the Schrödinger equation is well-defined on a Euclidean space $\mathscr{E}$, or, in other words, it is independent of the product structure of $\mathbb{R}^{3N} = (\mathbb{R}^3)^N$.

5.2 Riemannian Manifolds

In order to transfer (16) and (17) to Riemannian manifolds, we need only replace $\mathscr{E}$ by the tangent space $T_{Q(t)}\mathcal{Q}$. In this subsection, the wave functions we consider are W-valued functions on $\mathcal{Q}$, with W a Hermitian vector space.

We begin by recalling the definitions of the gradient and the Laplacian on Riemannian manifolds. By the gradient ∇f of a function $f : \mathcal{Q} \to \mathbb{R}$ we mean the tangent vector field on $\mathcal{Q}$ metrically equivalent (by "raising the index") to the 1-form df, the differential of f. For a function $\psi : \mathcal{Q} \to W$, the differential $d\psi$ is a W-valued 1-form, and thus $\nabla\psi(q) \in \mathbb{C}T_q\mathcal{Q} \otimes W$, where $\mathbb{C}T_q\mathcal{Q}$ denotes the complexified tangent space at q, and the tensor product $\otimes$ is, as always in the following, over the complex numbers. The Laplacian Δf of a function f is defined to be the divergence of ∇f, where the divergence of a vector field X is defined by

$$\operatorname{div} X = D_a X^a \tag{19}$$

with D the (standard) covariant derivative operator, corresponding to the Levi-Civita connection on the tangent bundle of $\mathcal{Q}$ arising from the metric g. Since $Dg = 0$, we can write

$$\Delta f = g^{ab} D_a D_b f\,, \tag{20}$$

where the second D, the one which is applied first, actually does not make use of the Levi-Civita connection. In other words, the Laplacian is the metric trace of the second (covariant) derivative. Another equivalent definition is $\Delta f = *d*df$ where d is the exterior derivative of differential forms and $*$ is the Hodge star operator (see, e.g., [21]).[8] For W-valued functions ψ the Laplacian $\Delta\psi$ is defined correspondingly as the divergence of the "W-valued vector field" $\nabla\psi$, or equivalently by

$$\Delta\psi = g^{ab} D_a D_b \psi \tag{21}$$

or by $\Delta\psi = *d*d\psi$, using the obvious extension of the exterior derivative to W-valued differential forms.

The time evolution of the state (Q_t, ψ_t) is simply given by the same formal equations as (16) and (17) with the appropriate interpretation of ∇ and Δ. We give the equations for future reference:

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t) \tag{22a}$$

$$i\hbar\frac{\partial\psi_t}{\partial t} = -\tfrac{\hbar^2}{2}\Delta\psi_t + V\psi_t\,, \tag{22b}$$

[8] The Hodge operator $*$ depends on the orientation of $\mathcal{Q}$ in such a way that a change of orientation changes the sign of the result. Thus, $*$ does not exist if $\mathcal{Q}$ is not orientable. However, it exists locally for any chosen local orientation, and since the Laplacian contains two Hodge operators, it is not affected by the sign ambiguity.

where the Bohmian velocity vector field v^ψ associated to the wave function ψ is

$$v^\psi := \hbar \,\mathrm{Im} \frac{(\psi, \nabla\psi)}{(\psi,\psi)}. \tag{23}$$

Thus, given $\mathcal{Q}$, W, and V, we have specified a Bohmian dynamics, the *immediate Bohmian dynamics*.[9] We introduce the notation $\mathscr{C}_0(\mathcal{Q}, W, V)$ for the set containing just this one dynamics. We also write $\mathscr{C}_0(\mathcal{Q}, V)$ for $\mathscr{C}_0(\mathcal{Q}, \mathbb{C}, V)$. (A rigorous definition of what is meant here by a "dynamics," avoiding the question of the existence of solutions, is given in [14]. For now, we simply proceed as if we have the global existence of solutions and say (a bit vaguely) that a "dynamics" is defined by a set of wave functions, in this case $C_0^\infty(\mathcal{Q}, W)$, and for every wave function ψ a set $\mathscr{S}^\psi$ of trajectories in $\mathcal{Q}$, in this case the solutions of (22a), together with a probability distribution ρ^ψ on $\mathscr{S}^\psi$, in this case given by

$$\rho^\psi(dQ) = \big(\psi_t(Q_t), \psi_t(Q_t)\big)\, dQ_t\,. \tag{24}$$

Due to equivariance, (24) is independent of t.)

5.3 An Example

An important case is that of several particles moving in a Riemannian manifold M, a possibly curved physical space. Then the configuration space for N distinguished particles is $\mathcal{Q} := M^N$. Let the masses of the particles be m_i and the metric of M be g. Then the relevant metric on M^N, the analogue of (14) and (15) acting on the tangent space $T_{(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_N)}M^N = \bigoplus_{i=1}^N T_{\boldsymbol{q}_i}M$, is

$$g^N(v_1 \oplus \cdots \oplus v_N, w_1 \oplus \cdots \oplus w_N) := \sum_{i=1}^N m_i g(v_i, w_i).$$

Using g^N allows us to write (23) and (22) instead of the equivalent equations

$$\frac{d\boldsymbol{Q}_k}{dt} = \frac{\hbar}{m_k} \mathrm{Im} \frac{(\psi, \nabla_k\psi)}{(\psi,\psi)}(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N), \quad k = 1, \ldots, N \tag{25}$$

[9]The question arises whether these equations possess unique solutions, for all times or at least for short times. For some Riemannian manifolds $\mathcal{Q}$ this may require the introduction of boundary conditions. Since the existence question is mathematically demanding and not our concern here, we make only a few remarks: For a discussion of the existence question of Bohmian trajectories in $\mathbb{R}^{3N}$, see [7, 39]. The existence of the evolution of the wave function amounts to defining the Hamiltonian H as a self-adjoint operator, i.e., as a self-adjoint extension of $H^0 = -\frac{\hbar^2}{2}\Delta + V$, with H^0 defined on $C_0^\infty(\mathcal{Q}, W)$, the space of smooth W-valued functions with compact support. As far as we know, it is not known in all cases whether a self-adjoint extension exists, and, when so, how many exist, and what the physical meaning of the different extensions is when there is more than one. When several extensions exist, they must perhaps be regarded as different possible Bohmian dynamics on $\mathcal{Q}$, and thus as further possibilities, not captured in the classes $\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2$ considered in this paper. We shall not pursue this idea further, and shall reason instead in terms of the "formal" dynamics.

$$i\hbar\frac{\partial\psi}{\partial t} = -\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\Delta_k\psi + V\psi, \tag{26}$$

where $\boldsymbol{Q}_k$, the k^{th} component of Q, lies in M, and ∇_k and Δ_k are the gradient and the Laplacian with respect to g, acting on the k^{th} factor of M^N. We take $W = \mathbb{C}$. Observe that (8) and (9) are special cases, corresponding to Euclidean space $M = \mathbb{R}^3$, of (25) and (26).

The configuration space of N identical particles in M is

$${}^N M := \{S | S \subseteq M, |S| = N\}, \tag{27}$$

which inherits a Riemannian metric from M, see [17].

5.4 Vector Bundles

Even more generally, we can consider a Bohmian dynamics for wave functions taking values in a complex vector bundle E over the Riemannian manifold $\mathcal{Q}$. That is, the value space then depends on the configuration, and wave functions become sections of the vector bundle.[10]

Such a case occurs for identical particles with spin s, where the bundle E of spin spaces over the configuration space $\mathcal{Q} = {}^N\mathbb{R}^3$ defined in (12) consists of the $(2s+1)^N$-dimensional spaces

$$E_q = \bigotimes_{\boldsymbol{q}\in q} \mathbb{C}^{2s+1}, \quad q \in \mathcal{Q}. \tag{28}$$

For a detailed discussion of this bundle, of why this is the right bundle, and of the notion of a tensor product over an arbitrary index set, see [17]. Vector bundles also occur for particles with spin in a curved physical space. In addition to their physical relevance, bundles are a natural mathematical generalization of our previous setting involving wave functions defined on manifolds. Finally, the approaches we use in [14, 16] for suggesting natural classes of Bohmian dynamics are based on considerations concerning vector bundles (even for spinless particles).

We introduce some notation and terminology. $C^\infty(E)$ will denote the set of smooth sections while $C_0^\infty(E)$ will be the set of smooth sections with compact support.

Definition 1. *A* Hermitian vector bundle*, or* Hermitian bundle*, is a finite-dimensional complex vector bundle E with a connection and a positive-definite, Hermitian local inner product $(\,\cdot\,,\,\cdot\,)_q$ on E_q which is parallel.*

Recall that a connection defines (and is defined by) a notion of parallel transport of vectors in E_q along curves β in $\mathcal{Q}$ from q to r, given by linear mappings $P_\beta : E_q \to E_r$. A section ψ of E is *parallel* if always $P_\beta\psi(q) = \psi(r)$. If β is a loop, $q = r$, the mapping

[10] Recall that a *section* (also known as *cross-section*) of E is a map $\psi : \mathcal{Q} \to E$ such that $\psi(q) \in E_q$, i.e. it maps a point q of $\mathcal{Q}$ to an element of the vector fiber over q. For example, a vector field on a manifold M is a section of the tangent bundle TM.

P_β is called the *holonomy endomorphism* h_β of E_q associated with β. A connection also defines (and is defined by) a covariant derivative operator D, which allows us to form the derivative $D\psi$ of a section ψ of E. A section ψ is parallel if and only if $D\psi = 0$. A bundle with connection is called *flat* if all holonomies of contractible loops are trivial, i.e., the identity endomorphism (this is the case if and only if the curvature of the connection vanishes everywhere).

Parallelity of the local inner product means that parallel transport preserves inner products; equivalently, $D(\psi, \phi) = (D\psi, \phi) + (\psi, D\phi)$ for all $\psi, \phi \in C^\infty(E)$. It follows in particular that holonomy endomorphisms are always unitary.

Our bundle, the one of which ψ is a section, will always be a Hermitian bundle. Note that since a Hermitian bundle consists of a vector bundle and a connection, it can be nontrivial even if the vector bundle is trivial: namely, if the connection is nontrivial. The *trivial Hermitian bundle* $\mathcal{Q} \times W$, in contrast, consists of the trivial vector bundle with the trivial connection, whose parallel transport P_β is always the identity on W. The case of a W-valued function $\psi : \mathcal{Q} \to W$ corresponds to the trivial Hermitian bundle $\mathcal{Q} \times W$.

The global inner product on the Hilbert space of wave functions is the local inner product integrated against the Riemannian volume measure associated with the metric g,

$$\langle \phi, \psi \rangle = \int_{\mathcal{Q}} dq\, (\phi(q), \psi(q)) \,.$$

The Hilbert space equipped with this inner product, denoted $L^2(\mathcal{Q}, E)$, contains the square-integrable, measurable (not necessarily smooth) sections of E modulo equality almost everywhere. In an obvious sense, $C_0^\infty(E) \subseteq L^2(\mathcal{Q}, E)$.

The covariant derivative $D\psi$ of a section ψ is an "E-valued 1-form," i.e., a section of $\mathbb{C}T\mathcal{Q}^* \otimes E$ (with $T\mathcal{Q}^*$ the cotangent bundle), while we write $\nabla\psi$ for the section of $\mathbb{C}T\mathcal{Q} \otimes E$ metrically equivalent to $D\psi$. To define the covariant derivative of $D\psi$, one uses the connection on $\mathbb{C}T\mathcal{Q}^* \otimes E$ that arises in an obvious way from the Levi-Civita connection on $\mathbb{C}T\mathcal{Q}^*$ and the given connection on E, with the defining property $D_{\mathbb{C}T\mathcal{Q}^* \otimes E}(\omega \otimes \psi) = (D_{\mathbb{C}T\mathcal{Q}^*}\omega) \otimes \psi + \omega \otimes (D_E \psi)$ for every 1-form ω and every section ψ of E. We take as the Laplacian $\Delta\psi$ of ψ the (Riemannian) metric trace of the second covariant derivative of ψ,

$$\Delta\psi = g^{ab} D_a D_b \psi \,, \tag{29}$$

where the second D, the one which is applied first, is the covariant derivative on E, and the first D is the covariant derivative on $\mathbb{C}T\mathcal{Q}^* \otimes E$.[11] Again, an equivalent definition is $\Delta\psi = {*d{*d\psi}}$, using the obvious extension (based on the connection of E) of the exterior derivative to E-valued differential forms, i.e., sections of $\mathbb{C}\Lambda^p T\mathcal{Q}^* \otimes E$.

[11] While this is the natural definition of the Laplacian of a section of a Hermitian bundle, we note that for differential p-forms with $p \geq 1$ there are two inequivalent natural definitions of the Laplacian: one is $\Delta = -(d^*d + dd^*)$ (sometimes called the de Rham Laplacian, with $d^* = (-1)^{(\dim \mathcal{Q})(p+1)+1} * d *$ on p-forms [21, p. 9]), the other is (29) for $E = \mathbb{C}\Lambda^p T\mathcal{Q}^*$ (sometimes called the Bochner Laplacian). They differ by a curvature term given by the Weitzenböck formula [21, p. 11].

The potential V is now a self-adjoint section of the endomorphism bundle $E \otimes E^*$ acting on the vector bundle's fibers.

The equations defining the Bohmian dynamics are the same as before. Explicitly, we define v^ψ, the Bohmian velocity vector field associated with a wave function ψ, by

$$v^\psi := \hbar \,\mathrm{Im} \frac{(\psi, \nabla\psi)}{(\psi, \psi)}. \tag{30}$$

The time evolution of the state (Q_t, ψ_t) is given by

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t) \tag{31a}$$

$$i\hbar \frac{\partial \psi_t}{\partial t} = -\tfrac{\hbar^2}{2}\Delta\psi_t + V\psi_t \tag{31b}$$

The class $\mathscr{C}_0(\mathcal{Q}, E, V)$ contains just this one dynamics, defined by (30) and (31). This agrees with the definition of $\mathscr{C}_0(\mathcal{Q}, W, V)$ given in Section 5.2 in the sense that $\mathscr{C}_0(\mathcal{Q}, W, V) = \mathscr{C}_0(\mathcal{Q}, E, V)$ when E is the trivial bundle $\mathcal{Q} \times W$.

Equivariance of the distribution $\rho = (\psi, \psi)$ is (on a formal level) obtained from the equations

$$\frac{\partial \rho_t}{\partial t} = -\nabla \cdot (\rho_t v^{\psi_t}) \tag{32}$$

for the distribution ρ_t of the configuration Q_t and

$$\frac{\partial |\psi_t|^2}{\partial t} = -\nabla \cdot (|\psi_t|^2 v^{\psi_t}), \tag{33}$$

which follow, since $(\,\cdot\,,\,\cdot\,)$ is parallel, from (30) and (31) just as (10) and (11) follow from (8) and (9).

6 Scalar Periodic Wave Functions on the Covering Space

We introduce now the Bohmian dynamics belonging to the class that we denote $\mathscr{C}_1$; in Section 8 we introduce the dynamics of class $\mathscr{C}_2$. To this end, we will consider wave functions on the universal covering space of $\mathcal{Q}$. This idea is rather standard in the literature on quantum mechanics in multiply-connected spaces [29, 11, 30, 31, 28]. However, the standard treatment lacks the precise justification that one can provide in Bohmian mechanics. Moreover, the complete classification of the possibilities that we give in Section 8 includes some, corresponding to what we call *holonomy-twisted representations* of $\pi_1(\mathcal{Q})$, that until recently [13] had not been considered. The possibilities considered so far correspond to unitary representations of $\pi_1(\mathcal{Q})$ on the value space of the wave function. Each possibility has locally the same Hamiltonian $-\frac{\hbar^2}{2}\Delta + V$, with the same potential V, and each possibility is equally well defined and equally reasonable. In this section all wave functions will be complex-valued; in Section 8 we consider wave functions with higher-dimensional value spaces.

6.1 The Circle, for Example

Let us start with the configuration space $\mathcal{Q} = S^1$, the circle. This space is multiply connected since only those loops that surround the circle as many times clockwise as counterclockwise can be shrunk to a point. It is convenient to write the wave function $\psi : S^1 \to \mathbb{C}$ as a function $\hat{\psi}(\theta)$ of the angle coordinate, with $\hat{\psi} : \mathbb{R} \to \mathbb{C}$ a 2π-periodic function. From $\hat{\psi}$ one obtains

$$\hat{v}^{\hat{\psi}} = \hbar \operatorname{Im} \frac{\nabla \hat{\psi}}{\hat{\psi}} \tag{34}$$

as a 2π-periodic function of θ. The relevant observation is that for (34) to be 2π-periodic, it is (sufficient but) not necessary that $\hat{\psi}$ be 2π-periodic. It would be sufficient as well to have a $\hat{\psi}$ that is merely periodic up to a phase shift,

$$\hat{\psi}(\theta + 2\pi) = \gamma \hat{\psi}(\theta), \tag{35}$$

where γ is a complex constant of modulus one, called a *topological phase factor*.

Another way of viewing this is to write ψ in the polar form $Re^{iS/\hbar}$, where $R \geq 0$ and the phase S is real, and find that the Bohmian velocity (23) is given by $v^\psi = \nabla S$. If we view the phase S as a function $\hat{S}(\theta)$ of the angle coordinate, we see that $\nabla \hat{S}$ will be 2π-periodic if

$$\hat{S}(\theta + 2\pi) = \hat{S}(\theta) + \beta \tag{36}$$

for some constant $\beta \in \mathbb{R}$. This corresponds to (35) with $\gamma := e^{i\beta/\hbar}$.

Since $\hat{v}^{\hat{\psi}}$ is 2π-periodic, it makes sense to write the equation of motion

$$\frac{dQ_t}{dt} = \hat{v}^{\hat{\psi}}(\theta(Q_t)) = \hbar \operatorname{Im} \frac{\nabla \hat{\psi}}{\hat{\psi}}(\theta(Q_t)) \tag{37}$$

where $\theta(Q_t)$ is any of the values of the angle coordinate that one can associate with Q_t. If we let $\hat{\psi}$ evolve by the Schrödinger equation on the real line with a 2π-periodic potential V,

$$i\hbar \frac{\partial \hat{\psi}_t}{\partial t} = -\tfrac{\hbar^2}{2} \Delta \hat{\psi}_t + V \hat{\psi}_t, \tag{38}$$

then the periodicity condition (35) is preserved by the evolution, thanks to the linearity of the Schrödinger equation. Thus, for any fixed complex γ of modulus one, (37), (38), and (35) together define a Bohmian dynamics, just as (23) and (22) do. This dynamics permits as many different wave functions as the one defined by (23) and (22), which corresponds to $\gamma = 1$.

Since $|\gamma| = 1$, so that $|\hat{\psi}|^2$ is 2π-periodic, this theory also has an equivariant probability distribution on the circle, with density $\rho = |\hat{\psi}|^2$. This is the reason why we restrict the possibilities to γ of modulus 1: otherwise we lose equivariance. (The trajectories may still exist globally even if $|\gamma| \neq 1$.)

We summarize the results of our reasoning.

Assertion 1. *For each potential V and each complex number γ of modulus one, there is a Bohmian dynamics on the circle, defined by* (35), (37), *and* (38).

According to the notation that we will define later, these dynamics form the class $\mathscr{C}_1(S^1, V)$, a one-parameter class parametrized by γ. What are the physical factors that determine which γ is to be used? It depends. The following subsection provides a concrete example.

6.2 Relation to the Aharonov–Bohm Effect

The additional possibilities associated with nontrivial phase shifts γ occur in physics even in the case of the circle. We describe here a simplified version of the Aharonov–Bohm effect [1, 12, 27, 34].

Consider a single particle confined to a loop in 3-space. Suppose there is a magnetic field $\boldsymbol{B}$ that vanishes at every point of the loop, but with field lines that pass through the interior of the loop. Thus, if D is a 2-dimensional surface bounded by the loop, there may be a nonzero flux of the magnetic field across D,

$$\Phi := \int_D \boldsymbol{B} \cdot \boldsymbol{n} \, dA, \tag{39}$$

where dA is the area element and $\boldsymbol{n}$ is the unit normal on the surface D. (Note that by Maxwell's equation $\nabla \cdot \boldsymbol{B} = 0$ and the Ostrogradski–Gauss integral formula, the value of Φ does not depend on the particular choice of the surface D.)

The appropriate quantum or Bohmian theory on S^1 corresponds, in the sense of Assertion 1, to the phase factor

$$\gamma = e^{-ie\Phi/\hbar}, \tag{40}$$

where e in the exponent is the charge of the particle, provided that the orientation of the loop and the surface agree in the sense that the direction of increasing θ and the direction of $\boldsymbol{n}$ satisfy a right-hand-rule. A justification of this description will be given in Section 7.1.

From this example we conclude that the dynamics of class $\mathscr{C}_1 \setminus \mathscr{C}_0$ *do actually occur* in a universe governed by Bohmian mechanics as the effective dynamics in a restricted configuration space.

6.3 Notation and Relevant Facts Concerning Covering Spaces

In the remainder of this section, we generalize the considerations of Section 6.1 to arbitrary $\mathcal{Q}$. The relevant notation is summarized by the table:

	Generic	Simplest example
configuration space	$\mathcal{Q}$	S^1
universal covering space	$\widehat{\mathcal{Q}}$	$\mathbb{R}$
points in $\mathcal{Q}$, $\widehat{\mathcal{Q}}$	q, $\hat{q}$	$e^{i\theta}$, θ
projection map	$\pi : \widehat{\mathcal{Q}} \to \mathcal{Q}$	$e^{i\cdot} : \mathbb{R} \to S^1$
covering fiber over q, $e^{i\theta}$	$\pi^{-1}(q)$	$\{\theta + 2\pi k \mid k \in \mathbb{Z}\}$
fundamental group of $\mathcal{Q}$	$\pi_1(\mathcal{Q})$	$\mathbb{Z}$
covering transformation	$\sigma : \widehat{\mathcal{Q}} \to \widehat{\mathcal{Q}}$	$\sigma_k : \theta \mapsto \theta + 2\pi k$
covering group	$Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$	$\{\sigma_k \mid k \in \mathbb{Z}\}$
character of fundamental group	γ_σ	$\gamma_k = \gamma^k$
bundle, lifted bundle	E, $\widehat{E}$	$S^1 \times \mathbb{C}$, $\mathbb{R} \times \mathbb{C}$

Again, $\mathcal{Q}$ is a Riemannian manifold with metric g, with universal covering space denoted by $\widehat{\mathcal{Q}}$. Recall that the universal covering space is, by definition, a simply connected space, endowed with a covering map (a local diffeomorphism) $\pi : \widehat{\mathcal{Q}} \to \mathcal{Q}$, also called the projection. The *covering fiber* for $q \in \mathcal{Q}$ is the set $\pi^{-1}(q)$ of points in $\widehat{\mathcal{Q}}$ that project to q under π. Every function or vector field on $\mathcal{Q}$ can be lifted to a function, respectively vector field, on $\widehat{\mathcal{Q}}$. The functions and vector fields on $\widehat{\mathcal{Q}}$ arising in this way are called *projectable*. A function $f : \widehat{\mathcal{Q}} \to \mathbb{C}$ is projectable if and only if $f(\hat{q}) = f(\hat{r})$ whenever $\pi(\hat{q}) = \pi(\hat{r})$. In that case it is the lift of $\tilde{f} : \mathcal{Q} \to \mathbb{C}$ given by $\tilde{f}(\pi(\hat{q})) := f(\hat{q})$, called the projection of f. A vector field w on $\widehat{\mathcal{Q}}$ is projectable if and only if, whenever $\pi(\hat{q}) = \pi(\hat{r})$, $\pi^* w(\hat{q}) = \pi^* w(\hat{r})$ where π^* is the (push-forward) action of π on tangent vectors.

We shall always take $\widehat{\mathcal{Q}}$ to be endowed with the lifted metric $\hat{g}$, which makes $\widehat{\mathcal{Q}}$ a Riemannian manifold as well and assures that π is a local isometry. As a consequence, if $\hat{f}$ is the lift of the function f, $\Delta_{\widehat{\mathcal{Q}}} \hat{f} = \widehat{\Delta_{\mathcal{Q}} f}$.

A *covering transformation* is an isometry σ mapping the covering space to itself which preserves the covering fibers, $\pi \circ \sigma = \pi$. The group of such transformations is the *covering group* and is denoted by $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. It acts freely and transitively on every covering fiber, i.e., for every $\hat{q}$ and $\hat{r}$ in the same fiber there is precisely one σ such that $\hat{r} = \sigma\hat{q}$. As a consequence, projectability of the vector field w on $\widehat{\mathcal{Q}}$ is equivalent to the condition that $w(\sigma\hat{q}) = \sigma^* w(\hat{q})$ for all $\hat{q} \in \widehat{\mathcal{Q}}$ and all $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$.

The *fundamental group at a point* q, denoted by $\pi_1(\mathcal{Q}, q)$, is the set of equivalence classes of closed loops through q, where the equivalence relation is that of homotopy, i.e. smoothly deforming one curve into the other. The product in this group is concatenation; more precisely, $\sigma\tau$ corresponds to the loop obtained by first following τ and then following σ. (This is in contrast to the common definition of the product, with the opposite order. We do it this way as it seems more natural for parallel transport.) The fundamental groups at different points are isomorphic to each other as well as to the covering group, but the isomorphisms are not canonical. However, for every given $\hat{q} \in \widehat{\mathcal{Q}}$ there is a canonical isomorphism $\varphi_{\hat{q}} : Cov(\widehat{\mathcal{Q}}, \mathcal{Q}) \to \pi_1(\mathcal{Q}, \pi(\hat{q}))$; for different choices of $\hat{q}$ in the same fiber, the different $\varphi_{\hat{q}}$'s are conjugate: they are related by $\varphi_{\sigma\hat{q}}(\tau) = \varphi_{\hat{q}}(\sigma^{-1}\tau\sigma) = \varphi_{\hat{q}}(\sigma)^{-1}\,\varphi_{\hat{q}}(\tau)\,\varphi_{\hat{q}}(\sigma)$. By *the fundamental group of*

$\mathcal{Q}$, written $\pi_1(\mathcal{Q})$, we shall mean any one of the fundamental groups $\pi_1(\mathcal{Q}, q)$.

A *character* of a group G is a unitary, 1-dimensional representation of G, i.e., a homomorphism $G \to U(1)$ where $U(1)$ is the multiplicative group of the complex numbers of modulus one. The characters of G form a group denoted by G^*.

6.4 Scalar Periodic Wave Functions

The motion of the configuration Q_t in $\mathcal{Q}$ is determined by a velocity vector field v_t on $\mathcal{Q}$, which may arise from a wave function ψ not on $\mathcal{Q}$ but instead on $\widehat{\mathcal{Q}}$ in the following way.

Suppose we are given a map $\gamma : Cov(\widehat{\mathcal{Q}}, \mathcal{Q}) \to \mathbb{C}$, and suppose that a wave function $\psi : \widehat{\mathcal{Q}} \to \mathbb{C}$ satisfies the *periodicity condition associated with the topological factors* γ, i.e.,

$$\psi(\sigma \hat{q}) = \gamma_\sigma \psi(\hat{q}) \tag{41}$$

for every $\hat{q} \in \widehat{\mathcal{Q}}$ and $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. (We no longer put the hat $\hat{}$ on top of ψ that served for emphasizing that ψ lives on the covering space.) For (41) to be possible for a ψ that does not identically vanish, γ must be a representation of the covering group, as was first emphasized in [11]. To see this, let σ_1, $\sigma_2 \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. Then we have the following equalities

$$\gamma_{\sigma_1\sigma_2}\psi(\hat{q}) = \psi(\sigma_1\sigma_2\hat{q}) = \gamma_{\sigma_1}\psi(\sigma_2\hat{q}) = \gamma_{\sigma_1}\gamma_{\sigma_2}\psi(\hat{q}). \tag{42}$$

We thus obtain the fundamental relation

$$\gamma_{\sigma_1\sigma_2} = \gamma_{\sigma_1}\gamma_{\sigma_2}, \tag{43}$$

establishing (since $\gamma_{\mathrm{Id}} = 1$) that γ is a representation.

The 1-dimensional representations of the covering group are, via the canonical isomorphisms $\varphi_{\hat{q}} : Cov(\widehat{\mathcal{Q}}, \mathcal{Q}) \to \pi_1(\mathcal{Q}, q)$, $\hat{q} \in \pi^{-1}(q)$, in canonical correspondence with the 1-dimensional representations of any fundamental group $\pi_1(\mathcal{Q}, q)$: The different isomorphisms $\varphi_{\hat{q}}$, $\hat{q} \in \pi^{-1}(q)$, will transform a representation of $\pi_1(\mathcal{Q}, q)$ into representations of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ that are conjugate. But the 1-dimensional representations are homomorphisms to the *abelian* multiplicative group of $\mathbb{C}$ and are thus invariant under conjugation.

From (41) it follows that $\nabla\psi(\sigma\hat{q}) = \gamma_\sigma\, \sigma^*\nabla\psi(\hat{q})$, where σ^* is the (push-forward) action of σ on tangent vectors, using that σ is an isometry. Thus, the velocity field $\hat{v}^\psi$ on $\widehat{\mathcal{Q}}$ associated with ψ according to

$$\hat{v}^\psi(\hat{q}) := \hbar\, \mathrm{Im}\, \frac{\nabla\psi}{\psi}(\hat{q}) \tag{44}$$

is projectable, i.e.,

$$\hat{v}^\psi(\sigma\hat{q}) = \sigma^*\hat{v}^\psi(\hat{q}), \tag{45}$$

and therefore gives rise to a velocity field v^ψ on $\mathcal{Q}$,

$$v^\psi(q) = \pi^* \, \hat{v}^\psi(\hat{q}) \tag{46}$$

where $\hat{q}$ is an arbitrary element of $\pi^{-1}(q)$.

If we let ψ evolve according to the Schrödinger equation on $\widehat{\mathcal{Q}}$,

$$i\hbar \frac{\partial \psi}{\partial t}(\hat{q}) = -\tfrac{\hbar^2}{2}\Delta\psi(\hat{q}) + \widehat{V}(\hat{q})\psi(\hat{q}) \tag{47}$$

with $\widehat{V}$ the lift of the potential V on $\mathcal{Q}$, then the periodicity condition (41) is preserved by the evolution, since, according to

$$i\hbar \frac{\partial \psi}{\partial t}(\sigma\hat{q}) \overset{(47)}{=} -\tfrac{\hbar^2}{2}\Delta\psi(\sigma\hat{q}) + \widehat{V}(\sigma\hat{q})\psi(\sigma\hat{q}) = -\tfrac{\hbar^2}{2}\Delta\psi(\sigma\hat{q}) + \widehat{V}(\hat{q})\psi(\sigma\hat{q}) \tag{48}$$

(note the different arguments in the potential), the functions $\psi \circ \sigma$ and $\gamma_\sigma \psi$ satisfy the same evolution equation (47) with, by (41), the same initial condition, and thus coincide at all times.

Therefore we can let the Bohmian configuration Q_t move according to v^{ψ_t},

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t) = \hbar\,\pi^*\Big(\mathrm{Im}\,\frac{\nabla\psi}{\psi}\Big)(Q_t) = \hbar\,\pi^*\Big(\mathrm{Im}\,\frac{\nabla\psi}{\psi}\Big|_{\hat{q}\in\pi^{-1}(Q_t)}\Big). \tag{49}$$

One can also view the motion in this way: Given Q_0, choose $\widehat{Q}_0 \in \pi^{-1}(Q_0)$, let $\widehat{Q}_t$ move in $\widehat{\mathcal{Q}}$ according to $\hat{v}^{\psi_t}$, and set $Q_t = \pi(\widehat{Q}_t)$. Then the motion of Q_t is independent of the choice of $\widehat{Q}_0$ in the fiber over Q_0, and obeys (49).

If, as we shall assume from now on, $|\gamma_\sigma| = 1$ for all $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, i.e., if γ is a *unitary* representation (in $\mathbb{C}$) or a *character*, then the motion (49) also has an equivariant probability distribution, namely

$$\rho(q) = |\psi(\hat{q})|^2. \tag{50}$$

To see this, note that we have

$$|\psi(\sigma\hat{q})|^2 \overset{(41)}{=} |\gamma_\sigma|^2 |\psi(\hat{q})|^2 = |\psi(\hat{q})|^2, \tag{51}$$

so that the function $|\psi(\hat{q})|^2$ is projectable to a function on $\mathcal{Q}$ which we call $|\psi|^2(q)$ in this paragraph. From (47) we have that

$$\frac{\partial|\psi_t(\hat{q})|^2}{\partial t} = -\nabla\cdot\Big(|\psi_t(\hat{q})|^2\,\hat{v}^{\psi_t}(\hat{q})\Big)$$

and, by projection, that

$$\frac{\partial|\psi_t|^2(q)}{\partial t} = -\nabla\cdot\Big(|\psi_t|^2(q)\,v^{\psi_t}(q)\Big),$$

which coincides with the transport equation for a probability density ρ on $\mathcal{Q}$,

$$\frac{\partial \rho_t(q)}{\partial t} = -\nabla \cdot \Big(\rho_t(q)\, v^{\psi_t}(q)\Big).$$

Hence,

$$\rho_t(q) = |\psi_t|^2(q) \tag{52}$$

for all times if it is so initially; this is equivariance.

This also makes clear that the relevant wave functions are those with

$$\int_{\mathcal{Q}} dq\, |\psi(\hat{q})|^2 = 1 \tag{53}$$

where the choice of $\hat{q} \in \pi^{-1}(q)$ is arbitrary by (51). The relevant Hilbert space, which we denote $L^2(\widehat{\mathcal{Q}}, \gamma)$, thus consists of the measurable functions ψ on $\widehat{\mathcal{Q}}$ (modulo changes on null sets) satisfying (41) with

$$\int_{\mathcal{Q}} dq\, |\psi(\hat{q})|^2 < \infty. \tag{54}$$

It is a Hilbert space with the scalar product

$$\langle \phi, \psi \rangle = \int_{\mathcal{Q}} dq\, \overline{\phi(\hat{q})}\, \psi(\hat{q}). \tag{55}$$

Note that the value of the integrand at q is independent of the choice of $\hat{q} \in \pi^{-1}(q)$ since, by (41) and the fact that $|\gamma_\sigma| = 1$,

$$\overline{\phi(\sigma\hat{q})}\, \psi(\sigma\hat{q}) = \overline{\gamma_\sigma\, \phi(\hat{q})}\, \gamma_\sigma\, \psi(\hat{q}) = \overline{\phi(\hat{q})}\, \psi(\hat{q}).$$

We summarize the results of our reasoning.

Assertion 2. *Given a Riemannian manifold $\mathcal{Q}$ and a smooth function $V : \mathcal{Q} \to \mathbb{R}$, there is a Bohmian dynamics in $\mathcal{Q}$ with potential V for each character γ of the fundamental group $\pi_1(\mathcal{Q})$; it is defined by* (41), (47), *and* (49), *where the wave function ψ_t lies in $L^2(\widehat{\mathcal{Q}}, \gamma)$ and has norm one.*

We define $\mathscr{C}_1(\mathcal{Q}, V)$ to be the class of Bohmian dynamics provided by Assertion 2. It contains as many elements as there are characters of $\pi_1(\mathcal{Q})$ because different characters $\gamma' \neq \gamma$ always define different dynamics; we give a proof of this fact in [14].[12]

[12]But, essentially, this is already clear for the same reason as why one can, in the Aharonov–Bohm effect, read off the phase shift from a shift in the interference pattern: If one splits a wave packet, located at q, into two pieces and, say, lets them move along curves β_1 and β_2 from q to r that are such that the loop $\beta_1^{-1}\beta_2$ is incontractible, then one obtains interference between the two packets at r in a way that depends on the phase shift associated with the loop $\beta_1^{-1}\beta_2$.

Interestingly, different characters can define the same dynamics when we consider, instead of complex-valued wave functions, sections of Hermitian bundles. Here is an example of such a non-trivial Hermitian bundle E: consider $\mathcal{Q} = {}^N\mathbb{R}^3$, whose fundamental group is the permutation group S_N with two characters, and the bundle $E = F \oplus B$ over ${}^N\mathbb{R}^3$, where F is what we call the fermionic line bundle (the unique flat Hermitian line bundle over ${}^N\mathbb{R}^3$ whose holonomy representation is the alternating character) and $B = {}^N\mathbb{R}^3 \times \mathbb{C}$, the trivial line bundle; see section 7.2 of [17] for a discussion.

6.5 Remarks

1. Since the law of motion (49) involves a derivative of ψ, the merely measurable functions in $L^2(\widehat{\mathcal{Q}}, \gamma)$ will of course not be adequate for defining trajectories. However, we will leave aside the question, from which dense subspace of $L^2(\widehat{\mathcal{Q}}, \gamma)$ should one choose ψ.

2. In our example $\mathcal{Q} = S^1$, the possible dynamics are precisely those mentioned in Section 6.1. The covering group is isomorphic to $\mathbb{Z}$, and every homomorphism γ is of the form $\gamma_k = \gamma_1^k$. Thus a character is determined by a complex number γ_1 of modulus one; the periodicity condition (41) reduces to (35).

3. For the trivial character $\gamma_\sigma = 1$, we obtain the immediate dynamics, as defined by (23) and (22). Thus, $\mathscr{C}_0(\mathcal{Q}, V) \subseteq \mathscr{C}_1(\mathcal{Q}, V)$.

4. Another example, or application of Assertion 2, is provided by identical particles without spin. The natural configuration space ${}^N\mathbb{R}^3$ for identical particles, defined in (12), has fundamental group S_N, the group of permutations of N objects, which possesses two characters, the trivial character, $\gamma_\sigma = 1$, and the alternating character, $\gamma_\sigma = \mathrm{sgn}(\sigma) = 1$ or -1 depending on whether $\sigma \in S_N$ is an even or an odd permutation. As explained in detail in [17], the Bohmian dynamics associated with the trivial character is that of bosons, while the one associated with the alternating character is that of fermions.

5. When $|\gamma_\sigma| \neq 1$ for some $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, in which case the equivariant distribution (50) is not defined, one could think of obtaining instead an equivariant distribution by setting
$$\rho(q) = \sum_{\hat{q} \in \pi^{-1}(q)} |\psi(\hat{q})|^2. \tag{56}$$
However, this ansatz does not work for providing an equivariant distribution in this case. Any σ for which $|\gamma_\sigma| \neq 1$ must be an element of infinite order, since otherwise γ_σ would have to be a root of unity. Thus $\pi_1(\mathcal{Q})$ is infinite, and so is the covering fiber $\pi^{-1}(q)$, which is in a canonical (given $\hat{q} \in \pi^{-1}(q)$) correspondence with $\pi_1(\mathcal{Q}, q)$, and the sum on the right hand side of (56) is divergent unless ψ vanishes everywhere on this covering fiber. (To see this, note that either $|\gamma_\sigma| > 1$ or $|\gamma_{\sigma^{-1}}| > 1$ since γ is a representation; without loss of generality we suppose $|\gamma_\sigma| > 1$. If $\psi(\hat{q}) \neq 0$ for some $\hat{q}$, then already the sum over just the fiber elements $\sigma^k \hat{q}$, $k = 1, 2, 3, \ldots$, is divergent, since by the periodicity condition (41), $\sum_k |\psi(\sigma^k \hat{q})|^2 = \sum_k |\gamma_{\sigma^k}|^2 |\psi(\hat{q})|^2 = |\psi(\hat{q})|^2 \sum_k |\gamma_\sigma|^{2k} = \infty$.)

6. As stated already in Section 3, topological factors can also be introduced into GRW theories, provided we start with a GRW theory of the following kind: Wave functions ψ are functions on a Riemannian manifold $\mathcal{Q}$, and collapses according to (7) occur with rate (5) with collapse rate operators $\Lambda(\boldsymbol{x})$ (where $\boldsymbol{x}$

is in for example $\mathbb{R}^3$) that are multiplication operators on configuration space:

$$\Lambda(\boldsymbol{x})\,\psi(q) = f_{\boldsymbol{x}}(q)\,\psi(q)\,. \tag{57}$$

Then we may define a GRW theory with topological factor given by the character γ of $\pi_1(\mathcal{Q})$ by using wave functions ψ on the covering space $\widehat{\mathcal{Q}}$ satisfying the periodicity condition (41) associated with γ, with collapse rate operators the lifted multiplication operators on $\widehat{\mathcal{Q}}$:

$$\Lambda(\boldsymbol{x})\,\psi(\hat{q}) = f_{\boldsymbol{x}}(\pi(\hat{q}))\,\psi(\hat{q})\,. \tag{58}$$

Collapse then maps periodic to periodic wave functions, with the same topological factor. Note, however, that the theory would work as well with aperiodic wave functions.

7 The Aharonov–Bohm Effect

We now give a more detailed treatment of the Aharonov–Bohm effect in the framework of Bohmian mechanics and its relation to topological phase factors. In doing so, we repeat various standard considerations on this topic.

7.1 Derivation of the Topological Phase Factor

To justify the dynamics described in Section 6.2, we consider a less idealized description of the Aharonov–Bohm effect. Consider a particle moving in $\mathbb{R}^3$ that cannot enter the solid cylinder

$$\mathcal{C} = \left\{\boldsymbol{q} = (q_1, q_2, q_3) \in \mathbb{R}^3 \,\middle|\, q_1^2 + q_2^2 \leq 1\right\} \tag{59}$$

because of, say, a potential V that goes to $+\infty$ as $\boldsymbol{q}$ approaches the cylinder from outside. The effective configuration space $\mathcal{Q} = \mathbb{R}^3 \setminus \mathcal{C}$ has the same fundamental group $\mathbb{Z}$ as the circle since it is diffeomorphic to $S^1 \times \mathbb{R}^+ \times \mathbb{R}$ (the fundamental group of a Cartesian product is the direct product of the fundamental groups, and the half plane $\mathbb{R}^+ \times \mathbb{R}$ is simply connected). The magnetic field $\boldsymbol{B}$, which vanishes outside $\mathcal{C}$ but not inside, is included in the equations by means of a vector potential $\boldsymbol{A}$ with $\nabla \times \boldsymbol{A} = \boldsymbol{B}$:

$$\frac{dQ_t}{dt} = v^{\psi}(Q_t) = \hbar\,\mathrm{Im}\frac{(\psi, (\nabla - \frac{ie}{\hbar}\boldsymbol{A})\psi)}{(\psi,\psi)}(Q_t) \tag{60a}$$

$$i\hbar\frac{\partial\psi_t}{\partial t} = -\tfrac{\hbar^2}{2}(\nabla - \tfrac{ie}{\hbar}\boldsymbol{A})^2\psi_t + V\psi_t \tag{60b}$$

with e the charge of the particle. These equations are in fact best regarded as instances of the immediate dynamics (31) on a Hermitian bundle for a nontrivial connection on the trivial vector bundle $\mathcal{Q} \times \mathbb{C}$, a point of view that we will discuss in Section 7.2.

For now we observe that the vector potential can be gauged away—not on $\mathcal{Q}$ but on the covering space $\widehat{\mathcal{Q}}$ (which is diffeomorphic to $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$). More precisely, the dynamics is unaffected by (i) lift to the covering space with $\gamma = 1$, and (ii) change of gauge.

Concerning (i), note that v^ψ is the projection of the vector field

$$\hat{v}^{\hat{\psi}} = \hbar \, \mathrm{Im} \frac{(\hat{\psi}, (\nabla - \frac{ie}{\hbar}\widehat{\boldsymbol{A}})\hat{\psi})}{(\hat{\psi}, \hat{\psi})} \tag{61}$$

where $\hat{\psi}$ is the lift of ψ (and thus a periodic wave function with $\gamma = 1$) and evolves according to the lift of (60b),

$$i\hbar \frac{\partial \hat{\psi}_t}{\partial t} = -\frac{\hbar^2}{2}(\nabla - \tfrac{ie}{\hbar}\widehat{\boldsymbol{A}})^2 \hat{\psi}_t + \widehat{V} \hat{\psi}_t \,. \tag{62}$$

We will now write again ψ, rather than $\hat{\psi}$, for the wave function on $\widehat{\mathcal{Q}}$.

Concerning (ii), a change of gauge means the simultaneous replacement

$$\widehat{\boldsymbol{A}}(\hat{q}) \to \widehat{\boldsymbol{A}}'(\hat{q}) = \widehat{\boldsymbol{A}}(\hat{q}) + \nabla f(\hat{q}) \,, \quad \psi(\hat{q}) \to \psi'(\hat{q}) = e^{ief(\hat{q})/\hbar}\psi(\hat{q}) \tag{63}$$

for an arbitrary function $f : \widehat{\mathcal{Q}} \to \mathbb{R}$. This does not change the dynamics. A vector field with vanishing curl, such as $\boldsymbol{A}$ on $\mathcal{Q}$ or $\widehat{\boldsymbol{A}}$ on $\widehat{\mathcal{Q}}$, is a gradient in every simply connected region; thus, while $\boldsymbol{A}$ is locally but not globally a gradient, $\widehat{\boldsymbol{A}}$ is globally a gradient,

$$\widehat{\boldsymbol{A}} = \nabla g \,. \tag{64}$$

The (not projectable) function $g : \widehat{\mathcal{Q}} \to \mathbb{R}$ is given by

$$g(\hat{q}) = \int_{\hat{q}_0}^{\hat{q}} \widehat{\boldsymbol{A}}(\hat{r}) \cdot d\hat{r} + C \tag{65}$$

for arbitrary $\hat{q}_0 \in \widehat{\mathcal{Q}}$, where the integration path is an arbitrary curve from $\hat{q}_0$ to $\hat{q}$ and C is a constant depending on $\hat{q}_0$. By setting $f = -g$, we can change the gauge in such a way that $\widehat{\boldsymbol{A}}'$ vanishes.

However, the change of gauge affects the periodicity of the wave function ψ: instead of $\psi(\sigma\hat{q}) = \psi(\hat{q})$ we have that

$$\psi'(\sigma\hat{q}) = \gamma_\sigma \, \psi'(\hat{q}) \tag{66}$$

with $\gamma_\sigma = \gamma^{k(\sigma)}$, where $\gamma = \exp(-ie\Phi/\hbar)$ as in (40), with Φ the magnetic flux given by (39), and $k(\sigma) \in \mathbb{Z}$ is the number of full counterclockwise rotations that the covering transformation σ induces on $\widehat{\mathcal{Q}}$.

To see this, note first that $\psi'(\sigma\hat{q}) = \exp(-ieg(\sigma\hat{q})/\hbar)\, \psi(\sigma\hat{q}) = \exp(-ieg(\sigma\hat{q})/\hbar)\, \psi(\hat{q})$. Since, by (65),

$$g(\sigma\hat{q}) = \int_{\hat{q}_0}^{\hat{q}} \widehat{\boldsymbol{A}}(\hat{q}) \cdot d\hat{r} + \int_{\hat{q}}^{\sigma\hat{q}} \widehat{\boldsymbol{A}}(\hat{r}) \cdot d\hat{r} + C = \int_{\hat{q}}^{\sigma\hat{q}} \widehat{\boldsymbol{A}}(\hat{r}) \cdot d\hat{r} + g(\hat{q})$$

for arbitrary integration paths with the indicated end points, we have that

$$\psi'(\sigma\hat{q}) = \exp\Big(-i\tfrac{e}{\hbar}\int_{\hat{\alpha}} \widehat{\boldsymbol{A}}(\hat{r})\cdot d\hat{r}\Big)\,\psi'(\hat{q}) =: \gamma_\sigma\,\psi'(\hat{q}) \tag{67}$$

for arbitrary path $\hat{\alpha}$ from $\hat{q}$ to $\sigma\hat{q}$. To evaluate the integral and show that it is independent of $\hat{q}$, note that it agrees with the corresponding integral along the projected path $\alpha = \pi(\hat{\alpha})$, a loop in $\mathcal{Q}$:

$$\int_{\hat{\alpha}} \widehat{\boldsymbol{A}}(\hat{r})\cdot d\hat{r} = \int_{\alpha} \boldsymbol{A}(r)\cdot dr\,, \tag{68}$$

which depends only on the homotopy class of α. Now consider for α a loop in $\mathcal{Q}$ that surrounds the cylinder once counterclockwise. By the Stokes integral formula, the last integral then agrees with the integral of $\boldsymbol{B}$ over any surface D in $\mathbb{R}^3$ bounded by α,

$$\int_{\hat{\alpha}} \widehat{\boldsymbol{A}}(\hat{r})\cdot d\hat{r} = \int_{D} \boldsymbol{B}\cdot\boldsymbol{n}\,dA = \Phi\,. \tag{69}$$

This completes the proof.

The dynamics can thus be described without a vector potential by a wave function on $\widehat{\mathcal{Q}}$ satisfying the periodicity condition (66). Ignoring the radial and q_3 coordinates, keeping only the circle S^1, yields the model of Section 6.2.

7.2 The Bundle View

We re-express our discussion in Section 7.1 of the Aharonov–Bohm effect in terms of Hermitian bundles.

The dynamics that fundamentally takes place in $\mathbb{R}^3$ is the immediate dynamics, given by (31), for the Hermitian bundle E^0 consisting of the trivial vector bundle $\mathbb{R}^3\times\mathbb{C}$, the local inner product $(\phi(q),\psi(q))_q = \overline{\phi(q)}\psi(q)$, and the nontrivial connection whose gradient operator is $\nabla = \nabla_{\text{trivial}} - \frac{ie}{\hbar}\boldsymbol{A}$. (The inner product is parallel iff $\mathrm{Im}\boldsymbol{A} = 0$.) The curvature of this connection is proportional to the magnetic field $\boldsymbol{B}$; therefore E^0 is curved but its restriction $E = E^0|_{\mathcal{Q}}$ to the effective configuration space $\mathcal{Q} = \mathbb{R}^3\setminus\mathcal{C}$ is flat.

Let, for the moment, E be any flat Hermitian line bundle over any Riemannian manifold $\mathcal{Q}$. Its lift $\widehat{E}$ is a trivial Hermitian bundle, like every flat bundle over a simply connected base manifold. We obtain the same dynamics (as from E) from periodic sections of $\widehat{E}$ with $\gamma = 1$. However, the description of $\widehat{E}$ in which the periodicity condition has trivial topological factor $\gamma = 1$, namely the description as the lift of E, is not the description in which the triviality of $\widehat{E}$ is manifest, namely the description relative to a trivialization, which may change the topological factor in the periodicity condition as follows.

A trivialization corresponds to a parallel choice of orthonormal basis in every fiber $\widehat{E}_{\hat{q}}$ (i.e., of an identification of $\widehat{E}_{\hat{q}}$ with $\mathbb{C}$), and thus to a parallel section ϕ of $\widehat{E}$ with

$(\phi_{\hat{q}}, \phi_{\hat{q}})_{\hat{q}} = 1$. Relative to this trivialization, a section ψ of $\widehat{E}$ corresponds to a function $\psi' : \widehat{\mathcal{Q}} \to \mathbb{C}$ according to

$$\psi(\hat{r}) = \psi'(\hat{r})\, \phi(\hat{r})\,. \tag{70}$$

If $\psi(\sigma\hat{q}) = \psi(\hat{q})$ corresponding to $\gamma = 1$ then ψ' satisfies the periodicity condition

$$\psi'(\sigma\hat{q}) = h_\alpha^{-1}\, \psi'(\hat{q})\,, \tag{71}$$

where α is any loop in $\mathcal{Q}$ based at $\pi(\hat{q})$ whose lift $\hat{\alpha}$ starting at $\hat{q}$ leads to $\sigma\hat{q}$, and h_α is the associated holonomy (which in this case, with rank $E = 1$, is a complex number of modulus 1). This follows from (70) by parallel transport along $\hat{\alpha}$, using that, by parallelity, $\phi(\sigma\hat{q}) = h_\alpha\, \phi(\hat{q})$.

As a consequence, every dynamics from $\mathscr{C}_0(\mathcal{Q}, E, V)$ for a flat Hermitian line bundle E exists also in $\mathscr{C}_1(\mathcal{Q}, V)$. In other words, we can avoid the use of a nontrivial flat Hermitian line bundle if we use a dynamics of class $\mathscr{C}_1 \setminus \mathscr{C}_0$ with suitable topological phase factor.

Let us return to the concrete bundle defined in the beginning of this section. It remains to determine the holonomy. For a loop α in $\mathcal{Q}$ that surrounds the cylinder once counterclockwise,

$$h_\alpha = \exp\Big(\tfrac{ie}{\hbar}\int_\alpha \boldsymbol{A}(r)\cdot dr\Big)\,. \tag{72}$$

Since the integral equals, according to our computation in Section 7.1, the magnetic flux Φ, the topological phase factor is given by $\gamma = \exp(-ie\Phi/\hbar)$.

8 Vector-Valued Periodic Wave Functions on the Covering Space

When the wave function is not a scalar but rather a mapping to a vector space W of dimension greater than 1, such as for a particle with spin, the topological factors can be matrices, forming a unitary representation of $\pi_1(\mathcal{Q})$, as we shall derive presently. The more complicated case in which ψ_t is a section of a vector bundle is discussed in Section 8.4. The possibility of topological factors given by representations more general than characters was first mentioned in [37], Notes to Section 23.3.

8.1 Vector Spaces

Suppose that the wave functions assume values in a Hermitian vector space W. Then in a periodicity condition analogous to (41),

$$\psi(\sigma\hat{q}) = \Gamma_\sigma\, \psi(\hat{q}), \tag{73}$$

we can allow the topological factor Γ_σ to be an endomorphism $W \to W$, rather than just a complex number. By the same argument as in the scalar case, using that $\psi(\hat{q})$ can be any element of W, Γ must be a representation of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ on W.

It follows from (73) that $\nabla\psi(\sigma\hat{q}) = (\sigma^* \otimes \Gamma_\sigma)\nabla\psi(\hat{q})$, where $\nabla\psi(\hat{q})$ is viewed as an element of $\mathbb{C}T_{\hat{q}}\widehat{\mathcal{Q}} \otimes W$. Assume now that, in addition, Γ is a *unitary* representation of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. Then the velocity field $\hat{v}^\psi$ on $\widehat{\mathcal{Q}}$ associated with ψ according to

$$\hat{v}^\psi(\hat{q}) := \hbar \,\mathrm{Im}\, \frac{(\psi, \nabla\psi)}{(\psi, \psi)}(\hat{q}) \tag{74}$$

is projectable, $\hat{v}^\psi(\sigma\hat{q}) = \sigma^* \hat{v}^\psi(\hat{q})$, and gives rise to a velocity field v^ψ on $\mathcal{Q}$. (While we used unitarity in the scalar case of Section 6.4 only for obtaining the equivariant probability density, we use it here already for having a projectable velocity field. For this purpose, we could have allowed Γ_σ to be, rather than unitary, a complex multiple of a unitary endomorphism; unitarity would then be required in order to obtain an equivariant density.)

The potential V can now assume values in the Hermitian endomorphisms of W; the space of endomorphisms can be written $W \otimes W^*$, so that V is a function $\mathcal{Q} \to W \otimes W^*$. We let ψ evolve according to the Schrödinger equation on $\widehat{\mathcal{Q}}$,

$$i\hbar \frac{\partial \psi}{\partial t}(\hat{q}) = -\tfrac{\hbar^2}{2}\Delta\psi(\hat{q}) + \widehat{V}(\hat{q})\,\psi(\hat{q}). \tag{75}$$

The periodicity condition (73) is preserved by the evolution (75) when and only when every Γ_σ commutes with every $V(q)$,

$$\Gamma_\sigma V(q) = V(q)\Gamma_\sigma \tag{76}$$

for all $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ and all $q \in \mathcal{Q}$. To see this, note that $\psi \circ \sigma$ is a solution of (75) if ψ is. Thus (73) is preserved if and only if $\Gamma_\sigma\psi$ satisfies (75), which is the case precisely when multiplication by Γ_σ commutes with the Hamiltonian. Since it trivially commutes with the Laplacian, the relevant condition is that Γ_σ commute with the potential $\widehat{V}(\hat{q})$ at every $\hat{q} \in \widehat{\mathcal{Q}}$, or, what amounts to the same, with $V(q)$ at every $q \in \mathcal{Q}$.

Given (76), we can let the configuration Q_t move according to v^{ψ_t},

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t) = \hbar\,\pi^*\Big(\mathrm{Im}\,\frac{(\psi, \nabla\psi)}{(\psi, \psi)}\Big)(Q_t). \tag{77}$$

Since Γ is a *unitary* representation of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, the motion (77) has an equivariant probability distribution, namely

$$\rho(q) = (\psi(\hat{q}), \psi(\hat{q})). \tag{78}$$

The right hand side does not depend on the choice of $\hat{q} \in \pi^{-1}(q)$ since, by (73), $(\psi(\sigma\hat{q}), \psi(\sigma\hat{q})) = (\Gamma_\sigma\psi(\hat{q}), \Gamma_\sigma\psi(\hat{q})) = (\psi(\hat{q}), \psi(\hat{q}))$. Equivariance can be established in the same way as in the scalar case.

We define the Hilbert space $L^2(\widehat{\mathcal{Q}}, W, \Gamma)$ to be the set of measurable functions $\psi : \widehat{\mathcal{Q}} \to W$ (modulo changes on null sets) satisfying (73) with

$$\int_{\mathcal{Q}} dq\, (\psi(\hat{q}), \psi(\hat{q})) < \infty, \tag{79}$$

endowed with the scalar product

$$\langle \phi, \psi \rangle = \int_{\mathcal{Q}} dq \, (\phi(\hat{q}), \psi(\hat{q})). \tag{80}$$

Again, the value of the integrand at q is independent of the choice of $\hat{q} \in \pi^{-1}(q)$.

We summarize the results of our reasoning.

Assertion 3. *Given a Riemannian manifold $\mathcal{Q}$, a Hermitian vector space W, and a Hermitian function $V : \mathcal{Q} \to W \otimes W^*$, there is a Bohmian dynamics for each unitary representation Γ of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ on W that commutes with all the endomorphisms $V(q)$; it is defined by* (73), (75)*, and* (77)*, where the wave function ψ_t lies in $L^2(\mathcal{Q}, W, \Gamma)$ and has norm 1.*

We define $\mathscr{C}_2(\mathcal{Q}, W, V)$ to be the class of Bohmian dynamics provided by Assertion 3.

The characters γ of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ (which are in a canonical one-to-one correspondence with the characters of $\pi_1(\mathcal{Q})$) are contained in Assertion 3 as special cases of unitary representations Γ by setting

$$\Gamma_\sigma = \gamma_\sigma \, \mathrm{Id}_W \, . \tag{81}$$

These are precisely those unitary representations Γ for which all Γ_σ are multiples of the identity. We define $\mathscr{C}_1(\mathcal{Q}, W, V)$ to be the class of those Bohmian dynamics from $\mathscr{C}_2(\mathcal{Q}, W, V)$ arising from a unitary representation Γ of the form (81), i.e., arising from a character. This class contains as many elements as there are characters of $\pi_1(\mathcal{Q})$, since different characters define different dynamics; we give a proof of this fact in [14, Sect. 6.4] (see also footnote 12). The definition of $\mathscr{C}_1(\mathcal{Q}, W, V)$ agrees with that of $\mathscr{C}_1(\mathcal{Q}, V)$ given in Section 6.4 in the sense that the latter is the special case $W = \mathbb{C}$, $\mathscr{C}_1(\mathcal{Q}, V) = \mathscr{C}_1(\mathcal{Q}, \mathbb{C}, V)$. Trivially, $\mathscr{C}_0(\mathcal{Q}, W, V) \subseteq \mathscr{C}_1(\mathcal{Q}, W, V) \subseteq \mathscr{C}_2(\mathcal{Q}, W, V)$.

8.2 Remarks

7. The condition that Γ be a representation of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ that commutes with V can alternatively be expressed by saying that Γ is a homomorphism $Cov(\widehat{\mathcal{Q}}, \mathcal{Q}) \to C(V)$ where $C(V)$ denotes the *centralizer* of V, i.e., the subgroup of $U(W)$ (the unitary group of W) containing all elements that commute with each $V(q)$.

8. The dynamics defined by W, V, and Γ is the same as the one defined by W', V', and Γ' (another vector space, a potential on W', and a representation on W') if there is a unitary isomorphism $U : W \to W'$ such that

$$V' = UVU^{-1} \tag{82}$$

and

$$\Gamma' = U\Gamma U^{-1} \, . \tag{83}$$

To see this, define a mapping $\psi \mapsto \psi'$, from $L^2(\widehat{\mathcal{Q}}, W, \Gamma)$ to $L^2(\widehat{\mathcal{Q}}, W', \Gamma')$, by $\psi'(\hat{q}) := U\psi(\hat{q})$. Here we use that

$$\psi'(\sigma\hat{q}) = U\psi(\sigma\hat{q}) = U\Gamma_\sigma\psi(\hat{q}) = U\Gamma_\sigma U^{-1}\psi'(\hat{q}) = \Gamma'_\sigma\psi'(\hat{q})\,.$$

Since $(-\frac{\hbar^2}{2}\Delta + \widehat{V}')\psi' = U(-\frac{\hbar^2}{2}\Delta + \widehat{V})\psi$, U intertwines the time evolutions on $L^2(\widehat{\mathcal{Q}}, W, \Gamma)$ and $L^2(\widehat{\mathcal{Q}}, W', \Gamma')$ based on V and V', i.e., $(\psi')_t = (\psi_t)'$. Since, moreover, at any fixed time ψ' and ψ lead to the same probability distribution ρ on $\mathcal{Q}$ and to the same velocity fields $\hat{v}^{\psi'} = \hat{v}^{\psi}$ and $v^{\psi'} = v^{\psi}$, ψ' and ψ lead to the same trajectories with the same probabilities. That is, the dynamics defined by W, V, Γ and the one defined by W', V', Γ' are the same.

9. As a consequence of the previous remark, we can use, in Assertion 3, representations of the *fundamental group* $\pi_1(\mathcal{Q})$ instead of representations of the *covering group* $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$.

   With a unitary representation $\tilde{\Gamma}$ of $\pi_1(\mathcal{Q}, q)$ on the vector space W (for any q) there are naturally associated several unitary representations $\Gamma(\hat{q})$ of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ on W, one for each $\hat{q} \in \pi^{-1}(q)$, defined by $\Gamma(\hat{q}) = \tilde{\Gamma} \circ \varphi_{\hat{q}}$, i.e., by $\Gamma_\tau(\hat{q}) = \tilde{\Gamma}_{\varphi_{\hat{q}}(\tau)}$, using the isomorphism $\varphi_{\hat{q}} : Cov(\widehat{\mathcal{Q}}, \mathcal{Q}) \to \pi_1(\mathcal{Q}, q)$ introduced in Section 6.3. However, these representations $\Gamma(\hat{q})$ lead to the same dynamics. To see this, consider two points $\hat{q}, \hat{r} \in \pi^{-1}(q)$ with $\hat{r} = \sigma\hat{q}$, $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. Then $\Gamma_\tau(\hat{r}) = \tilde{\Gamma}_{\varphi_{\hat{r}}(\tau)} = \tilde{\Gamma}_{\varphi_{\hat{q}}(\sigma^{-1}\tau\sigma)} = \Gamma_{\sigma^{-1}\tau\sigma}(\hat{q}) = \Gamma_\sigma(\hat{q})^{-1}\,\Gamma_\tau(\hat{q})\,\Gamma_\sigma(\hat{q}) = U\Gamma_\tau(\hat{q})U^{-1}$ with $U = \Gamma_\sigma(\hat{q})^{-1}$ a unitary endomorphism of W. Since $\tilde{\Gamma}$ commutes with V so does U, and by virtue of Remark 8, $W, V, \Gamma(\hat{q})$ defines the same dynamics as does $W, V, \Gamma(\hat{r})$.

10. As a further consequence of Remark 8, corresponding to the case in which $W' = W$ and $V' = V$, if $\Gamma' = U\Gamma U^{-1}$ for $U \in C(V)$ (so that $UVU^{-1} = V$) then W, V, and Γ' define the same dynamics as W, V, and Γ. Therefore, $\mathscr{C}_2(\mathcal{Q}, W, V)$ contains at most as many elements as there are homomorphisms $\tilde{\Gamma} : \pi_1(\mathcal{Q}) \to C(V)$ modulo conjugation by elements U of $C(V)$.

11. The characters—more precisely, the representations of the form (81)—commute with all endomorphisms of W, and are thus compatible with *every* potential. All other unitary representations Γ are compatible only with *some* potentials. (If Γ_σ is not a multiple of the identity, then there is a Hermitian endomorphism, which could occur as a $V(q)$ for some q, that does not commute with it.)

12. Moreover, characters are the only representations that commute with a potential V when (and, if $\pi_1(\mathcal{Q})$ has a nontrivial character, only when) the algebra $\mathrm{Alg}(V(\mathcal{Q}))$ generated by the $V(q)$ is the full endomorphism algebra $\mathrm{End}(W)$ of W, a condition that is satisfied for a generic potential. Thus, for a generic potential V, $\mathscr{C}_2(\mathcal{Q}, W, V) = \mathscr{C}_1(\mathcal{Q}, W, V)$.

13. Similarly, characters are the only representations that commute with several potentials $V_1, \ldots, V_m$ when the algebra $\mathrm{Alg}\big(V_1(\mathcal{Q}) \cup \ldots \cup V_m(\mathcal{Q})\big)$ generated by the $V_1(q), \ldots, V_m(q)$ is $\mathrm{End}(W)$.

14. Even for V such that $\mathrm{Alg}(V(\mathcal{Q})) \neq \mathrm{End}(W)$, generically only characters are necessary. This is because for a generic such V there will be a $q \in \mathcal{Q}$ such that $V(q)$ is nondegenerate. Since Γ_σ must commute with $V(q)$, Γ_σ and $V(q)$ must be simultaneously diagonalizable, and if $V(q)$ is nondegenerate we have that the representation Γ is diagonal, with diagonal entries $\gamma^{(i)}$ given by characters, in the basis $|i\rangle \in W$ of eigenvectors of $V(q)$. In other words, the representation is of the form

$$\Gamma_\sigma = \sum_i \gamma_\sigma^{(i)} P_{W^{(i)}}, \tag{84}$$

where $P_{W^{(i)}}$ is the projection onto the i-th eigenspace $W^{(i)} = \mathbb{C}|i\rangle$ of $V(q)$. Moreover, when the $\gamma^{(i)}$'s are all different, we then have that every $V(r)$ is diagonal in the basis $|i\rangle$ and the corresponding Schrödinger dynamics, of class $\mathscr{C}_2(\mathcal{Q}, W, V)$, can be decomposed into a direct sum of dynamics of class $\mathscr{C}_1(\mathcal{Q}, W^{(i)}, V^{(i)})$, given by characters, where $V^{(i)}$ is the action of V on $W^{(i)}$. Thus, the set of dynamics corresponding to representations Γ of the form (84) could be denoted

$$\bigoplus_i \mathscr{C}_1(\mathcal{Q}, W^{(i)}, V^{(i)}). \tag{85}$$

When the $\gamma^{(i)}$'s are not distinct, a similar decomposition holds, with the sum over i replaced by the sum over the distinct characters γ and with the $W^{(i)}$'s replaced by the spans of the $W^{(i)}$'s corresponding to the same γ.

15. In the situation described in the previous remark, the representation Γ on W is reducible, as it clearly is when it is given by a character (unless $\dim W = 1$). In fact, by Schur's lemma, Γ can be irreducible only when the potential V is a scalar, i.e., of the form $V(q) = \tilde{V}(q)\mathrm{Id}_W$ with $\tilde{V}(q) \in \mathbb{R}$.

16. We have so far considered the possible Bohmian dynamics associated with a configuration space $\mathcal{Q}$, a Hermitian vector space W and a Hermitian function $V : \mathcal{Q} \to W \otimes W^*$, and have argued that we have one such dynamics for each representation Γ of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ that commutes with V. Let us now consider the class $\mathscr{C}_2(\mathcal{Q}, W, \Gamma)$ of possible Bohmian dynamics associated with a configuration space $\mathcal{Q}$, a Hermitian vector space W, and a representation Γ of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. There is of course one such dynamics for every choice of V that commutes with Γ but there are more. In fact, in addition to these there is also a dynamics for every Hermitian function $V^* : \widehat{\mathcal{Q}} \to W \otimes W^*$ satisfying

$$V^*(\sigma\hat{q}) = \Gamma_\sigma V^*(\hat{q}) \Gamma_\sigma^{-1}, \tag{86}$$

a dynamics involving a potential $V^*(\hat{q})$ on $\widehat{\mathcal{Q}}$ that need not be the lift of any potential V on $\mathcal{Q}$. If Γ is given by a character then V^* must in fact be the lift of a potential V on $\mathcal{Q}$, $V^*(\hat{q}) = V(\pi(q))$, and we obtain nothing new, but when Γ is not given by a character, many new possibilities occur.

8.3 Examples

Let us give an example of matrices as topological factors: a higher-dimensional version of the Aharonov–Bohm effect. We may replace the vector potential in the Aharonov–Bohm setting by a non-abelian gauge field (à la Yang–Mills) whose field strength (curvature) vanishes outside the cylinder $\mathcal{C}$ but not inside; the value space W (now corresponding not to spin but to, say, quark color) has dimension greater than one, and the difference between two wave packets that have passed the cylinder $\mathcal{C}$ on different sides is in general, rather than a phase, a unitary endomorphism Γ of W.

A more practical version is provided by the Aharonov–Casher variant [2] of the Aharonov–Bohm effect, according to which a neutral spin-1/2 particle that carries a magnetic moment μ acquires a nontrivial phase while encircling a charged wire $\mathcal{C}$. Start with the Dirac equation for a neutral particle with nonzero magnetic moment μ (such as a neutron),

$$i\hbar\gamma^\mu\partial_\mu\psi = m\psi + \tfrac{1}{2}\mu F^{\mu\nu}\sigma_{\mu\nu}\psi\,, \tag{87}$$

where $\psi : \mathbb{R}^4 \to \mathbb{C}^4$, γ^μ are the four Dirac matrices, $F^{\mu\nu}$ is the field tensor of the external electromagnetic field, and $\sigma_{\mu\nu} = \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu$. The last term in (87) should be regarded as phenomenological. Consider now the nonrelativistic limit, in which the wave function assumes values in spin space $W = \mathbb{C}^2$, acted upon by the vector $\boldsymbol{\sigma}$ of spin matrices. Suppose that the magnetic field is zero and the electric field $\boldsymbol{E}$ is generated by a charge distribution $\varrho(\boldsymbol{q})$ inside $\mathcal{C}$ which is invariant under translations in the direction $\boldsymbol{e} \in \mathbb{R}^3$, $\boldsymbol{e}^2 = 1$ in which the wire is oriented. Then the charge per unit length λ is given by the integral

$$\lambda = \int_D \varrho(\boldsymbol{q})\, dA \tag{88}$$

over the cross-section disk D in any plane perpendicular to $\boldsymbol{e}$. The Hamiltonian is [2]

$$H = -\tfrac{\hbar^2}{2m}\big(\nabla - \tfrac{i\mu}{\hbar}\boldsymbol{E}\times\boldsymbol{\sigma}\big)^2 - \tfrac{\mu^2}{m}\boldsymbol{E}^2\,, \tag{89}$$

where $\times$ denotes the vector product in $\mathbb{R}^3$. This looks like a Hamiltonian $-\frac{\hbar^2}{2}\Delta + V$ based on a nontrivial connection $\nabla = \nabla_{\text{trivial}} - \frac{i\mu}{\hbar}\boldsymbol{E}\times\boldsymbol{\sigma}$ on the vector bundle $\mathbb{R}^3\times\mathbb{C}^2$. The restriction of this connection, outside of $\mathcal{C}$, to any plane Σ orthogonal to the wire turns out to be flat[13] so that its restriction to the intersection $\mathcal{Q}$ of $\mathbb{R}^3\setminus\mathcal{C}$ with

[13] The curvature is $\Omega = d_{\text{trivial}}\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega}$ with $\boldsymbol{\omega} = -i\frac{\mu}{\hbar}\boldsymbol{E}\times\boldsymbol{\sigma}$. The 2-form Ω is dual to the vector $\nabla_{\text{trivial}}\times\boldsymbol{\omega} + \boldsymbol{\omega}\times\boldsymbol{\omega} = i\frac{\mu}{\hbar}(\nabla\cdot\boldsymbol{E})\boldsymbol{\sigma} - i\frac{\mu}{\hbar}(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E} - 2i(\frac{\mu}{\hbar})^2(\boldsymbol{\sigma}\cdot\boldsymbol{E})\boldsymbol{E}$. Outside the wire, the first term vanishes and, noting that $\boldsymbol{E}\cdot\boldsymbol{e} = 0$, the other two terms have vanishing component in the direction of $\boldsymbol{e}$ and thus vanish when integrated over any region within an orthogonal plane.

the orthogonal plane can be replaced, as in the Aharonov–Bohm case, by the trivial connection if we introduce a periodicity condition on the wave function with the topological factor

$$\Gamma_1 = \exp\Big(-\frac{4\pi i\mu\lambda}{\hbar}\,\boldsymbol{e}\cdot\boldsymbol{\sigma}\Big)\,. \tag{90}$$

In this way we obtain a representation $\Gamma : \pi_1(\mathcal{Q}) \to SU(2)$ that is not given by a character. (For further discussion of the link between gauge connections and topological factors, see [14].)

Though the Γ's are matrices in the above examples, the representation is still abelian since $\pi_1(\mathcal{Q}) \cong \mathbb{Z}$ is an abelian group. To obtain a non-abelian representation, let $\mathcal{Q}$ be $\mathbb{R}^3$ minus two disjoint solid cylinders; its fundamental group is isomorphic to the non-abelian group $\mathbb{Z} * \mathbb{Z}$ where $*$ denotes the free product of groups; it is generated by two loops, σ_1 and σ_2, each surrounding one of the cylinders. Using again non-abelian gauge fields, one can arrange that the matrices Γ_{σ_i} corresponding to σ_i, $i = 1, 2$, do not commute with each other.

Another example, concerning a system of N spin 1/2 fermions: $\mathcal{Q} = {}^N\mathbb{R}^3$ (whose fundamental group is the permutation group S_N), $W = \bigotimes_{i=1}^N \mathbb{C}^2$, and $\Gamma_\sigma = \mathrm{sgn}(\sigma) R_\sigma$, where R_σ is the natural action of permutations on the tensor product. The Pauli spin interaction is well defined on $\widehat{\mathcal{Q}}$ (but not on $\mathcal{Q}$) for this W (unlike for the natural spin bundle (28)). It is given by

$$V^*(\hat q) = -\mu \sum_i \boldsymbol{B}(\boldsymbol{q}_i)\cdot\boldsymbol{\sigma}_i \tag{91}$$

with $\boldsymbol{\sigma}_i$ the vector of spin matrices acting on the i-th component of the tensor product. V^* is not the lift of any potential V on $\mathcal{Q}$ (since there is no continuous section of the bundle, over $\mathcal{Q} = {}^N\mathbb{R}^3$, of maps $q \mapsto \{1, \ldots, N\}$). This is an example of class $\mathscr{C}_2(\mathcal{Q}, W, \Gamma)$, see Remark 16.

8.4 Vector Bundles

We now consider wave functions that are sections of vector bundles. The topological factors will be expressed as *periodicity sections*, i.e., parallel unitary sections of the endomorphism bundle indexed by the covering group and satisfying a certain composition law, or, equivalently, as *twisted representations* of $\pi_1(\mathcal{Q})$.

If E is a vector bundle over $\mathcal{Q}$, then the lift of E, denoted by $\widehat{E}$, is a vector bundle over $\widehat{\mathcal{Q}}$; the fiber space at $\hat q$ is defined to be the fiber space of E at q, $\widehat{E}_{\hat q} := E_q$, where $q = \pi(\hat q)$. It is important to realize that with this construction, it makes sense to ask whether $v \in \widehat{E}_{\hat q}$ is equal to $w \in \widehat{E}_{\hat r}$ whenever $\hat q$ and $\hat r$ are elements of the same covering fiber. Equivalently, $\widehat{E}$ is the pull-back of E through $\pi : \widehat{\mathcal{Q}} \to \mathcal{Q}$. As a particular example, the lift of the tangent bundle of $\mathcal{Q}$ to $\widehat{\mathcal{Q}}$ is canonically isomorphic to the tangent bundle of $\widehat{\mathcal{Q}}$. Sections of E or $E \otimes E^*$ can be lifted to sections of $\widehat{E}$ respectively $\widehat{E} \otimes \widehat{E}^*$.

If E is a Hermitian vector bundle, then so is $\widehat{E}$. The wave function ψ that we consider here is a section of $\widehat{E}$, so that the $\hat{q}$-dependent Hermitian vector space $\widehat{E}_{\hat{q}}$ replaces the fixed Hermitian vector space W of the previous subsection. V is a section of the bundle $E \otimes E^*$, i.e., $V(q)$ is an element of $E_q \otimes E_q^*$. To indicate that every $V(q)$ is a Hermitian endomorphism of E_q, we say that V is a Hermitian section of $E \otimes E^*$.

Since $\psi(\sigma\hat{q})$ and $\psi(\hat{q})$ lie in the same space $E_q = \widehat{E}_{\hat{q}} = \widehat{E}_{\sigma\hat{q}}$, a periodicity condition can be of the form

$$\psi(\sigma\hat{q}) = \Gamma_\sigma(\hat{q})\,\psi(\hat{q}) \tag{92}$$

for $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, where $\Gamma_\sigma(\hat{q})$ is an endomorphism $E_q \to E_q$. By the same argument as in (42), the condition for (92) to be possible, if $\psi(\hat{q})$ can be any element of $\widehat{E}_{\hat{q}}$, is the composition law

$$\Gamma_{\sigma_1\sigma_2}(\hat{q}) = \Gamma_{\sigma_1}(\sigma_2\hat{q})\,\Gamma_{\sigma_2}(\hat{q}). \tag{93}$$

Note that this law differs from the one $\Gamma(\hat{q})$ would satisfy if it were a representation, which reads $\Gamma_{\sigma_1\sigma_2}(\hat{q}) = \Gamma_{\sigma_1}(\hat{q})\,\Gamma_{\sigma_2}(\hat{q})$, since in general $\Gamma(\sigma\hat{q})$ need not be the same as $\Gamma(\hat{q})$.

For periodicity (92) to be preserved under the Schrödinger evolution,

$$i\hbar\frac{\partial\psi}{\partial t}(\hat{q}) = -\tfrac{\hbar^2}{2}\Delta\psi(\hat{q}) + \widehat{V}(\hat{q})\,\psi(\hat{q}), \tag{94}$$

we need that multiplication by $\Gamma_\sigma(\hat{q})$ commute with the Hamiltonian. Observe that

$$[H,\Gamma_\sigma]\psi(\hat{q}) = -\tfrac{\hbar^2}{2}(\Delta\Gamma_\sigma(\hat{q}))\psi(\hat{q}) - \hbar^2(\nabla\Gamma_\sigma(\hat{q}))\cdot(\nabla\psi(\hat{q})) + [\widehat{V}(\hat{q}),\Gamma_\sigma(\hat{q})]\,\psi(\hat{q}). \tag{95}$$

Since we can choose ψ such that, for any one particular $\hat{q}$, $\psi(\hat{q}) = 0$ and $\nabla\psi(\hat{q})$ is any element of $\mathbb{C}T_{\hat{q}}\widehat{\mathcal{Q}} \otimes E_q$ we like, we must have that

$$\nabla\Gamma_\sigma(\hat{q}) = 0 \tag{96}$$

for all $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ and all $\hat{q} \in \widehat{\mathcal{Q}}$, i.e., that Γ_σ is parallel. . Inserting this in (95), the first two terms on the right hand side vanish. Since we can choose for $\psi(\hat{q})$ any element of E_q we like, we must have that

$$[\widehat{V}(\hat{q}),\Gamma_\sigma(\hat{q})] = 0 \tag{97}$$

for all $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ and all $\hat{q} \in \widehat{\mathcal{Q}}$. Conversely, assuming (96) and (97), we obtain that Γ_σ commutes with H for every $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, so that the periodicity (92) is preserved.

From (92) and (96) it follows that $\nabla\psi(\sigma\hat{q}) = (\sigma^* \otimes \Gamma_\sigma(\hat{q}))\nabla\psi(\hat{q})$. If every $\Gamma_\sigma(\hat{q})$ is *unitary*, as we assume from now on, the velocity field $\hat{v}^\psi$ on $\widehat{\mathcal{Q}}$ associated with ψ according to

$$\hat{v}^\psi(\hat{q}) := \hbar\,\mathrm{Im}\,\frac{(\psi,\nabla\psi)}{(\psi,\psi)}(\hat{q}) \tag{98}$$

is projectable, $\hat{v}^\psi(\sigma\hat{q}) = \sigma^*\hat{v}^\psi(\hat{q})$, and gives rise to a velocity field v^ψ on $\mathcal{Q}$. We let the configuration move according to v^{ψ_t},

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t) = \hbar\,\pi^*\Big(\mathrm{Im}\,\frac{(\psi,\nabla\psi)}{(\psi,\psi)}\Big)(Q_t). \tag{99}$$

Definition 2. *Let E be a Hermitian bundle over the manifold $\mathcal{Q}$. A* periodicity section Γ *over E is a family indexed by $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ of unitary parallel sections Γ_σ of $\widehat{E} \otimes \widehat{E}^*$ satisfying the composition law* (93).

Since $\Gamma_\sigma(\hat{q})$ is unitary, one sees as before that the probability distribution

$$\rho(q) = (\psi(\hat{q}), \psi(\hat{q})) \tag{100}$$

does not depend on the choice of $\hat{q} \in \pi^{-1}(q)$ and is equivariant.

As usual, we define for any periodicity section Γ the Hilbert space $L^2(\widehat{\mathcal{Q}}, \widehat{E}, \Gamma)$ to be the set of measurable sections ψ of $\widehat{E}$ (modulo changes on null sets) satisfying (92) with

$$\int_{\mathcal{Q}} dq\, (\psi(\hat{q}), \psi(\hat{q})) < \infty, \tag{101}$$

endowed with the scalar product

$$\langle \phi, \psi \rangle = \int_{\mathcal{Q}} dq\, (\phi(\hat{q}), \psi(\hat{q})). \tag{102}$$

As before, the value of the integrand at q is independent of the choice of $\hat{q} \in \pi^{-1}(q)$.

We summarize the results of our reasoning.

Assertion 4. *Given a Hermitian bundle E over the Riemannian manifold $\mathcal{Q}$ and a Hermitian section V of $E \otimes E^*$, there is a Bohmian dynamics for each periodicity section Γ commuting (pointwise) with $\widehat{V}$ (cf. (97)); it is defined by* (92), (94), *and* (99), *where the wave function ψ_t lies in $L^2(\widehat{\mathcal{Q}}, \widehat{E}, \Gamma)$ and has norm 1.*

The situation of Section 8.1, where the wave function assumed values in a fixed Hermitian space W instead of a bundle, corresponds to the trivial Hermitian bundle $E = \mathcal{Q} \times W$ (i.e., with the trivial connection, for which parallel transport is the identity on W). Then, parallelity (96) implies that $\Gamma_\sigma(\hat{r}) = \Gamma_\sigma(\hat{q})$ for any $\hat{r}, \hat{q} \in \widehat{\mathcal{Q}}$, or $\Gamma_\sigma(\hat{q}) = \Gamma_\sigma$, so that (93) becomes the usual composition law $\Gamma_{\sigma_1\sigma_2} = \Gamma_{\sigma_1}\Gamma_{\sigma_2}$. As a consequence, Γ is a unitary representation of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, and Assertion 3 is a special case of Assertion 4.

Every character γ of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ (or of $\pi_1(\mathcal{Q})$) defines a periodicity section by setting

$$\Gamma_\sigma(\hat{q}) := \gamma_\sigma \mathrm{Id}_{\widehat{E}_{\hat{q}}}. \tag{103}$$

It commutes with every potential V. Conversely, a periodicity section Γ that commutes with every potential must be such that every $\Gamma_\sigma(\hat{q})$ is a multiple of the identity, $\Gamma_\sigma(\hat{q}) = \gamma_\sigma(\hat{q})\, \mathrm{Id}_{\widehat{E}_{\hat{q}}}$. By unitarity, $|\gamma_\sigma| = 1$; by parallelity (96), $\gamma_\sigma(\hat{q}) = \gamma_\sigma$ must be constant; by the composition law (93), γ must be a homomorphism, and thus a character.

We define $\mathscr{C}_2(\mathcal{Q}, E, V)$ to be the class of Bohmian dynamics provided by Assertion 4. We define $\mathscr{C}_1(\mathcal{Q}, E, V)$ to be the class of those Bohmian dynamics from

$\mathscr{C}_2(\mathcal{Q}, E, V)$ arising from characters: The class $\mathscr{C}_1(\mathcal{Q}, E, V)$ contains at most[14] as many elements as there are characters of $\pi_1(\mathcal{Q})$. These definitions agree with the definitions of $\mathscr{C}_1(\mathcal{Q}, W, V)$ and $\mathscr{C}_2(\mathcal{Q}, W, V)$ given in Section 8.1 in the sense that $\mathscr{C}_1(\mathcal{Q}, W, V) = \mathscr{C}_1(\mathcal{Q}, E, V)$ and $\mathscr{C}_2(\mathcal{Q}, W, V) = \mathscr{C}_2(\mathcal{Q}, E, V)$ when E is taken to be the trivial bundle $\mathcal{Q} \times W$.

We briefly indicate how a periodicity section Γ corresponds to something like a representation of $\pi_1(\mathcal{Q})$, in fact to a (holonomy-) twisted representation of $\pi_1(\mathcal{Q})$. Fix a $\hat{q} \in \widehat{\mathcal{Q}}$. Then $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ can be identified with $\pi_1(\mathcal{Q}) = \pi_1(\mathcal{Q}, \pi(\hat{q}))$ via $\varphi_{\hat{q}}$. Since the sections Γ_σ of $\widehat{E} \otimes \widehat{E}^*$ are parallel, $\Gamma_\sigma(\hat{r})$ is determined for every $\hat{r}$ by $\Gamma_\sigma(\hat{q})$. (Note in particular that the parallel transport $\Gamma_\sigma(\tau\hat{q})$ of $\Gamma_\sigma(\hat{q})$ from $\hat{q}$ to $\tau\hat{q}, \tau \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, may differ from $\Gamma_\sigma(\hat{q})$.) Thus, the periodicity section Γ is completely determined by the endomorphisms $\Gamma_\sigma := \Gamma_\sigma(\hat{q})$ of E_q, $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, which satisfy the composition law

$$\Gamma_{\sigma_1\sigma_2} = h_{\alpha_2} \Gamma_{\sigma_1} h_{\alpha_2}^{-1} \Gamma_{\sigma_2}\,, \tag{104}$$

where α_2 is any loop in $\mathcal{Q}$ based at $\pi(\hat{q})$ whose lift starting at $\hat{q}$ leads to $\sigma_2\hat{q}$, and h_{α_2} is the associated holonomy endomorphism of E_q. Since (104) is not the composition law $\Gamma_{\sigma_1\sigma_2} = \Gamma_{\sigma_1}\Gamma_{\sigma_2}$ of a representation, the Γ_σ form, not a representation of $\pi_1(\mathcal{Q})$, but what we call a *twisted representation.* See [14] for further discussion.

8.5 Further Remarks

17. The dynamics defined by E, V, and Γ is the same as the one defined by E', V', and Γ' (another Hermitian bundle, a potential on E', and a periodicity section over E') if there is a unitary parallel section U of $\widehat{E}' \otimes \widehat{E}^*$ such that

$$\widehat{V}'(\hat{q}) = U(\hat{q})\, \widehat{V}(\hat{q})\, U(\hat{q})^{-1} \tag{105}$$

and

$$\Gamma'_\sigma(\hat{q}) = U(\sigma\hat{q})\, \Gamma_\sigma(\hat{q})\, U(\hat{q})^{-1}. \tag{106}$$

To see this, define a mapping $\psi \mapsto \psi'$, from $L^2(\widehat{\mathcal{Q}}, \widehat{E}, \Gamma)$ to $L^2(\widehat{\mathcal{Q}}, \widehat{E}', \Gamma')$, by $\psi'(\hat{q}) := U(\hat{q})\, \psi(\hat{q})$. Here we use that

$$\begin{aligned}\psi'(\sigma\hat{q}) &= U(\sigma\hat{q})\, \psi(\sigma\hat{q}) = U(\sigma\hat{q})\, \Gamma_\sigma(\hat{q})\, \psi(\hat{q}) \\ &= U(\sigma\hat{q})\, \Gamma_\sigma(\hat{q})\, U(\hat{q})^{-1}\, \psi'(\hat{q}) = \Gamma'_\sigma(\hat{q})\, \psi'(\hat{q})\,.\end{aligned}$$

Since, by the parallelity of U, $(-\frac{\hbar^2}{2}\Delta + \widehat{V}')\psi' = U(-\frac{\hbar^2}{2}\Delta + \widehat{V})\psi$, U intertwines the time evolutions on $L^2(\widehat{\mathcal{Q}}, \widehat{E}, \Gamma)$ and $L^2(\widehat{\mathcal{Q}}, \widehat{E}', \Gamma')$ based on V and V', i.e., $(\psi')_t = (\psi_t)'$. Since, moreover, at any fixed time ψ' and ψ lead to the same probability distribution ρ on $\mathcal{Q}$ (by the unitarity of U) and to the same velocity fields $\hat{v}^{\psi'} = \hat{v}^{\psi}$ and $v^{\psi'} = v^{\psi}$ (by the parallelity and unitarity of U), ψ' and ψ lead to the same trajectories with the same probabilities. That is, the dynamics defined by E, V, Γ and the one defined by E', V', Γ' are the same.

[14]For nontrivial Hermitian bundles, different characters can lead to the same dynamics; we gave an example in footnote 12.

18. As a consequence of the previous remark, we can use, in Assertion 4, twisted representations $\tilde{\Gamma}$ of the fundamental group $\pi_1(\mathcal{Q})$, satisfying

$$\tilde{\Gamma}_{\alpha_1\alpha_2} = h_{\alpha_2}\tilde{\Gamma}_{\alpha_1}h_{\alpha_2}^{-1}\tilde{\Gamma}_{\alpha_2}, \tag{107}$$

instead of periodicity sections (twisted representations of the covering group $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$).

19. As a further consequence of Remark 17, corresponding to the case in which $E' = E$ and $V' = V$, if $\Gamma'_\sigma(\hat{q}) = U(\sigma\hat{q})\,\Gamma_\sigma(\hat{q})\,U(\hat{q})^{-1}$ for a unitary parallel section U of $\widehat{E}' \otimes \widehat{E}^*$ that commutes with $\widehat{V}$ then E, V, and Γ' define the same dynamics as E, V, and Γ. Therefore, $\mathscr{C}_2(\mathcal{Q}, E, V)$ contains at most as many elements as there are twisted representations $\tilde{\Gamma}$ of $\pi_1(\mathcal{Q})$, i.e., periodicity sections Γ over E, that commute with $\widehat{V}$, modulo conjugation by such U's.

20. As we have already seen, the characters—the periodicity sections of the form (103)—are compatible with every potential, and all other periodicity sections are compatible only with some potentials.

21. A potential V does not commute with any periodicity section save the characters when (and, if $\pi_1(\mathcal{Q})$ has a nontrivial character, only when) for arbitrary $q \in \mathcal{Q}$,

$$\mathrm{Alg}\big(V(\mathcal{Q})_q \cup \Theta_q\big) = \mathrm{End}(E_q)\,, \tag{108}$$

where $V(\mathcal{Q})_q = \big\{P_\beta^{-1}V(r)P_\beta : r \in \mathcal{Q}, \beta \text{ a curve from } q \text{ to } r\big\}$ with $P_\beta : E_q \to E_r$ denoting parallel transport, and $\Theta_q = \{h_\alpha : \alpha \text{ a contractible loop based at q}\}$ with $h_\alpha = P_\alpha$, the holonomy of α. This follows from the fact that a periodicity section, by parallelity, must commute with Θ_q. The condition (108) holds, for example, for the potential occurring in the Pauli equation for N identical particles with spin,

$$V(q) = -\mu \sum_{\boldsymbol{q}\in q} \boldsymbol{B}(\boldsymbol{q}) \cdot \boldsymbol{\sigma}_{\boldsymbol{q}} \tag{109}$$

on the spin bundle (28) over ${}^N\mathbb{R}^3$, with $\boldsymbol{\sigma}_{\boldsymbol{q}}$ the vector of spin matrices acting on the spin space of the particle at $\boldsymbol{q}$, provided merely that the magnetic field $\boldsymbol{B}$ is not parallel. Thus, for a generic potential V, $\mathscr{C}_2(\mathcal{Q}, E, V) = \mathscr{C}_1(\mathcal{Q}, E, V)$.

22. A periodicity section Γ defining a Bohmian dynamics of class $\mathscr{C}_2(\mathcal{Q}, E, V)$ can be irreducible only when the potential V is a scalar. (When Γ is reducible, its decomposition may involve sub-bundles $E^{(i)}$ of $\widehat{E}$ that are not the lifts of any sub-bundles of E.)

23. Consider the class $\mathscr{C}_2(\mathcal{Q}, E, \Gamma)$ of possible Bohmian dynamics associated with a Riemannian manifold $\mathcal{Q}$, a Hermitian bundle E over $\mathcal{Q}$, and a periodicity section Γ over E. There is one such dynamics for every choice of Hermitian section V^* of $\widehat{E} \otimes \widehat{E}^*$ satisfying

$$V^*(\sigma\hat{q}) = \Gamma_\sigma(\hat{q})V^*(\hat{q})\Gamma_\sigma(\hat{q})^{-1}, \tag{110}$$

a dynamics involving a potential $V^*(\hat{q})$ on $\widehat{\mathcal{Q}}$ that need not be the lift of any potential V on $\mathcal{Q}$.

24. For a *generic* (curved) Hermitian bundle E, and any fixed V we have that $\mathscr{C}_2(\mathcal{Q}, E, V) = \mathscr{C}_1(\mathcal{Q}, E, V)$; in other words, there are no more possibilities than the characters. This follows from the fact that, generically, for every unitary endomorphism U of E_q there is a contractible curve α based at q whose holonomy is U. That is, Θ_q is the full unitary group of E_q, and thus, by (108), all periodicity sections correspond to characters.

25. Consider a dynamics of class $\mathscr{C}_2(\mathcal{Q}, W, \Gamma)$ or class $\mathscr{C}_2(\mathcal{Q}, E, \Gamma)$, given by a potential V^* on $\widehat{\mathcal{Q}}$ satisfying (86), respectively (110). We show in [14] that there is a Hermitian bundle E' over $\mathcal{Q}$ that is locally isomorphic to $\mathcal{Q} \times W$, respectively E, such that this dynamics, of class $\mathscr{C}_2$, coincides with the dynamics of class $\mathscr{C}_0(\mathcal{Q}, E', V)$, i.e., the immediate dynamics for $\mathcal{Q}$, E', and a potential V on $\mathcal{Q}$. For example, the dynamics associated with $\mathcal{Q} = {}^N\mathbb{R}^3$, $W = \bigotimes_{i=1}^N \mathbb{C}^{2s+1}$, and V^* on $\widehat{\mathcal{Q}}$ given by (91) coincides with the dynamics associated with ${}^N\mathbb{R}^3$, the natural spin bundle E' (28) over ${}^N\mathbb{R}^3$, and the Pauli interaction V on $\mathcal{Q}$ given by (109). (By "dynamics" here we refer to the evolution of the configuration.)

8.6 Examples Involving Vector Bundles

We close this section with two examples of topological factors on vector bundles.

The most important example is provided by identical particles *with spin*. In fact, for this case, Assertion 4 entails the same conclusions we arrived at in Remark 4, the alternative between bosons and fermions, even for particles with spin. To understand how this comes about, consider the potential occurring in the Pauli equation (109) for N identical particles with spin, on the spin bundle (28) over ${}^N\mathbb{R}^3$, and observe that the algebra generated by $\{V(q)\}$ arising from all possible choices of the magnetic field $\boldsymbol{B}$ is $\mathrm{End}(E_q)$. Thus the only holonomy-twisted representations that define a dynamics for all magnetic fields (or even for a single magnetic field provided only that it is not parallel, see Remark 21) are those given by a character.

Our last example involves a holonomy-twisted representation Γ that is not a representation in the ordinary sense. Consider N fermions, each as in the examples at the beginning of Section 8.3, moving in $M = \mathbb{R}^3 \setminus \cup_i \mathcal{C}_i$, where $\mathcal{C}_i$ are one or more disjoint solid cylinders. More generally, consider N fermions, each having 3-dimensional configuration space M and value space W (which may incorporate spin or "color" or both). Then the configuration space $\mathcal{Q}$ for the N fermions is the set ${}^NM$ of all N-element subsets of M, with universal covering space $\widehat{\mathcal{Q}} = \widehat{{}^NM} = \widehat{M}^N \setminus \Delta$ with Δ the extended diagonal, the set of points in $\widehat{M}^N$ whose projection to M^N lies in its coincidence set. Every diffeomorphism $\sigma \in Cov(\widehat{{}^N M}, {}^N M)$ can be expressed as a product

$$\sigma = p\tilde{\sigma} \tag{111}$$

where $p \in S_N$ and $\tilde{\sigma} = (\sigma^{(1)}, \ldots, \sigma^{(N)}) \in Cov(\widehat{M}, M)^N$ and these act on $\hat{q} = (\hat{\boldsymbol{q}}_1, \ldots, \hat{\boldsymbol{q}}_N) \in \widehat{M}^N$ as follows:

$$\tilde{\sigma}\hat{q} = (\sigma^{(1)}\hat{\boldsymbol{q}}_1, \ldots, \sigma^{(N)}\hat{\boldsymbol{q}}_N) \tag{112}$$

and

$$p\hat{q} = (\hat{\boldsymbol{q}}_{p^{-1}(1)}, \ldots, \hat{\boldsymbol{q}}_{p^{-1}(N)}). \tag{113}$$

Thus

$$\sigma\hat{q} = (\sigma^{(p^{-1}(1))}\hat{\boldsymbol{q}}_{p^{-1}(1)}, \ldots, \sigma^{(p^{-1}(N))}\hat{\boldsymbol{q}}_{p^{-1}(N)}). \tag{114}$$

Moreover, the representation (111) of σ is unique. Thus, since

$$\sigma_1\sigma_2 = p_1\tilde{\sigma}_1 p_2\tilde{\sigma}_2 = (p_1p_2)(p_2^{-1}\tilde{\sigma}_1 p_2\tilde{\sigma}_2) \tag{115}$$

with $p_2^{-1}\tilde{\sigma}_1 p_2 = (\sigma_1^{(p_2(1))}, \ldots, \sigma_1^{(p_2(N))}) \in Cov(\widehat{M}, M)^N$, we find that $Cov(\widehat{{}^N M}, {}^N M)$ is a semidirect product of S_N and $Cov(\widehat{M}, M)^N$, with product given by

$$\sigma_1\sigma_2 = (p_1, \tilde{\sigma}_1)(p_2, \tilde{\sigma}_2) = (p_1p_2, p_2^{-1}\tilde{\sigma}_1 p_2\tilde{\sigma}_2). \tag{116}$$

Wave functions for the N fermions are sections of the lift $\widehat{E}$ to $\widehat{\mathcal{Q}}$ of the bundle E over $\mathcal{Q}$ with fiber

$$E_q = \bigotimes_{\boldsymbol{q}\in q} W \tag{117}$$

and (nontrivial) connection inherited from the trivial connection on $M \times W$. If the dynamics for $N = 1$ involves wave functions on $\widehat{M}$ obeying (92) with topological factor $\Gamma_\sigma(\hat{\boldsymbol{q}}) = \Gamma_\sigma$ given by a unitary representation of $\pi_1(M)$ (i.e., independent of $\hat{\boldsymbol{q}}$), then the N fermion wave function obeys (92) with topological factor

$$\Gamma_\sigma(\hat{q}) = \mathrm{sgn}(p) \bigotimes_{\boldsymbol{q}\in\pi(\hat{q})} \Gamma_{\sigma^{(i_{\hat{q}}(\boldsymbol{q}))}} \equiv \mathrm{sgn}(p)\Gamma_{\tilde{\sigma}}(\hat{q}) \tag{118}$$

where for $\hat{q} = (\hat{\boldsymbol{q}}_1, \ldots, \hat{\boldsymbol{q}}_N)$, $\pi(\hat{q}) = \{\pi_M(\hat{\boldsymbol{q}}_1), \ldots, \pi_M(\hat{\boldsymbol{q}}_N)\}$ and $i_{\hat{q}}(\pi_M(\hat{\boldsymbol{q}}_j)) = j$. Since

$$\Gamma_{\tilde{\sigma}_1\tilde{\sigma}_2}(\hat{q}) = \Gamma_{\tilde{\sigma}_1}(\hat{q})\,\Gamma_{\tilde{\sigma}_2}(\hat{q}) \tag{119}$$

we find, using (116) and (119), that

$$\Gamma_{\sigma_1\sigma_2}(\hat{q}) = \mathrm{sgn}(p_1p_2)\Gamma_{p_2^{-1}\tilde{\sigma}_1 p_2\tilde{\sigma}_2}(\hat{q}) \tag{120a}$$

$$= \mathrm{sgn}(p_1)\Gamma_{p_2^{-1}\tilde{\sigma}_1 p_2}(\hat{q})\mathrm{sgn}(p_2)\Gamma_{\tilde{\sigma}_2}(\hat{q}) \tag{120b}$$

$$= P_2\Gamma_{\sigma_1}(\hat{q})P_2^{-1}\Gamma_{\sigma_2}(\hat{q}), \tag{120c}$$

which agrees with (104) since the holonomy on the bundle E is given by permutations P acting on the tensor product (117).

9 The Character Quantization Principle

We have seen that for a Riemannian manifold $\mathcal{Q}$ that is multiply connected, there are additional possibilities for a Bohmian dynamics beyond the usual ones. These new possibilities correspond to (twisted) representations of $\pi_1(\mathcal{Q})$, the most important of which are given by the characters. In fact, unless the potential V is very special, the characters are the only representations that define a possible dynamics involving that potential.

We summarize our discussion so far, invoking the special status of the characters, in the

Character Quantization Principle. *Consider a quantum system whose configuration space is given by the Riemannian manifold $\mathcal{Q}$ and whose value space for the wave function is given by the the Hermitian vector space W [or the Hermitian bundle E over $\mathcal{Q}$]. Then for every character γ of the fundamental group $\pi_1(\mathcal{Q})$, there is a family $\mathcal{B}_\gamma = \{\mathcal{B}_\gamma(V)\}$ of Bohmian dynamics, one for each potential, i.e., Hermitian function $V : \mathcal{Q} \to W \otimes W^*$ [or Hermitian section V of $E \otimes E^*$]. The dynamics $\mathcal{B}_\gamma(V)$ associated with the potential V can be taken to be defined by*

$$\psi(\sigma\hat{q}) = \gamma_\sigma \psi(\hat{q}) \,, \tag{121a}$$

$$i\hbar \frac{\partial \psi}{\partial t}(\hat{q}) = -\tfrac{\hbar^2}{2} \Delta\psi(\hat{q}) + \widehat{V}(\hat{q})\, \psi(\hat{q}) \,, \tag{121b}$$

$$\frac{dQ_t}{dt} = \hbar\, \pi^* \Big(\mathrm{Im} \, \frac{(\psi, \nabla\psi)}{(\psi, \psi)} \Big)(Q_t) \tag{121c}$$

with $\psi \in L^2(\widehat{\mathcal{Q}}, W, \gamma)$ [or $\psi \in L^2(\widehat{\mathcal{Q}}, \widehat{E}, \gamma)$].

Equations (121) are identical with (41), (94), and (99). Recall that the characters of $\pi_1(\mathcal{Q})$ are canonically identified with those of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$.

The Character Quantization Principle corresponds to the symmetrization postulate for the case of N identical particles in $\mathbb{R}^3$: in this case the natural configuration space $\mathcal{Q} = {}^N\mathbb{R}^3$ and the fundamental group $\pi_1(\mathcal{Q}) = S_N$, the group of permutations of N elements, which has two characters, the trivial character, corresponding to bosons, and the alternating character, corresponding to fermions.

We now wish to elaborate upon why the theories given by characters deserve special attention, and are, arguably, the only possibilities for theories that can be regarded as fundamental. There are at least four crucial considerations: (i) freedom, (ii) genericity, (iii) theoretical stability, and (iv) irreducibility.

It seems within our power to expose a physical system, for example a system of N identical particles (which has the multiply-connected configuration space ${}^N\mathbb{R}^3$), to a wide variety of potentials. As we have noted already in Remarks (11), (12), (13), [and (20)], if we can arrange any potential we like, or if the potentials we can arrange are sufficient to generate together the algebra $\mathrm{End}(W)$ [respectively $\mathrm{End}(E_q)$], then the (twisted) representations defining the dynamics must be given by a character. For example, as we show in [17], if we can expose a system of N identical particles to arbitrary magnetic fields $\boldsymbol{B}$ then the potentials (109) on the natural spin bundle (28) over ${}^N\mathbb{R}^3$, which occur in the Pauli equation, generate $\mathrm{End}(E_q)$.

The second consideration is based on the hypothesis that, to the extent that a Hamiltonian defining a fundamental physical theory can be regarded as a Schrödinger operator $-\frac{\hbar^2}{2}\Delta + V$, the potential V is rather generic, or at least not too special. But generically we have that $\mathrm{Alg}(V(\mathcal{Q})) = \mathrm{End}(W)$ [and that $\mathrm{Alg}(V(\mathcal{Q})_q \cup \Theta_q) = \mathrm{End}(E_q)$]. It then follows, as we have pointed out in Remarks 12 and 21, that the dynamics belongs to $\mathscr{C}_1$. And even systems that we can describe to a very good degree of approximation by special (e.g., scalar) potentials [and, if appropriate, special (e.g., flat) Hermitian bundles] then still cannot have a dynamics from $\mathscr{C}_2 \setminus \mathscr{C}_1$.

The third consideration concerns the stability of the theory under perturbations. The idea is that the theoretical description of a system (such as, again, N identical particles) should not be so delicately contrived as to make sense only for a single potential V, but should also make sense for all potentials close to V. (One reason why one might require theoretical stability is the idea that our theoretical descriptions may be idealized, for example when we take physical space to be Euclidean $\mathbb{R}^3$ or a magnetic field to vanish, neglecting small perturbations.) This implies that the theory should be well defined for a generic potential, allowing only dynamics of class $\mathscr{C}_1$.

Finally, and perhaps most importantly, it seems reasonable to demand of a fundamental physical theory that it be suitably irreducible. But it follows from Remark (12) [respectively (21)] that a (twisted) representation can fail to be given by a character only when $\mathrm{Alg}(V(\mathcal{Q})) \neq \mathrm{End}(W)$ [respectively when $\mathrm{Alg}(V(\mathcal{Q})_q \cup \Theta_q) \neq \mathrm{End}(E_q)$], and in this case the Schrödinger dynamics is decomposable into a direct sum of dynamics corresponding to subspaces of the value-space W [or to sub-bundles]. One might wonder, in this case, why the full value-space [or bundle] was involved to begin with.

These considerations are of course related. For example, a generic potential clearly corresponds to an irreducible dynamics. Freedom relies on genericity in the following way. Since our one universe has in fact just one Hamiltonian and thus just one potential $V = V_{\mathrm{univ}}$, what must be meant when one speaks of exposing a system to various potentials V_{sys} is that

$$V_{\mathrm{sys}}(q_{\mathrm{sys}}) = V_{\mathrm{univ}}(q_{\mathrm{sys}}, Q_{\mathrm{env}})\,, \tag{122}$$

for all configurations q_{sys} of the system, where Q_{env} is the actual configuration of the environment of the system (i.e., the rest of the universe), which we can control to a certain extent. In words, V_{sys} is the *conditional potential* of a subsystem of the universe. Thus, the diversity of potentials that we can arrange for a system is inherited from the richness of the potential of the universe: if V_{univ} were scalar, we would be unable to arrange potentials V_{sys} other than scalars. Thus, the origin of the freedom of potentials must lie in genericity. On the other hand, freedom, since it requires the genericity hypothesis, lends support to it.

10 Conclusions

We have studied the possible quantum theories on a topologically nontrivial configuration space $\mathcal{Q}$ from the point of view of Bohmian mechanics, which is fundamentally concerned with the motion of matter in physical space, represented by the evolution of a point in configuration space.

Our goal was to find, define, and classify all Bohmian dynamics in $\mathcal{Q}$, where the wave functions may be sections of a Hermitian vector bundle E. What "all" Bohmian dynamics means is not obvious; we have followed one approach to what it can mean; other approaches are described in [14, 15, 16]. The present approach uses wave functions ψ that are defined on the universal covering space $\widehat{\mathcal{Q}}$ of $\mathcal{Q}$ and satisfy a periodicity condition ensuring that the Bohmian velocity vector field on $\widehat{\mathcal{Q}}$ defined in terms of ψ can be projected to $\mathcal{Q}$. We have arrived in this way at two natural classes $\mathscr{C}_1 \subseteq \mathscr{C}_2$ of Bohmian dynamics beyond the immediate Bohmian dynamics. A dynamics from $\mathscr{C}_1$ is defined by a potential and some topological information encoded in a character (one-dimensional unitary representation) of the fundamental group of the configuration space, $\pi_1(\mathcal{Q})$. A dynamics from $\mathscr{C}_2$ is defined by a potential and a more general algebraic-geometrical object, a "periodicity section" Γ.

The dynamics of $\mathscr{C}_2 \setminus \mathscr{C}_1$ exist only for special potentials. Those of $\mathscr{C}_1$, however, are compatible with *every* potential, as one would desire for what could be considered a version of quantum mechanics in $\mathcal{Q}$. We have thus arrived at the known fact that for every character of $\pi_1(\mathcal{Q})$ there is a version of quantum mechanics in $\mathcal{Q}$; we have formulated this in terms of Bohmian mechanics as the "Character Quantization Principle." A consequence, which will be discussed in detail in a sister paper [17], is the symmetrization postulate for identical particles. These different quantum theories emerge naturally when one contemplates the possibilities for defining a Bohmian dynamics in $\mathcal{Q}$.

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