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QUANTUM EQUILIBRIUM AND THE ORIGIN OF ABSOLUTE UNCERTAINTY

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Dedicated to the memory of J. S. Bell.

Abstract. The quantum formalism is a "measurement" formalism—a phenomenological formalism describing certain macroscopic regularities. We argue that it can be regarded, and best be understood, as arising from Bohmian mechanics, which is what emerges from Schrödinger's equation for a system of particles when we merely insist that "particles" means particles. While distinctly non-Newtonian, Bohmian mechanics is a fully deterministic theory of particles in motion, a motion choreographed by the wave function. We find that a Bohmian universe, though deterministic, evolves in such a manner that an *appearance* of randomness emerges, precisely as described by the quantum formalism and given, for example, by "$\rho = |\psi|^2$." A crucial ingredient in our analysis of the origin of this randomness is the notion of the effective wave function of a subsystem, a notion of interest in its own right and of relevance to any discussion of quantum theory. When the quantum formalism is regarded as arising in this way, the paradoxes and perplexities so often associated with (nonrelativistic) quantum theory simply evaporate.

KEY WORDS: Quantum randomness; quantum uncertainty; hidden variables; effective wave function; collapse of the wave function; the measurement problem; Bohm's causal interpretation of quantum theory; pilot wave; foundations of quantum mechanics.

1. Introduction

> I am, in fact, rather firmly convinced that the essentially statistical character of contemporary quantum theory is solely to be ascribed to the fact that this (theory) operates with an incomplete description of physical systems. (Einstein, in ref. 50, p. 666)

What is randomness? probability? certainty? knowledge? These are old and difficult questions, and we shall not focus on them here. Nonetheless, we shall obtain sharp, striking conclusions concerning the relationship between these concepts.

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Our primary concern in this paper lies with the status and origin of randomness in quantum theory. According to the quantum formalism, measurements performed on a quantum system with definite wave function ψ typically yield random results. Moreover, even the specification of the wave function of the composite system including the apparatus for performing the measurement will not generally diminish this randomness. However, the quantum dynamics governing the evolution of the wave function over time, at least when no measurement is being performed, and given, say, by Schrödinger's equation, is completely deterministic. Thus, insofar as the particular physical processes which we call measurements are governed by the same fundamental physical laws that govern all other processes,[1] one is naturally led to the hypothesis that the origin of the randomness in the results of quantum measurements lies in random initial conditions, in our ignorance of the complete description of the system of interest—including the apparatus—of which we know only the wave function.

But according to orthodox quantum theory, and most nonorthodox interpretations as well, *the complete description* of a system is provided by its wave function alone, and there is no property of the system beyond its wave function (our ignorance of) which might account for the observed quantum randomness. Indeed, it used to be widely claimed, on the authority of von Neumann [56], that such properties, the so called hidden variables, are impossible, that as a matter of mathematics, averaging over ignorance cannot reproduce statistics compatible with the predictions of the quantum formalism. And this claim is even now not uncommon, despite the fact that a widely discussed counterexample, the quantum theory of David Bohm [13,14], has existed for almost four decades.[2]

We shall call this theory, which will be "derived" and described in detail in Section 3, **Bohmian mechanics**. Bohmian mechanics is a new mechanics, a completely deterministic—but distinctly non-Newtonian—theory of particles in motion, with the wave

[1] And it is difficult to believe that this is not so; the very notion of measurement itself seems too imprecise to allow such a distinction within a fundamental theory, even if we were otherwise somehow attracted by the granting to measurement of an extraordinary status.

[2] For an analysis of why von Neumann's and related "impossibility proofs" are not nearly so physically relevant as frequently imagined, see Bell's article [2]. (See also the celebrated article of Bell [3] for an "impossibility proof" which does have physical significance. See as well [6].) For a recent, and comprehensive, account of Bohm's ideas see [20].

function itself guiding this motion. (Thus the "hidden variables" for Bohmian mechanics are simply the particle positions themselves.) Moreover, while its *formulation* does not involve the notion of quantum observables, as given by self-adjoint operators—so that its relationship to the quantum formalism may at first appear somewhat obscure—it can in fact be shown that Bohmian mechanics not only accounts for quantum phenomena [14,15,18], but also embodies the quantum formalism itself as the very expression of its empirical import [29]. (The analysis in the present paper establishes agreement between Bohmian mechanics and the quantum formalism without addressing the question of how the *detailed* quantum formalism naturally emerges—how and why specific operators, such as the energy, momentum, and angular momentum operators, end up playing the roles they do, as well as why "observables" should rather generally be identified with self-adjoint operators. We shall answer these questions in [29], in which a general analysis of measurement from a Bohmian perspective is presented. We emphasize that the present paper is not at all concerned directly with measurement per se, not even of positions.) That this is so is for the most part quite straightforward, but it does involve a crucial subtlety which, so far as we know, has never been dealt with in a completely satisfactory manner.

The subtlety to which we refer concerns the origin of the very randomness so characteristic of quantum phenomena. The predictions of Bohmian mechanics concerning the results of a quantum experiment can easily be seen to be precisely those of the quantum formalism, *provided* it is assumed that prior to the experiment the positions of the particles of the systems involved are randomly distributed according to Born's statistical law, i.e., according to the probability distribution given by $|\psi|^2$. And the difficulty upon which we shall focus here concerns the status—the justification and significance—of this assumption within Bohmian mechanics: not just why it should be satisfied, but also, and perhaps more important, what—in a completely deterministic theory—it could possibly mean!

In Section 2 we provide some background to Bohmian mechanics, describing its relationship to other approaches to quantum mechanics and how in fact it emerges from an analysis of these alternatives. This section, which presents a rather personal perspective on these matters, will play no role in the detailed analysis of the later sections and may be skipped on a first reading of this paper.

The crucial concepts in our analysis of Bohmian mechanics are those of **effective wave function** (Section 5) and **quantum equilibrium** (Sections 4, 6, 13, and 14). The latter is a concept analogous to, but quite distinct from, thermodynamic equilibrium. In particular, quantum equilibrium provides us with a precise and natural notion of **typicality** (Section 7), a concept which frequently arises in the analysis of "large systems" and of the "long time behavior" of systems of any size. For a universe governed by Bohmian mechanics it is of course true that, given the initial wave function and the initial positions of all particles, *everything* is completely determined and nothing whatsoever is actually random. Nonetheless, we show that typical initial configurations, for the universe as a whole, evolve in such a way as to give rise to the *appearance* of randomness, with **empirical distributions** (Sections 7 and 10) in agreement with the predictions of the quantum formalism.

From a general perspective, perhaps the most noteworthy consequence of our analysis concerns **absolute uncertainty** (Section 11). In a universe governed by Bohmian mechanics there are sharp, precise, and irreducible limitations on the possibility of obtaining knowledge, limitations which can in no way be diminished through technological progress leading to better means of measurement.

This absolute uncertainty is in precise agreement with Heisenberg's uncertainty principle. But while Heisenberg used uncertainty to argue for the meaninglessness of particle trajectories, we find that, with Bohmian mechanics, absolute uncertainty arises as a necessity, emerging as a remarkably clean and simple consequence of the existence of trajectories. Thus quantum uncertainty, regarded as an experimental fact, is *explained* by Bohmian mechanics, rather than *explained away* as it is in orthodox quantum theory.

Our analysis covers all of nonrelativistic quantum mechanics. However, since our concern here is mainly conceptual, we shall for concreteness and simplicity consider only particles without spin, and shall ignore indistinguishability and the exclusion principle. Spin and permutation symmetry arise naturally in Bohmian mechanics [2,30,13,46,35], and an analysis *explicitly* taking them into account would differ from the one given here in no essential way.

In fact, our analysis really depends only on rather general qualitative features of the structure of abstract quantum theory, not on the details of any specific quantum theory—such as nonrelativistic quantum mechanics or a quantum field theory. In particular, the analysis does not require a particle ontology; a field ontology, for example, would do just as well.

Our analysis is, however, fundamentally nonrelativistic. It may well be the case that a fully relativistic generalization of the kind of physics explored here requires new concepts [28,9,16,55]—if not new mathematical structures. But if one has not first understood the nonrelativistic case, one could hardly know where to begin for the relativistic one.

Perhaps this paper should be read in the following spirit: In order to grasp the essence of Quantum Theory, one must first completely understand *at least one* quantum theory.

2. Reality and the role of the wave function

> For each measurement one is required to ascribe to the ψ-function a characteristic, quite sudden change, which *depends on the measurement result obtained,* and so *cannot be forseen;* from which alone it is already quite clear that this second kind of change of the ψ-function has nothing whatever in common with its orderly development *between* two measurements. The abrupt change by measurement...is the most interesting point of the entire theory....For *this* reason one can *not* put the ψ-function directly in place of...the physical thing...because in the realism point of view observation is a natural process like any other and cannot *per se* bring about an interruption of the orderly flow of natural events. (Schrödinger [51])

The conventional wisdom that the wave function provides a complete description of a quantum system is certainly an attractive possibility: other things being equal, monism—the view that there is but one kind of reality—is perhaps more alluring than pluralism. But the problem of the origin of quantum randomness, described at the beginning of Section 1, already suggests that other things are not, in fact, equal.

Moreover, wave function monism suffers from another serious defect, to which the problem of randomness is closely related: Schrödinger's evolution tends to produce spreading over configuration space, so that the wave function ψ of a macroscopic system will typically evolve to one supported by distinct, and vastly different, macroscopic configurations, to a grotesque macroscopic superposition, even if ψ were originally quite prosaic. This is precisely what happens during a measurement, over the course of which the wave function describing the measurement process will become a superposition of components corresponding to the various apparatus readings to which the quantum formalism assigns nonvanishing probability. And the difficulty with this conception, of a world *completely* described by such an exotic wave function, is not even so much that it is extravagantly bizarre, but rather that this conception—or better our place in it, as well as that of the random events which the quantum formalism is supposed to govern—is exceedingly obscure.[3]

What has just been said supports, not the impossibility of wave function monism, but rather its incompatibility with the Schrödinger evolution. And the allure of wave function monism is so strong that most interpretations of quantum mechanics in fact involve the abrogation of Schrödinger's equation. This abrogation is often merely implicit and, indeed, is often presented as if it were compatible with the quantum dynamics. This is the case, for example, when the measurement postulates, regarded as embodying "collapse of the wave packet," are simply combined with Schrödinger's equation in the formulation of quantum theory. The "measurement problem" is merely an expression of this inconsistency.

There have been several recent proposals—for example, by Wigner [63], by Leggett [42], by Stapp [55], by Weinberg [57] and by Penrose [48]—suggesting explicitly that the quantum evolution is not of universal validity, that under suitable conditions, encompassing those which prevail during measurements, the evolution of the wave function is not governed by Schrödinger's equation (see also [59]). A common suggestion is that the quantum dynamics should be replaced by some sort of "nonlinear" (possibly nondeterministic) modification, to which, on the microscopic level, it is but an extremely good approximation. One of the most concrete proposals along these lines is that of Ghirardi, Rimini, and Weber [33].

The theory of GRW modifies Schrödinger's equation by the incorporation of a random "quantum jump," to a macroscopically localized wave function. As an explanation of the origin of quantum randomness it is thus not very illuminating, accounting, as it does, for the randomness in a rather ad hoc manner, essentially by fiat. Nonetheless this theory

[3] What we have just described is often presented more colorfully as the paradox of Schrödinger's cat [51].

should be commended for its precision, and for the light it sheds on the relationship between Lorentz invariance and nonlocality (see [9]).

A related, but more serious, objection to proposals for the modification of Schrödinger's equation is the following: The quantum evolution embodies a deep mathematical beauty, which proclaims "Do not tamper! Don't degrade my integrity!" Thus, in view of the fact that (the relativistic extension of) Schrödinger's equation, or, better, the quantum theory, in which it plays so prominent a role, has been verified to a remarkable—and unprecedented—degree, these proposals for the modification of the quantum dynamics appear at best dubious, based as they are on purely conceptual, philosophical considerations.

But is wave function monism really so compelling a conception that we must struggle to retain it in the face of the formidable difficulties it entails? Certainly not! In fact, we shall argue that even if there were no such difficulties, even in the case of "other things being equal," a strong case can be made for the superiority of pluralism.

According to (pre-quantum-mechanical) scientific precedent, when new mathematically abstract theoretical entities are introduced into a theory, the physical significance of these entities, their very meaning insofar as physics is concerned, arises from their dynamical role, from the role they play in (governing) the evolution of the more primitive—more familiar and less abstract—entities or dynamical variables. For example, in classical electrodynamics the *meaning* of the electromagnetic field derives solely from the Lorentz force equation, i.e., from the field's role in governing the evolution of the positions of charged particles, through the specification of the forces, acting upon these particles, to which the field gives rise; while in general relativity a similar statement can be made for the gravitational metric tensor. That this should be so is rather obvious: Why would these abstractions be introduced in the first place, if not for their relevance to the behavior of *something else*, which somehow already has physical significance?

Indeed, it should perhaps be thought astonishing that the wave function was not also introduced in this way—insofar as it is a field on configuration space rather than on physical space, the wave function is an abstraction of even higher order than the electromagnetic field.

But, in fact, it was! The concept of the wave function originated in 1924 with de

Broglie [24], who—intrigued by Einstein's idea of the "Gespensterfeld"—proposed that just as electromagnetic waves are somehow associated with particles, the photons, so should material particles, in particular electrons, be accompanied by waves. He conceived of these waves as "pilot waves," somehow governing the motion of the associated particles in a manner which he only later, in the late 1920's, made explicit [25]. However, under an onslaught of criticism by Pauli, he soon abandoned his pilot wave theory, only to return to it more than two decades later, after his ideas had been rediscovered, extended, and vastly refined by David Bohm [13,14].

Moreover, in a paper written shortly after Schrödinger invented wave mechanics, Born too explored the hypothesis that the wave function might be a "guiding field" for the motion of the electron [23]. As consequences of this hypothesis, Born was led in this paper both to his statistical interpretation of the wave function and to the creation of scattering theory. Born did not explicitly specify a guiding law, but he did insist that the wave function should somehow determine the motion of the electron only statistically, that deterministic guiding is impossible. And, like de Broglie, he later quickly abandoned the guiding field hypothesis, in large measure owing to the unsympathetic reception of Heisenberg, who insisted that physical theories be formulated directly in terms of observable quantities, like spectral lines and intensities, rather than in terms of microscopic trajectories.

The Copenhagen interpretation of quantum mechanics can itself be regarded as giving the wave function a role in the behavior of something else, namely of certain macroscopic objects, called "measurement instruments," during "quantum measurements" [12,40]. Indeed, the most modest attitude one could adopt towards quantum theory would appear to be that of regarding it as a phenomenological formalism, roughly analogous to the thermodynamic formalism, for the description of certain *macroscopic* regularities. But it should nonetheless strike the reader as somewhat odd that the wave function, which appears to be the fundamental theoretical entity of the fundamental theory of what we normally regard as microscopic physics, should be assigned a role on the level of the macroscopic, itself an imprecise notion, and specifically in terms, even less precise, of measurements, rather than on the microscopic level.

Be that as it may, the modest position just described is not a stable one: It raises the question of how this phenomenological formalism arises from the behavior of the microscopic constituents of the macroscopic objects with which it is concerned. Indeed, this very question, in the context of the thermodynamic formalism, led to the development of statistical mechanics by Boltzmann and Gibbs, and, with some help from Einstein, eventually to the (almost) universal acceptance of the atomic hypothesis.

Of course, the Copenhagen interpretation is not quite so modest. It goes further, insisting upon the *impossibility* of just such an explanation of the (origin of the) quantum formalism. On behalf of this claim—which is really quite astounding in that it raises to a universal level the personal failure of a generation of physicists to find a satisfactory *objective* description of microscopic processes—the arguments which have been presented are not, in view of the rather dramatic conclusions that they are intended to establish, as compelling as might have been expected. Nonetheless, the very acceptance of these arguments by several generations of physicists should lead us to expect that, if not impossible, it should at best be extraordinarily difficult to account for the quantum formalism in objective microscopic terms.

Exhortations to the contrary notwithstanding, suppose that we do seek a microscopic origin for the quantum formalism, and that we do this by trying to find a role on the microscopic level for the wave function, relating it to the behavior of something else. How are we to proceed? A modest proposal: First try the obvious! Then proceed to the less obvious and, as is likely to be necessary, eventually to the not-the-least-bit-obvious. We shall implement this proposal here, and shall show that we need nothing but the obvious![4]

What we regard as the obvious choice of primitive ontology—the basic kinds of entities that are to be the building blocks of everything else[5]—should by now be clear: Particles, described by their positions in space, changing with time—some of which, owing to the dynamical laws governing their evolution, perhaps combine to form the familiar macroscopic objects of daily experience.

However, the *specific* role the wave function should play in governing the motion of the particles is perhaps not so clear, but for this, too, we shall find that there is a rather obvious

[4] Insofar as nonrelativistic quantum mechanics is concerned.

[5] Except, of course, the wave function.

choice, which when combined with Schrödinger's equation becomes Bohmian mechanics. (That an abstraction such as the wave function, for a many-particle system a field that is not on physical space but on configuration space, should be a fundamental theoretical entity in such a theory appears quite natural—as a compact expression of dynamical principles governing an evolution of *configurations*.[6])

3. Bohmian mechanics

> ...in physics the only observations we must consider are position observations, if only the positions of instrument pointers. It is a great merit of the de Broglie-Bohm picture to force us to consider this fact. If you make axioms, rather than definitions and theorems, about the 'measurement' of anything else, then you commit redundancy and risk inconsistency. (Bell [8])

Consider a quantum system of N particles, with masses $m_1, \ldots, m_N$ and position coordinates $\mathbf{q}_1, \ldots, \mathbf{q}_N$, whose wave function $\psi = \psi(\mathbf{q}_1, \ldots, \mathbf{q}_N, t)$ satisfies Schrödinger's equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \mathbf{\Delta}_k \psi + V\psi, \tag{3.1}$$

where $\mathbf{\Delta}_k = \nabla_k \cdot \nabla_k = \dfrac{\partial^2}{\partial {\mathbf{q}_k}^2}$ and $V = V(\mathbf{q}_1, \ldots, \mathbf{q}_N)$ is the potential energy of the system.

Suppose that the wave function ψ does not provide a complete description of the system, that the most basic ingredient of the description of the state at a given time t is provided by the positions $\mathbf{q}_1, \ldots, \mathbf{q}_N$ of its particles at that time, and that the wave function governs the *evolution* of (the positions of) these particles. Insofar as first derivatives are simpler than higher derivatives, the simplest possibility would appear to be that the wave function determine the *velocities* $\mathbf{v}_1^\psi, \ldots, \mathbf{v}_N^\psi$ of all the particles. Here $\mathbf{v}_k^\psi \equiv \mathbf{v}_k^\psi(\mathbf{q}_1, \ldots, \mathbf{q}_N)$ is a velocity vector field, on *configuration space*, for the k-th particle, i.e.,

$$\frac{d\mathbf{q}_k}{dt} = \mathbf{v}_k^\psi(\mathbf{q}_1, \ldots, \mathbf{q}_N). \tag{3.2}$$

[6] However, *with wave function monism,* without such a role and, indeed, without particle positions from which to form configurations, how can we make sense of a field on the *space of configurations*? We might well ask *"What configurations?"* (And the wave function really is on configuration space—it is in this representation that quantum mechanics assumes its simplest form!)

Since (3.1) and (3.2) are first order differential equations, it would then follow that the state of the system is indeed given by ψ and $q \equiv (\mathbf{q}_1, \ldots, \mathbf{q}_N)$—the specification of these variables at any time would determine them at all times.

Since two wave functions of which one is a nonzero constant multiple of the other should be physically equivalent, we demand that $\mathbf{v}_k^\psi$ be homogeneous of degree 0 as a function of ψ,

$$\mathbf{v}_k^{c\psi} = \mathbf{v}_k^\psi \tag{3.3}$$

for any constant $c \neq 0$.

In order to arrive at a form for $\mathbf{v}_k^\psi$ we shall use symmetry as our main guide. Consider first a single free particle of mass m, whose wave function $\psi(\mathbf{q})$ satisfies the free Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \mathbf{\Delta} \psi. \tag{3.4}$$

We wish to choose $\mathbf{v}^\psi$ in such a way that the system of equations given by (3.4) and

$$\frac{d\mathbf{q}}{dt} = \mathbf{v}^\psi(\mathbf{q}) \tag{3.5}$$

is Galilean and time-reversal invariant.[7] Rotation invariance, with the requirement that $\mathbf{v}^\psi$ be homogeneous of degree 0, yields the form

$$\mathbf{v}^\psi = \alpha \frac{\nabla \psi}{\psi},$$

where α is a constant scalar, as the simplest possibility.

This form will not in general be real, so that we should perhaps take real or imaginary parts. Time-reversal is implemented on ψ by the involution $\psi \to \psi^*$ of complex conjugation, which renders Schrödinger's equation time reversal invariant. If the full system, including (3.5), is also to be time-reversal invariant, we must thus have that

$$\mathbf{v}^{\psi^*} = -\mathbf{v}^\psi, \tag{3.6}$$

[7] Note that a first-order (Aristotelian) Galilean invariant theory of particle motion may *appear* to be an oxymoron.

which selects the form

$$\mathbf{v}^{\psi} = \alpha \operatorname{Im} \frac{\nabla \psi}{\psi} \tag{3.7}$$

with α real.

Moreover the constant α is determined by requiring full Galilean invariance: Since $\mathbf{v}^{\psi}$ must transform like a velocity under boosts, which are implemented on wave functions by $\psi \mapsto e^{i\frac{m}{\hbar}\mathbf{v}_0\cdot\mathbf{q}}\psi$, invariance under boosts requires that $\alpha = \frac{\hbar}{m}$, so that (3.7) becomes

$$\mathbf{v}^{\psi} = \frac{\hbar}{m} \operatorname{Im} \frac{\nabla \psi}{\psi}. \tag{3.8}$$

For a general N-particle system, with general potential energy V, we define the velocity vector field by requiring (3.8) for each particle, i.e., by letting

$$\mathbf{v}_k^{\psi} = \frac{\hbar}{m_k} \operatorname{Im} \frac{\nabla_k \psi}{\psi}, \tag{3.9}$$

so that (3.2) becomes

$$\frac{d\mathbf{q}_k}{dt} = \frac{\hbar}{m_k} \operatorname{Im} \frac{\nabla_k \psi}{\psi}(\mathbf{q}_1, \ldots, \mathbf{q}_N). \tag{3.10}$$

We've arrived at **Bohmian mechanics**: for our system of N particles the state is given by

$$(q, \psi) \tag{3.11}$$

and the evolution by

$$\begin{aligned} \frac{d\mathbf{q}_k}{dt} &= \frac{\hbar}{m_k} \operatorname{Im} \frac{\nabla_k \psi}{\psi}(\mathbf{q}_1, \ldots, \mathbf{q}_N) \\ i\hbar\frac{\partial \psi}{\partial t} &= -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \boldsymbol{\Delta}_k \psi + V\psi. \end{aligned} \tag{3.12}$$

We note that Bohmian mechanics is time-reversal invariant, and that it is Galilean invariant whenever V has this property, e.g., when V is the sum of a pair interaction of the usual form,

$$V(\mathbf{q}_1, \ldots, \mathbf{q}_N) = \sum_{i<j} \phi(|\mathbf{q}_i - \mathbf{q}_j|). \tag{3.13}$$

However, our analysis will not depend on the form of V.

Note also that Bohmian mechanics depends only upon the Riemannian structure $g = (g_{ij}) = (m_i \delta_{ij})$ defined by the masses of the particles: In terms of this Riemannian structure, the evolution equations (3.1) and (3.10) of Bohmian mechanics become

$$\begin{aligned} \frac{dq}{dt} &= \hbar \, \mathrm{Im} \frac{\mathrm{grad}\, \psi}{\psi}(q) \\ i\hbar \frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2}\Delta \psi + V\psi, \end{aligned} \tag{3.14}$$

where $q = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ is the configuration, and Δ and grad are, respectively, the Laplace-Beltrami operator and the gradient on the configuration space equipped with this Riemannian structure.

While Bohmian mechanics shares Schrödinger's equation with the usual quantum formalism, it might appear that they have little else in common. After all, the former is a theory of particles in motion, albeit of an apparently highly nonclassical, non-Newtonian character; while the observational content of the latter derives from a calculus of noncommuting "observables," usually regarded as implying radical epistemological innovations. Indeed, if the coefficient in the first equation of (3.12) were other than $\frac{\hbar}{m_k}$, i.e., for general constants α_k, the corresponding theory would have little else in common with the quantum formalism. But for the particular choice of α_k, of the coefficient in (3.12), which defines Bohmian mechanics, the quantum formalism itself emerges as a phenomenological consequence of this theory.

What makes the choice $\alpha_k = \frac{\hbar}{m_k}$ special—apart from Galilean invariance, which plays little or no role in the remainder of this paper—is that with this value, the probability distribution on configuration space given by $|\psi(q)|^2$ possesses the property of equivariance, a concept to which we now turn.

Note well that ψ on the right hand side of (3.2) or (3.10) is a solution to Schrödinger's equation (3.1) and is thus time-dependent, $\psi = \psi(t)$. It follows that the vector field $\mathbf{v}_k^\psi$, the right hand side of (3.10), will in general be (explicitly) time-dependent. Therefore, given a solution ψ to Schrödinger's equation, we cannot in general expect the evolution on configuration space defined by (3.10) to possess a stationary probability distribution, an object which very frequently plays an important role in the analysis of a dynamical system.

However, the distribution given by $|\psi(q)|^2$ plays a role similar to that of—and for all practical purposes is just as good as—a stationary one: Under the evolution $\rho(q,t)$ of probability densities, of ensemble densities, arising from (3.10), given by the continuity equation

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho v^{\psi}) = 0 \tag{3.15}$$

with $v^{\psi} = (\mathbf{v}_1^{\psi}, \ldots, \mathbf{v}_N^{\psi})$ the configuration space velocity arising from ψ and div the divergence on configuration space, the density $\rho = |\psi|^2$ is stationary *relative to* ψ, i.e., $\rho(t)$ retains its form as a functional of $\psi(t)$. In other words,

$$\textit{if } \rho(q,t_0) = |\psi(q,t_0)|^2 \textit{ at some time } t_0, \textit{ then } \rho(q,t) = |\psi(q,t)|^2 \textit{ for all } t. \tag{3.16}$$

We say that such a distribution is **equivariant**.[8]

To see that $|\psi|^2$ is, in fact, equivariant observe that

$$J^{\psi} = |\psi|^2 v^{\psi} \tag{3.17}$$

where $J^{\psi} = (\mathbf{J}_1^{\psi}, \ldots, \mathbf{J}_N^{\psi})$ is the quantum probability current,

$$\mathbf{J}_k^{\psi} = \frac{\hbar}{2im_k}(\psi^*\nabla_k\psi - \psi\nabla_k\psi^*); \tag{3.18}$$

[8] More generally, and more precisely, we say that a functional $\psi \to \mu^{\psi}$, from wave functions to finite measures on configuration space, is **equivariant** if the diagram

$$\begin{array}{ccc} \psi & \longrightarrow & \mu^{\psi} \\ {\scriptstyle U_t}\big\downarrow & & \big\downarrow{\scriptstyle F_t^{\psi}} \\ \psi_t & \longrightarrow & \mu^{\psi_t} \end{array}$$

is commutative, where $U_t = e^{-\frac{i}{\hbar}tH}$, with Hamiltonian $H = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k}\boldsymbol{\Delta}_k\psi + V\psi$, is the solution map for Schrödinger's equation and F_t^{ψ} is the solution map for the natural evolution on measures which arises from (3.10), with initial wave function ψ. ($F_t^{\psi}(\mu)$ is the measure to which μ evolves in t units of time when the initial wave function is ψ.)

thus $\rho(q,t) = |\psi(q,t)|^2$ satisfies (3.15).

Now consider a quantum measurement, involving an interaction between a system "under observation" and an apparatus which performs the "observation." Let ψ be the wave function and $q = (q_{sys}, q_{app})$ the configuration of the composite system of system and apparatus. Suppose that prior to the measurement, at time t_i, q is random, with probability distribution given by $\rho(q,t_i) = |\psi(q,t_i)|^2$. When the measurement has been completed, at time t_f, the configuration at this time will, of course, still be random, as will typically be the outcome of the measurement, as given by appropriate apparatus variables, for example, by the orientation of a pointer on a dial or by the pattern of ink marks on paper. Moreover, by equivariance, the distribution of the configuration q at time t_f will be given by $\rho(q,t_f) = |\psi(q,t_f)|^2$, in agreement with the prediction of the quantum formalism for the distribution of q at this time. In particular, Bohmian mechanics and the quantum formalism then agree on the statistics for the outcome of the measurement.[9]

4. The problem of quantum equilibrium

> Then for instantaneous macroscopic configurations the pilot-wave theory gives the same distribution as the orthodox theory, insofar as the latter is unambiguous. However, this question arises: what is the good of *either* theory, giving distributions over a hypothetical ensemble (of worlds!) when we have only one world. (Bell [7])

Suppose a system has wave function ψ. We shall call the probability distribution on configuration space given by $\rho = |\psi|^2$ the **quantum equilibrium** distribution. And we shall say that a system is **in quantum equilibrium** when its coordinates are "randomly distributed" according to the quantum equilibrium distribution. As we have seen, when a system and apparatus are in quantum equilibrium the results of "measurement" arising

[9] This argument appears to leave open the possibility of disagreement when the outcome of the measurement is not configurationally grounded, i.e., when the apparatus variables which express this outcome are not functions of q_{app}. However, the reader should recall Bohr's insistence that the outcome of a measurement be describable in classical terms, as well as note that results of measurements must always be at least *potentially* grounded configurationally, in the sense that we can arrange that they be recorded in configurational terms *without affecting the result.* Otherwise we could hardly regard the process leading to the original result as a completed measurement.

from the interaction between system and apparatus will conform with the predictions of the quantum formalism for such a measurement.

More precisely(!), we say that a system is in quantum equilibrium when the quantum equilibrium distribution is appropriate for its description. It is a major goal of this paper to explain what exactly this might mean and to show that, indeed, when understood properly, it is *typically* the case that systems are in quantum equilibrium. In other words, our goal here is to clarify and justify the **quantum equilibrium hypothesis**:

When a system has wave function ψ, the distribution ρ of its coordinates satisfies

$$\rho = |\psi|^2. \tag{4.1}$$

We shall do this in the later sections of this paper. In the rest of this section we will elaborate on the *problem* of quantum equilibrium.

From a dynamical systems perspective, it would appear natural to attempt to justify (4.1) using such notions as "convergence to equilibrium," "mixing," or "ergodicity"—suitably generalized. And if it were in fact necessary to establish such properties for Bohmian mechanics in order to justify the quantum equilibrium hypothesis, we could not reasonably expect to succeed, at least not with any degree of rigor. The problem of establishing good ergodic properties for nontrivial dynamical systems is extremely difficult, even for highly simplified, less than realistic, models.

It might seem that Bohmian mechanics rather trivially *fails* to possess good ergodic properties, if one considers the motion arising from the standard energy eigenstates of familiar systems. However, quantum systems attain such simple wave functions only through complex interactions, for example with an apparatus during a measurement or preparation procedure, during which time they are not governed by a *simple* wave function. Thus the question of the ergodic properties of Bohmian mechanics refers to the motion under generic, more complex, wave functions.

We shall show, however, that establishing such properties is neither necessary nor sufficient for our purposes: That it is not necessary follows from the analysis in the later sections of this paper, and that it would not be sufficient follows from the discussion to which we now turn.

The reader may wonder why the quantum equilibrium hypothesis should present any difficulty at all. Why can we not regard it as an additional postulate, on say initial conditions (in analogy with equilibrium statistical mechanics, where the Gibbs distribution is often uncritically accepted as axiomatic)? Then, by equivariance, it will be preserved by the dynamics, so that we obtain the quantum equilibrium hypothesis for all times. In fact, when all is said and done, we shall find that this is an adequate description of the situation *provided the quantum equilibrium hypothesis is interpreted in the appropriate way.* But for the quantum equilibrium hypothesis as so far formulated, such an account would be grossly inadequate.

Note first that the quantum equilibrium hypothesis relates objects belonging to rather different conceptual categories: The right hand side of (4.1) refers to a dynamical object, which from the perspective of Bohmian mechanics is of a thoroughly objective character; while the left refers to a probability distribution—an object whose *physical* significance remains mildly obscure and moderately controversial, and which often is regarded as having a strongly subjective aspect. Thus, some explanation or justification is called for.

One very serious difficulty with (4.1) is that it *seems* to be demonstrably false in a great many situations. For example, the wave function—of system and apparatus—after a measurement (arising from Schrödinger's equation) is supported by the set of all configurations corresponding to the *possible* outcomes of the measurement, while the probability distribution at this time is supported only by those configurations corresponding to the *actual* outcome, e.g., given by a specific pointer position, a main point of measurement being to obtain the information upon which this probability distribution is grounded.

This difficulty is closely related to an ambiguity in the domain of physical applicability of Bohmian mechanics. In order to avoid inconsistency we must regard Bohmian mechanics as describing the entire universe, i.e., our system should consist of all particles in the universe: The behavior of parts of the universe, of subsystems of interest, must arise from the behavior of the whole, evolving according to Bohmian mechanics. It turns out, as we shall show, that subsystems are themselves, in fact, frequently governed by Bohmian mechanics. But if we *postulate* that subsystems must obey Bohmian mechanics, we "commit redundancy and risk inconsistency."

Note also that the very nature of our concerns—the origin and justification of (local) randomness—forces us to consider the universal level: Local systems are not (always and are never entirely) isolated. Recall that cosmological considerations similarly arise in connection with the problem of the origin of irreversibility (see R. Penrose [49]).

Thus, strictly speaking, for Bohmian mechanics only the universe has a wave function, since the *complete* state of an N particle universe at any time is given by *its* wave function ψ and the configuration $q = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ of its particles. Therefore the right hand side of the quantum equilibrium hypothesis (4.1) is also obscure as soon as it refers to a system smaller than the entire universe—and the systems to which (4.1) is normally applied are very small indeed, typically microscopic.

Suppose, as suggested earlier, we consider (4.1) for the entire universe. Then the right hand side is clear, but the left is completely obscure: Focus on (4.1) for THE INITIAL TIME. What physical significance can be assigned to a probability distribution on the initial configurations for the entire universe? What can be the relevance to physics of such an ensemble of universes? After all, we have at our disposal only the particular, actual universe of which we are a part. Thus, even if we could make sense of the right hand side of (4.1), and in such a way that (4.1) remains a consequence of the quantum equilibrium hypothesis at THE INITIAL TIME, we would still be far from our goal, appearances to the contrary notwithstanding.

Since the inadequacy of the quantum equilibrium hypothesis regarded as describing an ensemble of universes is a crucial point, we wish to elaborate. For each choice of initial universal wave function ψ and configuration q, a "history"—past, present, and future—is completely determined. In particular, the results of all experiments, including quantum measurements, are determined.

Consider an ensemble of universes initially satisfying (4.1), and suppose that it can be shown that for this ensemble the outcome of a particular experiment is randomly distributed with distribution given by the quantum formalism. This would tell us only that if we were to repeat the *very same* experiment—whatever this might mean—many times, sampling from our ensemble of universes, we would obtain the desired distribution. But this is both impossible and devoid of physical significance: While we *can* perform many *similar* experiments, differing, however, at the very least, by location or time, we cannot perform the very same experiment more than once.

What we need to know about, if we are to make contact with physics, is *empirical distributions*—actual relative frequencies within an ensemble of actual events—arising from repetitions of similar experiments, performed at different places or times, within a single sample of the universe—the one we are in. In other words, what is physically relevant is not sampling across an ensemble of universes—across (initial) q's—but sampling across space and time within a single universe, corresponding to a fixed (initial) q (and ψ).

Thus, to demonstrate the compatibility of Bohmian mechanics with the predictions of the quantum formalism, we must show that for at least some choice of initial universal ψ and q, the evolution (3.12) leads to an apparently random pattern of events, with empirical distribution given by the quantum formalism. In fact, we show much more.

We prove that for *every* initial ψ, this agreement with the predictions of the quantum formalism is obtained for *typical*—i.e., for the overwhelming majority of—choices of initial q. And the sense of typicality here is with respect to the only mathematically natural—because equivariant—candidate at hand, namely, quantum equilibrium.

Thus, on the universal level, the physical significance of quantum equilibrium is as a measure of typicality, and the ultimate justification of the quantum equilibrium hypothesis is, as we shall show, in terms of the statistical behavior arising from a typical initial configuration.

According to the usual understanding of the quantum formalism, when a system has wave function ψ, (4.1) is satisfied *regardless of whatever additional information we might have*. When we claim to have established agreement between Bohmian mechanics and the predictions of the quantum formalism, we mean to include this statement among those predictions. We are thus claiming to have established that in a universe governed by Bohmian mechanics it is in principle impossible to know more about the configuration of any subsystem than what is expressed by (4.1)—despite the fact that for Bohmian mechanics the actual configuration is an *objective* property, beyond the wave function.

This may appear to be an astonishing claim, particularly since it refers to knowledge, a concept both vague and problematical, in an essential way. More astonishing still is this: This uncertainty, of an absolute and precise character, emerges with complete ease, the structure of Bohmian mechanics being such that it allows for the formulation and clean demonstration of statistical statements of a purely *objective* character which nonetheless imply our claims concerning the irreducible limitations on possible knowledge *whatever this "knowledge" may precisely mean, and however we might attempt to obtain this knowledge*, provided it is consistent with Bohmian mechanics. We shall therefore call this limitation on what can be known *absolute uncertainty*.

5. The effective wave function

> *No one can understand this theory until he is willing to think of ψ as a real objective field rather than just a 'probability amplitude.' Even though it propagates not in 3-space but in $3N$-space.* (Bell [7])

We now commence our more detailed analysis of the behavior of an N-particle nonrelativistic universe governed by Bohmian mechanics, focusing in this section on the notion of the effective wave function of a subsystem. We begin with some notation.

We shall use Ψ as the variable for the universal wave function, reserving ψ for the effective wave function of a subsystem, the definition and clarification of which is the aim of this section. By Ψ_t we shall denote the universal wave function at time t. We shall use $q = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ as the *generic* configuration space variable, which, to avoid confusion, we shall usually distinguish from the *actual* configuration of the particles, for which we shall usually use capitals. Thus we shall write $\Psi = \Psi(q)$ and shall denote the configuration of the universe at time t by Q_t.

We remind the reader that according to Bohmian mechanics the state (Q_t, Ψ_t) of the universe at time t evolves via

$$\begin{aligned} \frac{dQ_t}{dt} &= v^{\Psi_t}(Q_t) \\ i\hbar\frac{d\Psi_t}{dt} &= -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k}\mathbf{\Delta}_k\Psi_t + V\Psi_t, \end{aligned} \tag{5.1}$$

where $v^\Psi = (\mathbf{v}_1^\Psi, \ldots, \mathbf{v}_N^\Psi)$ with $\mathbf{v}_k^\Psi$ defined by (3.9).

For any given subsystem of particles we obtain a splitting

$$q = (x, y), \tag{5.2}$$

with x the generic variable for the configuration of the subsystem and y the generic variable for the configuration of the complementary subsystem, formed by the particles not in the given subsystem. We shall call the given subsystem the *x-system*, and we shall sometimes call its complement—the *y-system*—the *environment* of the x-system.[10]

Of course, for any splitting (5.2) we have a splitting

$$Q = (X, Y) \tag{5.3}$$

for the actual configuration. And for the wave function Ψ we may write $\Psi = \Psi(x, y)$.

Frequently the subsystem of interest naturally decomposes into smaller subsystems. For example, we may have

$$x = (x_{sys}, x_{app}), \tag{5.4}$$

for the composite formed by system and apparatus, or

$$x = (x_1, \ldots, x_M), \tag{5.5}$$

for the composite formed from M disjoint subsystems. And, of course, any of the x_i in (5.5) could be of the form (5.4).

Consider now a subsystem with associated splitting (5.2). We wish to explore the circumstances under which we may reasonably regard this subsystem as "itself having a wave function." This will serve as motivation for our definition of the effective wave function of this subsystem. To this end, suppose first that the universal wave function factorizes so that

$$\Psi(x, y) = \psi(x)\Phi(y). \tag{5.6}$$

[10] While we have in mind the situation in which the x-system consists of a set of particles selected by their labels, what we say would not be (much) affected if the x-system consisted, say, of all particles in a given region. In fact the splitting (5.2) could be more general than one based upon what we would normally regard as a division into complementary systems of particles; for example, the x-system might include the center of mass of some collection of particles, while the y-system includes the relative coordinates for this collection.

Then we obtain the splitting

$$v^{\Psi} = (v^{\psi}, v^{\Phi}), \tag{5.7}$$

and, in particular, we have that

$$\frac{dX}{dt} = v^{\psi}(X) \tag{5.8}$$

for as long as (5.6) is satisfied. Moreover, to the extent that the interaction between the x-system and its environment can be ignored, i.e., that the Hamiltonian

$$H = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \mathbf{\Delta}_k + V \tag{5.9}$$

in (5.1) can be regarded as being of the form

$$H = H^{(x)} + H^{(y)} \tag{5.10}$$

where $H^{(x)}$ and $H^{(y)}$ are the contributions to H arising from terms involving only the particle coordinates of the x-system, respectively, the y-system[11], the form (5.6) is preserved by the evolution, with ψ, in particular, evolving via

$$i\hbar \frac{d\psi}{dt} = H^{(x)}\psi. \tag{5.11}$$

It must be emphasized, however, that the factorization (5.6) is extremely unphysical. After all, interactions between system and environment, which tend to destroy the factorization (5.6), are commonplace. In particular, they occur whenever a measurement is performed on the x-system. Thus, the universal wave function Ψ should now be of an extremely complex form, involving intricate "quantum correlations" between x-system and y-system, however simple it may have been originally!

[11] The sense of the approximation expressed by (5.10) is somewhat delicate. In particular, (5.10) should not be regarded as a condition on H (or V) so much as a condition on (the supports of) the factors ψ and Φ of the wave function Ψ whose evolution is governed by H; namely, that these supports be sufficiently well separated so that all contributions to V involving both particle coordinates in the support of ψ and particle coordinates in the support of Φ are so small that they can be neglected when H is applied to such a Ψ.

Note, however, that if

$$\Psi = \Psi^{(1)} + \Psi^{(2)} \tag{5.12}$$

with the wave functions on the right having (approximately[12]) disjoint supports, then (approximately)

$$v^{\Psi}(Q) = v^{\Psi^{(i)}}(Q) \tag{5.13}$$

for Q in the support of $\Psi^{(i)}$. Of course, by mere linearity, if Ψ is of the form (5.12) at some time τ, it will be of the same form

$$\Psi_t = \Psi_t^{(1)} + \Psi_t^{(2)} \tag{5.14}$$

for all t, where $\Psi_t^{(i)}$ is the solution agreeing with $\Psi^{(i)}$ at time τ of the second equation of (5.1). Moreover, if the supports of $\Psi^{(1)}$ and $\Psi^{(2)}$ are "sufficiently disjoint" at this time, we should expect the approximate disjointness of these supports, and hence the approximate validity of (5.13), to persist for a "substantial" amount of time.

Finally, we note that according to orthodox quantum measurement theory [56,12,60,62], after a measurement, or preparation, has been performed on a quantum system, the wave function for the composite formed by system and apparatus is of the form

$$\sum_{\alpha} \psi_\alpha \otimes \phi_\alpha \tag{5.15}$$

with the different ϕ_α supported by the macroscopically distinct (sets of) configurations corresponding to the various possible outcomes of the measurement, e.g., given by apparatus pointer positions. Of course, for Bohmian mechanics the terms of (5.15) are not all on the same footing: one of them, and only one, is selected, or more precisely supported, by the outcome—corresponding, say, to α_0—which *actually* occurs. To emphasize this we may write (5.15) in the form

$$\psi \otimes \phi + \Psi^{\perp} \tag{5.16}$$

[12] in an appropriate sense, of course. Note in this regard that the simplest metrics d on the projective space of rays $\{c\Psi\}$ are of the form $d(\Psi,\Psi') = ||\frac{\nabla\Psi}{\Psi} - \frac{\nabla\Psi'}{\Psi'}||$, where "$||\ ||$" is a norm on the space of complex vector fields on configuration space. Moreover the metric d is preserved by the space-time symmetries (when "$||\ ||$" is translation and rotation invariant).

where $\psi = \psi_{\alpha_0}$, $\phi = \phi_{\alpha_0}$, and $\Psi^\perp = \sum_{\alpha \neq \alpha_0} \psi_\alpha \otimes \phi_\alpha$.

Motivated by these observations, we say that a subsystem, with associated splitting (5.2), has **effective wave function** ψ (at a given time) if the universal wave function $\Psi = \Psi(x, y)$ and the actual configuration $Q = (X, Y)$ (at that time) satisfy

$$\Psi(x, y) = \psi(x)\Phi(y) + \Psi^\perp(x, y) \tag{5.17}$$

with Φ and $\Psi^\perp$ having macroscopically disjoint y-supports, and

$$Y \in \operatorname{supp} \Phi. \tag{5.18}$$

Here, by the macroscopic disjointness of the y-supports of Φ and $\Psi^\perp$ we mean not only that their supports are disjoint but that there is a macroscopic function of y—think, say, of the orientation of a pointer—whose values for y in the support of Φ differ by a *macroscopic* amount from its values for y in the support of $\Psi^\perp$.

The reader familiar with quantum measurement theory should convince himself (see (5.15) and (5.16)) that our definition of effective wave function coincides with the usual practice of the quantum formalism in ascribing wave functions to systems *whenever the latter does assign a wave function*. In particular, whenever a system has a wave function for orthodox quantum theory, it has an effective wave function for Bohmian mechanics.[13] However, there may well be situations in which a system has an effective wave function according to Bohmian mechanics, but the standard quantum formalism has nothing to say. (We say "may well be" because the usual quantum formalism is too imprecise and too controversial insofar as these questions—for which "collapse of the wave packet" must in some ill-defined manner be invoked—are concerned to allow for a more definite statement.) Readers who are not familiar with quantum measurement theory can—as a consequence of our later analysis—simply replace whatever vague notion they may have of the wave function of a system with the more precise notion of effective wave function.

[13] Note that the x-system will not have an effective wave function—even approximately—when, for example, it belongs to a larger microscopic system whose effective wave function does not factorize in the appropriate way. Note also that the *larger* the environment of the x-system, the *greater* is the potential for the existence of an effective wave function for this system, owing in effect to the greater abundance of "measurement-like" interactions with a larger environment (see, for example, Point 20 of the Appendix and the references therein).

Despite the slight vagueness in the definition of effective wave function, arising from its reference to the imprecise notion of the macroscopic, the effective wave function, when it exists, is unambiguous. In fact, it is given by[14] the **conditional wave function**

$$\psi(x) = \Psi(x, Y), \tag{5.19}$$

which, moreover, is (almost) always defined (assuming continuity, which, of course, we must). In fact, the main result of this paper, concerning the *statistical* properties of subsystems, remains valid when the notion of effective wave function is replaced by the completely precise, and less restrictive, formulation provided by the conditional wave function (5.19).[15]

Note that by virtue of the first equation of (5.1), the velocity vector field for the x-system is generated by its conditional wave function. However, the conditional wave function will not in general evolve (even approximately) according to Schrödinger's equation, even when the x-system is dynamically decoupled from its environment. Thus (5.19) by itself lacks the central *dynamical* implications, as suggested by the preliminary discussion, of our definition (5.17), (5.18). And it is of course from these dynamical implications that the wave function of a system derives much of its physical significance.[16]

Note well that the notion of effective wave function, or conditional wave function, is made possible by the existence of the *actual* configuration $Q = (X, Y)$ as well as Ψ! (In particular, the effective—or conditional—wave function is *objective,* while a related notion in Everett's Many-Worlds or Relative State interpretation of quantum theory [26] is merely

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[14] We identify wave functions related by a nonzero constant factor.

[15] We therefore need not be be too concerned here by the fact that our definition is also somewhat unrealistic, in the sense that in situations where we would in practice say that a system has wave function ψ, the terms on the right hand side of (5.17) are only approximately disjoint, or, what amounts to the same thing, the first term on the right is only approximately of the product from, though to an enormously good degree of approximation.

[16] In this regard note the following: Let $W^Y(x) = V_I(x, Y)$, where V_I is the contribution to V arising from the terms which represent interactions between the x-system and the y-system, i.e., $H = H^x + H^{(y)} + V_I$. Suppose that W^Y does not depend upon Y for Y in the support of Φ, $W^Y = W$ for $Y \in \operatorname{supp} \Phi$. Then the effective wave function ψ satisfies $i\hbar \dfrac{d\psi}{dt} = (H^{(x)} + W)\psi$. The reader should think, for example, of a gas confined by the walls of a box, or of a particle moving among obstacles. The interaction of the gas or the particle with the walls or the obstacles—which after all are part of the environment—is expressed thru W.

relative.[17]) Note also that the conditional wave function is the function of x most naturally arising from Ψ and Y.[18]

We emphasize that the effective wave function—as well as the conditional wave function—is, like any honest to goodness attribute or objective property, a functional of state description, here a function-valued functional of Ψ and $Q = (X, Y)$ which depends on Q only through Y. We shall sometimes write

$$\psi = \psi^{Y,\Psi} \tag{5.20}$$

to emphasize this relationship. For the conditional or effective wave function at time t we shall sometimes write

$$\psi_t = \psi^{Y_t,\Psi_t} \equiv \psi_t^{Y_t}, \tag{5.21}$$

suppressing the dependence upon Ψ.

Note that though we speak of ψ as a property of the x-system, it depends not upon the coordinates of the x-system but only upon the environment, a distinctly peculiar situation from a classical perspective. In fact, it is precisely because of this that the effective wave function behaves like a degree of freedom for the x-system which is independent of its configuration X.

Consider now a composite $x = (x_1, \ldots, x_M)$ of *microscopic* subsystems, with M not too large, i.e., not "macroscopically large." Suppose that (simultaneously) each x_i-system has effective wave function ψ_i. Then the x-system has effective wave function

$$\psi(x) = \psi_1(x_1)\psi_2(x_2)\cdots\psi_M(x_M), \tag{5.22}$$

[17] For an incisive critique of the Many-Worlds interpretation, as well as a detailed comparison with Bohmian mechanics, see Bell [4,7].

[18] For particles with spin our definition (5.17), (5.18) needs no essential modification. However, (5.19) would have to be replaced by $\Psi(x, Y) = \psi(x) \otimes \Phi$, where "$\otimes$" here denotes the tensor product over the spin degrees of freedom. In particular, for particles with spin, a subsystem need not have even a conditional wave function.

in agreement with the quantum formalism.[19] To see this, note that for each i we have that

$$\Psi = \psi_i(x_i)\Phi_i(y_i) + \Psi_i^{\perp}(x_i, y_i) \tag{5.23}$$

with Φ_i and $\Psi_i^{\perp}$ having macroscopically disjoint y_i-supports and hence, because the x_i-systems are microscopic, having disjoint y-supports as well.[20] Moreover,

$$Y \in \operatorname{supp} \Phi_1 \cap \operatorname{supp} \Phi_2 \cap \cdots \cap \operatorname{supp} \Phi_M, \tag{5.24}$$

and for all such Y we have

$$\Psi(x_1, \ldots, x_M, Y) = \psi_i(x_i)\Phi_i(\hat{x}_i, Y) \tag{5.25}$$

for all i, where $\hat{x}_i = (x_1, \ldots, x_M)$ with x_i missing. It follows by separation of variables, writing

$$\Psi(x, Y) = \psi_1(x_1) \cdots \psi_M(x_M)\Phi(x, Y) \tag{5.26}$$

and dividing by $\prod_i \psi_i$, that for Y satisfying (5.24)

$$\Psi(x, Y) = \psi_1(x_1) \cdots \psi_M(x_M)\Phi(Y) \tag{5.27}$$

and, indeed, that the x-system has an effective wave function, given by the product (5.22).

Note that this result would not in general be valid for conditional wave functions. In fact, the derivation of (5.22), which is used for the equal-time analysis of Section 7, is the only place where more than (5.19) is required for our results, and even here only the more

[19] As far as the quantum formalism is concerned, recall that from a purely operational perspective, whatever procedure simultaneously prepares each system in the corresponding quantum state is a preparation of the product state for the composite. Moreover, an analysis of such a simultaneous preparation in terms of quantum measurement theory would, of course, lead to the same conclusion. Note also that if the x-system is described by a density matrix whose reduced density matrix for each x_i-system is given by the wave function ψ_i, then this density matrix is itself, in fact, given by the corresponding product wave function.

[20] It is at this point that the condition that M not be "too large"—so large that x can be used to form a macroscopic variable—becomes relevant. And while the problematical situation which worries us here may seem far fetched, it is not as far fetched as it initially might appear to be. It may be that SQUIDs, superconducting quantum interference devices, can be regarded as giving rise to a situation just like the one with which we are concerned, in which lots of microscopic systems have, say, the same effective wave function, but the composite does not have the corresponding product as effective wave function. See, however, the comment following the proof of (5.22).

precise consequence (5.25) is needed. Moreover, our more general, multitime analysis (see Sections 8–10) does not appeal to (5.22) and requires only (5.19).

We wish to point out that while the qualifications under which we have established (5.22) are so mild that in practice they exclude almost nothing, (5.22) is nonetheless valid in much greater generality. In fact, whenever it is "known" that the subsystems have the ψ_i as their respective effective wave functions—by investigators, by devices, or by any records or traces whatsoever—*insofar as this "knowledge" is grounded in the environment of the composite system,* i.e., is reflected in y, (5.22) follows without further qualification.

Nonetheless, in order better to appreciate the significance of the qualification "microscopic" for (5.22), the reader should consider the following unrealistic but instructive example: Consider a pair of macroscopic systems with the composite system having effective wave function $\psi(x) = \psi_L(x_1)\psi_L(x_2) + \psi_R(x_1)\psi_R(x_2)$, where ψ_L is a wave function supported by configurations in which a macroscopic coordinate is "on the left," and similarly for ψ_R. Suppose that X_1 and X_2 are "on the left." Then each system has effective wave function ψ_L.

What wave function would the quantum formalism assign to, say, system 1 in the previous example? Though we can imagine many responses, we believe that the best answer is, perhaps, that while the quantum formalism is for all practical purposes unambiguous, we are concerned here with one of those "impractical purposes" for which the usual quantum formalism is not sufficiently precise to allow us to make any definite statement on its behalf. In this regard, see Bell [11].

We shall henceforth often say "wave function" instead of "effective wave function."

6. The fundamental conditional probability formula

> The intellectual attractiveness of a mathematical argument, as well as the considerable mental labor involved in following it, makes mathematics a powerful tool of intellectual prestidigitation—a glittering deception in which some are entrapped, and some, alas, entrappers. Thus, for instance, the delicious ingenuity of the Birkhoff ergodic theorem has created the general impression that it must play a central role in the foundations of statistical mechanics.... The Birkhoff theorem does us the service of establishing its own inability to be more than a questionably relevant superstructure upon [the] hypothesis [of absolute continuity]. (Schwartz [53])

We are ready to begin the detailed analysis of the quantum equilibrium hypothesis (4.1). We shall find that by employing, purely as a mathematical device, the quantum equilibrium distribution on the universal scale, at, say, THE INITIAL TIME, we obtain the quantum equilibrium hypothesis in the sense of empirical distributions for all scales at all times. The key ingredient in the analysis is an elementary conditional probability formula.

Let us now denote the initial universal wave function by Ψ_0 and the initial universal configuration by Q, and for definiteness let us take THE INITIAL TIME to be $t = 0$. For the purposes of our analysis we shall regard Ψ_0 as fixed and Q as random. More precisely, for given fixed Ψ_0 we equip the space $\mathcal{Q} = \{Q\}$ of initial configurations with the quantum equilibrium probability distribution $\mathbf{P}(dQ) = \mathbf{P}^{\Psi_0}(dQ) = |\Psi_0(Q)|^2 dQ$. Q_t is then a random variable on the probability space $\{\mathcal{Q}, \mathbf{P}\}$, since it is determined via (5.1) by the initial condition given by $Q_0 = Q$ and Ψ_0. Thus, for any subsystem, with associated splitting (5.2), X_t, Y_t, and ψ_t are also random variables on $\{\mathcal{Q}, \mathbf{P}\}$, where $Q_t = (X_t, Y_t)$ is the splitting of Q_t arising from (5.2), and ψ_t is the (conditional) wave function of the x-system at time t (see equation (5.21)).[21]

We wish again to emphasize that, taking into account the discussion in Section 4, we regard the quantum equilibrium distribution $\mathbf{P}$, at least for the time being, solely as a mathematical device, facilitating the extraction of *empirical* statistical regularities

[21] The reader may wonder why we don't also treat Ψ_0 as random. First of all, we don't have to—we are able to establish our results for *every* initial Ψ_0, without having to invoke in any way any randomness in Ψ_0. Moreover, if it had proven necessary to invoke randomness in Ψ_0, the results so obtained would be of dubious physical significance, since to account for the nonequilibrium character of our world, the initial wave function must be a nonequilibrium, i.e., "atypical," wave function. See the discussion in Sections 12–14.

from Bohmian mechanics[22], and otherwise *devoid of physical significance.* (However, as a *consequence* of our analysis, the reader, if he so wishes, can safely also regard $\mathbf{P}$ as providing a measure of *subjective* probability for the initial configuration Q. After all, $\mathbf{P}$ could in fact be *somebody's* subjective probability for Q.)

Note that by equivariance the distribution of the random variable Q_t is given by $|\Psi_t|^2$. It thus follows directly from (5.17), and even more directly from (5.19), that for the conditional probability distribution of the configuration of a subsystem, given the configuration of its environment, we have the *fundamental conditional probability formula*[23]

$$\mathbf{P}(X_t \in dx|Y_t) = |\psi_t(x)|^2\, dx, \tag{6.1}$$

where $\psi_t = \psi_t^{Y_t}$ is the (conditional) wave function of the subsystem at time t. In particular, this conditional distribution on the configuration of a subsystem depends on the configuration of its environment only through its wave function—an object of quite independent dynamical significance. In other words, *X_t and Y_t are conditionally independent given ψ_t.* The entire *empirical* statistical content of Bohmian mechanics flows from (6.1) with remarkable ease.

We wish to emphasize that (6.1) involves conditioning on the detailed microscopic configuration of the environment—far more information than could ever be remotely accessible. Thus (6.1) is extremely strong. Note that it implies in particular that

$$\mathbf{P}(X_t \in dx|\psi_t) = |\psi_t(x)|^2\, dx, \tag{6.2}$$

which involves conditioning on what we would be minimally expected to know if we were testing Born's statistical law (4.1). However, it would be very peculiar to know *only* this—to know no more than the wave function of the system of interest. But (6.1) suggests—and we shall show, see Section 11—that whatever additional information we might have can be of no relevance whatsoever to the possible value of X_t.[24]

[22] in a manner roughly analagous to the use of ergodicity in deriving the *pointwise* behavior of time averages for dynamical systems.

[23] ψ is to be understood as normalized whenever we write $|\psi|^2$.

[24] It immediately follows from (6.1) that for random Ψ_0 we have that

$$\mathbf{P}(X_t \in dx|Y_t, \Psi_0) = |\psi_t(x)|^2\, dx,$$

7. Empirical distributions

> ...a single configuration of the world will show statistical distributions over its different parts. Suppose, for example, this world contains an actual ensemble of similar experimental set-ups....it follows from the theory that the 'typical' world will approximately realize quantum mechanical distributions over such approximately independent components. The role of the hypothetical ensemble is precisely to permit definition of the word 'typical.' (Bell [7])

In this section we present the simplest application of (6.1), to the empirical distribution on configurations arising from a large collection of subsystems, all of which have the "same" wave function at a common time. This is the situation relevant to an equal-time test of Born's statistical law. In practice the subsystems in our collection would be widely separated, perhaps even in different laboratories.

Consider M subsystems, with configurations $x_1, \ldots, x_M$, where x_i are coordinates relative to a frame of reference convenient for the i-th subsystem. Suppose that with respect to these coordinates each subsystem has at time t the same wave function ψ, with the composite $x = (x_1, \ldots, x_M)$ having the corresponding product

$$\psi_t(x) = \psi(x_1) \cdots \psi(x_M) \tag{7.1}$$

as its wave function at that time. Then applying the fundamental conditional probability formula to the x-system, we obtain

$$\mathbf{P}\big(X_t \in dx \bigm| Y_t = Y\big) = |\psi(x_1)|^2 \cdots |\psi(x_M)|^2 \, dx_1 \cdots dx_M, \tag{7.2}$$

where $Y_t = Y$ is the configuration of the environment at this time. In other words, we find that relative to the conditional probability distribution $\mathbf{P}_t^Y(dQ) \equiv \mathbf{P}(dQ|Y_t = Y)$ given the configuration of the environment of the composite system at time t, the (actual) coordinates $X_1, \ldots, X_M$ of the subsystems at this time form a collection of independent random variables, identically distributed, with common distribution $\rho_{qe} = |\psi|^2$.

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where now $\mathbf{P}(dQ, d\Psi_0) = |\Psi_0(Q)|^2 dQ \, \mu(d\Psi_0)$ with μ any probability measure whatsoever on initial wave functions. Moreover (6.2) remains valid.

In any test of the quantum equilibrium hypothesis (4.1), it is the **empirical distribution**

$$\rho_{emp}(z) = \frac{1}{M}\sum_{i=1}^{M}\delta(z - X_i) \tag{7.3}$$

of $(X_1, \ldots, X_M)$ which is *directly* observed—so that the operational significance of the quantum equilibrium hypothesis is that ρ_{emp} be (approximately) given by ρ_{qe}. Notice that ρ_{emp} is a (distribution-valued) random variable on $(\mathcal{Q}, \mathbf{P})$, and that $\rho_{emp}(\Gamma) \equiv \int_\Gamma \rho_{emp}(z)\,dz$ is the relative frequency in our ensemble of subsystems of the event "$X_i \in \Gamma$".

It now follows from the weak law of large numbers that when the number M of subsystems is large, ρ_{emp} is very close to ρ_{qe} for $(\mathbf{P}_t^Y$-)most initial configurations $Q \in \mathcal{Q}_t^Y \equiv \{Q \in \mathcal{Q} \mid Y_t = Y\}$, the *fiber* of $\mathcal{Q}$ for which $Y_t = Y$: For any bounded function $f(z)$, and any $\epsilon > 0$, let the *"agreement set"* $\mathbf{A}(M, f, \epsilon, t) \subset \mathcal{Q}_t^Y$ be the set of initial configurations $Q \in \mathcal{Q}_t^Y$ for which

$$\begin{aligned} \|\rho_{emp} - \rho_{qe}\|_f &\equiv \left| \int (\rho_{emp}(z) - \rho_{qe}(z))\, f(z)\, dz \right| \\ &= \left| \frac{1}{M}\sum_{i=1}^{M} f(X_i) - \int f(z)\, |\psi(z)|^2\, dz \right| \\ &\leq \epsilon. \end{aligned} \tag{7.4}$$

(We suppress the dependence of $\mathbf{A}$ upon Y and on the subsystems under consideration.) Then by the weak law of large numbers

$$\mathbf{P}_t^Y\big(\mathbf{A}(M, f, \epsilon, t)\big) = 1 - \delta(M, f, \epsilon) \tag{7.5}$$

where $\delta \to 0$ as $M \to \infty$.

For a single function f, $\|\ \|_f$ cannot provide a very good measure of closeness. Therefore, consider any finite collection $\mathbf{f} = (f_\alpha)$ of bounded functions, corresponding for example to a coarse graining of value space, and let

$$\begin{aligned} \mathbf{A}(M, \mathbf{f}, \epsilon, t) &\equiv \bigcap_\alpha \mathbf{A}(M, f_\alpha, \epsilon, t) \\ &\equiv \left\{ Q \in \mathcal{Q}_t^Y \,\middle|\, \|\rho_{emp} - \rho_{qe}\|_{\mathbf{f}} \equiv \sup_\alpha \|\rho_{emp} - \rho_{qe}\|_{f_\alpha} \leq \epsilon \right\}. \end{aligned} \tag{7.6}$$

It follows from (7.5) that

$$\mathbf{P}_t^Y\big(\mathbf{A}(M,\mathbf{f},\epsilon,t)\big) = 1-\delta(M,\mathbf{f},\epsilon) \tag{7.7}$$

where $\delta(M,\mathbf{f},\epsilon)\le\sum_\alpha \delta(M,f_\alpha,\epsilon)$.

The empirical distribution ρ_{emp} does not probe in a significant way the joint distribution (7.2), i.e., the independence, of $X_1,\dots,X_M$—the law of large numbers is valid under conditions far more general than independence. To explore independence one might employ pair functions $f(X_i,X_j)$, or functions of several variables, in a manner analogous to that of the preceding analysis. Rather than proceeding in this way, we merely note—more generally—the following:

For any decision regarding the joint distribution of the X_i, we have at our disposal only the values which happen to occur. On the basis of some feature of these values, we must arrive at a (possibly rather tentative) conclusion. With any such feature we may associate a subset $\mathcal{T}$ of the space $\mathbb{R}^{DM}=\big\{(x_1,\dots,x_M)\big\}$ of possible joint values, where $D=\dim(X_i)$ is the dimension of our subsystems.

Let $\mathcal{T}\subset\mathbb{R}^{DM}$ be a *statistical test* for the hypothesis that $X_1,\dots,X_M$ are independent, with distribution $|\psi|^2$. This means that the failure to occur of the event $(X_1,\dots,X_M)\in\mathcal{T}$ can be regarded as a strong indication that $X_1,\dots,X_M$ are not generated by such a joint distribution; in other words, it means that

$$\mathbb{P}(\mathcal{T}) = 1-\delta(\mathcal{T}) \tag{7.8}$$

with $\delta\ll 1$, where $\mathbb{P}(dx_1,\dots,dx_M)=|\psi(x_1)|^2\cdots|\psi(x_M)|^2\,dx_1\cdots dx_M$ is the joint distribution under examination. $1-\delta(\mathcal{T})$ is a measure of the reliability of the test $\mathcal{T}$.

Let

$$\mathbf{A}(\mathcal{T},t)=\big\{Q\in\mathcal{Q}_t^Y \;\big|\; X_t\equiv(X_1,\dots,X_M)\in\mathcal{T}\big\} \tag{7.9}$$

Then, trivially,

$$\mathbf{P}_t^Y\big(\mathbf{A}(\mathcal{T},t)\big)=1-\delta(\mathcal{T}); \tag{7.10}$$

i.e., the $\mathbf{P}_t^Y$-size of the set of initial configurations in $\mathcal{Q}_t^Y$ for which the test is passed matches precisely the reliability of the test. (We remind the reader that the *existence* of useful tests, analogous to, but more general than, the one defined for example by (7.4), is a consequence of the weak law of large numbers.) In particular, the size of M required for δ in (7.7) to be "sufficiently" small is precisely the size required for the corresponding test

$$\mathcal{T} = \left\{ (x_1, \ldots, x_M) \in \mathbb{R}^{DM} \,\middle|\, \sup_{\alpha} \left| \frac{1}{M} \sum_{i=1}^{M} f_\alpha(x_i) - \int f_\alpha(z) \left|\psi(z)\right|^2 dz \right| \le \epsilon \right\} \tag{7.11}$$

to be "sufficiently" reliable.[25]

Equations (7.5), (7.7), and (7.10) are valid only for Y as described, i.e., when the x-system has (conditional) wave function $\psi_t \equiv \psi^{Y,\Psi_t}$ of the form (7.1), with which we are primarily concerned. We remark, however, that for a general Y these equations remain valid, provided the agreement sets which appear in them are sensibly defined in terms of the conditional distribution $\mathbb{P}_t^Y(dx) = |\psi^{Y,\Psi_t}(x)|^2\, dx$ of X_t given $Y_t = Y$. For example, we may let

$$\mathbf{A}(Y,t) = \left\{ Q \in \mathcal{Q}_t^Y \mid X_t \in \mathcal{T}(\mathbb{P}_t^Y) \right\}, \tag{7.12}$$

where, for any distribution $\mathbb{P}$ (on $\mathbb{R}^{DM}$), $\mathcal{T} = \mathcal{T}(\mathbb{P})$ is a test for $\mathbb{P}$, satisfying (7.8) with $\delta(\mathcal{T}) \ll 1$.

In terms of such *conditioned agreement sets* $\mathbf{A}(Y,t)$, we may define an *unconditioned agreeement set* $\mathbf{A}(t)$ by requiring that

$$\mathbf{A}(t) \cap \mathcal{Q}_t^Y = \mathbf{A}(Y,t); \tag{7.13}$$

directly in terms of the tests $\mathcal{T}$,

$$\mathbf{A}(t) = \left\{ Q \in \mathcal{Q} \mid X_t \in \mathcal{T}(\mathbb{P}_t^{Y_t}) \right\}. \tag{7.14}$$

Corresponding to equations (7.5), (7.7), and (7.10) we then have that

$$\mathbf{P}(\mathbf{A}(t)) = 1 - \delta(t) \tag{7.15}$$

where

$$\delta(t) = \int \delta(Y_t, t)\, d\mathbf{P} \ll 1 \tag{7.16}$$

[25] See Point 12 of the Appendix.

with $\delta(Y,t) \equiv \delta(\mathcal{T}(\mathbb{P}_t^Y))$.

Having said this, we wish to emphasize that equations (7.5), (7.7), and (7.10) (for a general Y), expressing the "largeness" of the conditioned agreement sets, are much stronger and much more relevant than the equations (7.15), (7.16) which we have just obtained: The original equations demand that the *disagreement set* $\mathbf{B}(t) = \mathbf{A}(t)^c \equiv \mathcal{Q} \setminus \mathbf{A}(t)$ be "small," not just for *"most"* fibers $\mathcal{Q}_t^Y$ corresponding to the possible environments Y at time t, but for *all* such fibers. Insofar as the actual environment Y_t at time t might be rather special—for example, because it describes a world containing (human) life—the fact that "disagreement" has "insignificant probability" for *every* environment, regardless of how special, is quite important.[26] Indeed, it is the crucial element in our analysis of absolute uncertainty in Section 11.

We may summarize the conclusion at which we have so far arrived with the assertion that for Bohmian mechanics *typical* initial configurations lead to empirical statistics at time t which are governed by the quantum formalism (see the last paragraph of Section 3). Typicality is to be here understood in the sense of quantum equilibrium: something is true for *typical* initial configurations if the set of initial configurations for which it is false is small in the sense provided by the quantum equilibrium distribution $\mathbf{P}$ (and the appropriate conditional quantum equilibrium distributions $\mathbf{P}_t^Y$ arising from $\mathbf{P}$).

We wish to emphasize the role of equivariance in our analysis. Notice that equations (7.6), (7.7) would remain valid—with δ small—if, for example, ρ_{qe} were replaced by $|\psi|^4$, *provided* the sense $\mathbf{P}$ of typicality were given, not by $|\Psi|^4$ (which is not equivariant), but by the density to which $|\Psi_t|^4$ would (backwards) evolve as the time decreases from t to THE INITIAL TIME 0. This distribution, this sense of typicality, would presumably be extravagantly complicated and exceedingly artificial.

More important, it would depend upon the time t under consideration, while equivariance provides a notion of typicality that works for all t. In fact, because of this time independence of typicality for quantum equilibrium, we immediately obtain the typicality

[26] Note, in particular, that for any condition $\mathcal{C}$ on environments implying, among other things, that the wave function of the x-system at time t is of the form (7.1), we have the same statement of the "smallness" of the disagreement set with respect to the conditional distribution given $Y_t \in \mathcal{C}$.

of joint agreement for a not-too-large collection of times $t_1, \dots, t_J$

$$\mathbf{P}\left(\bigcup_j \mathbf{B}(t_j)\right) \ll 1, \tag{7.17}$$

as well as the typicality of joint agreement at *most* times of a collection of any size. We shall not go into this in more detail here because equivariance in fact yields results far more powerful than these, covering the empirical distribution for configurations $X_1, \dots, X_M$ referring to times $t_1, \dots, t_M$ which may all be different, to which we now turn. We shall find that in exploring this general situation, further novelties of the quantum domain emerge.

8. Multitime experiments: the problem[27]

In the previous section we analyzed the joint distribution of the simultaneous configurations $X_1, \dots, X_M$ of M (distinct and disjoint) subsystems, each of which has the same wave function ψ. We would now like to consider the more general, and more realistic, situation in which $X_1, \dots, X_M$ refer to any M subsystems, some or all of which might in fact be the same, at respective times $t_1, \dots, t_M$, which might all be different. And we would again like to conclude that suitably conditioned, $X_1, \dots, X_M$ are independent, each with distribution given by $|\psi|^2$; this would imply, precisely as in Section 7, the corresponding results about empirical distributions and tests.

We shall find, however, that this multitime situation requires considerably more care than we have so far needed; in particular, what we might think at first glance we would like to be true, in fact turns out to be in general false!

To begin to appreciate the difficulty, consider configurations X_1 and X_2 referring to the *same* system but at different times $t_1 < t_2$, and suppose this system has wave function ψ at both of these times. Can we conclude that X_1 and X_2 are independent? Of course not! For example, if the system is suitably isolated between the times t_1 and t_2, so that its configuration undergoes an autonomous evolution, then X_2 will in fact be a function of

[27] Sections 8–10 should perhaps be skipped at first reading.

X_1; in the simplest case, when the wave function ψ is a ground state, we will in fact have that $X_2 = X_1$.

What has just been described is not, however, an instance of disagreement with the quantum formalism, which concerns only the results of *observation*—and in the previous example observation would destroy the isolation upon which the strong correlation between X_1 and X_2 was based. Moreover, the particular difficulty just described is easily remedied by taking "observation" into account. However, it is perhaps worth noting that for the equal-time analysis it was not necessary in any way to take observation directly into account to obtain agreement with the quantum formalism—$X_1, \dots, X_M$ had the distribution given by the quantum formalism regardless of whether these variables were observed.

A much more serious, and subtle, difficulty arises from the fact that the wave function ψ_t of a system at time t is itself a random variable (see (5.21)), while we wish to consider situations in which our systems each have the same (non-random) wave function ψ. In the equal-time case this consideration led to no difficulty—and was barely noticed—since ψ_t is *nonrandom relative* to the environment Y_t upon which we there conditioned. For the multitime case, however, it is at first glance by no means clear how we should capture the stipulation that our systems each have wave function ψ.

One possibility would be to treat this stipulation as further conditioning, i.e., to consider the conditional distribution of $X_1, \dots, X_M$ given, among other things, that the wave functions ψ_{t_i} of our respective systems at the respective times $t_1, \dots, t_M$ satisfy $\psi_{t_i} = \psi$ for all i. This would be a bad idea! The conditioning just described can affect the distribution of the configurations $X_1, \dots, X_M$ in surprising, and uncontrollable, ways.

For example, suppose that when the result of an observation of X_1 is "favorable," the happy experimenter proceeds somehow to prepare the second system in state ψ at time t_2, while if the result is "unfavorable," the depressed experimenter requires some extra time to recuperate, and prepares the second system in state ψ at time $t_2' > t_2$. In this situation X_1 need not be independent of ψ_{t_2}, so that conditioning on ψ_{t_2} may bias the distribution of X_1.

Moreover, we believe that this example is not nearly so artificial as it may at first appear. In the real world, of which the experimenters and their equipment are a part, which experiments get performed where and when can, and typically will, be correlated with the results of previous experiments, with each other, and with any number of other factors, such as, for example, the weather, which we would not normally take into account. Therefore, stochastic conditioning can be a very tricky business here, yielding conditional distributions of a surprising, and thoroughly unwanted, character.

What has just been said suggests that our multitime formulation is, while nonetheless inadequate, also perhaps not as general as we might want. The times at which our experiments are performed, and indeed the subsystems upon which they are performed, may themselves be random, and a more general formulation, like the one we shall give, should take this into account. However, we wish to emphasize that, as we shall see, the primary value of such a "random system" formulation is not increased generality. Rather, it is first of all simply the case that, strictly speaking, the systems upon which experiments get performed are, in fact, themselves random—not just the results, or the state of the system, but the time of the experiment as well as the specific system, the particular collection of particles, upon which we focus and act. Furthermore, when we properly take *this* into account, the difficulty we have been discussing vanishes!

9. Random systems

Consider a pair $\sigma = (\pi, T)$, where $T \in \mathbb{R}$ (with $T \geq 0$ if THE INITIAL TIME is 0) and π is a splitting

$$q = (x, y) \equiv (\pi q, \pi^{\perp} q) \tag{9.1}$$

(see Section 5); we identify π with the projection $\mathcal{Q} \equiv \mathbb{R}^{3N} \to \mathbb{R}^{3m}$ onto the configuration of the (m-particle) x-system, with the components of $x \equiv \pi q$ ordered, say, as in q. π comes together with $\pi^{\perp}$, the complementary projection, onto the coordinates of the environment (also ordered as in q). Thus we may identify π with the subset of $\{1, \ldots, N\}$ corresponding to the particles of the x-system. σ specifies a subsystem at a given time, for example, the system upon which we experiment and the time at which the experiment begins.[28]

[28] If indistinguishability were taken into account, our identification of π would have to be modified accordingly. We might then associate it, for example, with a subset of $\mathbb{R}^3$. (See footnote 15.)

Now allow both T and π to be random, i.e., allow T to be a real-valued, and π to be a projection-valued, function on the space $\mathcal{Q}$ of initial configurations. (π may thus be identified with a random subset of $\{1, \ldots, N\}$.) For $\sigma = (\pi, T)$ we write

$$X_\sigma = \pi Q_T \tag{9.2}$$

for the configuration of the system and

$$Y_\sigma = \pi^\perp Q_T \tag{9.3}$$

for the configuration of its environment.[29]

We say that a pair

$$\sigma = (\pi, T), \tag{9.5}$$

consisting of a random projection and a random time as described, is a **random system** provided

$$\{\sigma = \sigma_0\} \in \mathcal{F}(Y_{\sigma_0}) \tag{9.6}$$

for any (nonrandom) $\sigma_0 = (\pi_0, t)$.[30] Here we use the notation $\mathcal{A} \in \mathcal{F}(W_1, W_2, \ldots)$ to convey that $I_\mathcal{A}$, the indicator function of the event $\mathcal{A} \subset \mathcal{Q}$, is a function of $W_1, W_2, \ldots$.[31]

We emphasize that for a random system σ, the configuration X_σ (Y_σ) of the system (of its environment) is *doubly random*—σ is itself random, and for a given value σ_0 of σ, X_{σ_0} (Y_{σ_0}) is, of course, still random.

[29] More explicitly, when π and T are random, X_σ is the random variable

$$X_\sigma(Q) = \pi(Q)\,(Q_{T(Q)}) \tag{9.4}$$

and similarly for Y_σ.

[30] The condition (9.6), which is formally what we need, technically suffers from "measure-0 defects"—since a random time T will typically be a continuous random variable, the event $\{\sigma = \sigma_0\}$ will typically have measure 0, while conditional probabilities, for which (9.6) is formally utilized, are strictly defined only up to sets of measure 0. This defect can be eliminated by replacing (9.6) by the condition that for any t there exist a number $\epsilon_0(t) > 0$ such that

$$\{\pi = \pi_0\,,\, t - \epsilon \le T \le t\} \in \mathcal{F}(Y_{(\pi_0, t)})$$

for all $0 < \epsilon < \epsilon_0(t)$, using which our formal analysis becomes rigorous via standard continuity-density arguments. (Of course, if time were discrete no such technicalities would arise.)

[31] More precisely, $\mathcal{F}(W_1, W_2, \ldots)$ denotes the sigma-algebra generated by the random variables $W_1, W_2, \ldots$.

The condition (9.6) says that the value of a random system, i.e., the identity of the particular subsystem and time that it happens to specify, is reflected in its environment. In practice, this value is expressed by the state of the experimenters, their devices and records, and whatever other features of the environment form the basis of its *selection*. It is for this reason that we usually fail to notice that our systems are random: relative to "ourselves," which we naturally don't think of as random, they are completely determined. Notice also that (9.6) fits nicely with the notion of the wave function of a subsystem, as expressed, e.g., by (5.19).[32]

We shall write ψ_σ for the (effective or conditional) wave function of the random system σ—given $Q \in \mathcal{Q}$, the wave function at time $T(Q)$ of the system defined by $\pi(Q)$. Using the notation of equation (5.21), we have that

$$\psi_\sigma = \psi^{Y_\sigma}_{T,\pi}, \tag{9.8}$$

where the subscript π makes explicit the dependence of ψ^Y_t upon the splitting $q = (x, y)$. Note that ψ_σ is a functional of both σ and Y_σ.

The crucial ingredient in our multitime analysis is the observation that the fundamental conditional probability formula (6.1) remains valid for random systems: For any random system σ[33]

$$\mathbf{P}(X_\sigma \in dx | Y_\sigma, \sigma) = |\psi_\sigma(x)|^2 dx, \tag{9.9}$$

which can in a sense be regarded as the most compact expression of the entire quantum

[32] While the preceding informal description may not appear to discriminate between (9.6) and the perhaps equally natural condition

$$\sigma \in \mathcal{F}(Y_\sigma),$$

which we may formally write as

$$\{\sigma = \sigma_0\} \in \mathcal{F}(Y_\sigma), \tag{9.7}$$

a careful reading should convey (9.6). The conditions (9.6) and (9.7) are not, in fact, equivalent, nor even comparable. In practice both are satisfied, the validity of (9.7) deriving mainly from the existence of "clocks." We have defined the notion of random system using only (9.6) because this is what turns out to be relevant for our analysis. (Note also that, trivially, $\sigma \in \mathcal{F}(Y_\sigma, \sigma)$.)

[33] The conditioning here on σ can of course be removed if $\sigma \in \mathcal{F}(Y_\sigma)$ or, more generally, if $\psi_\sigma \in \mathcal{F}(Y_\sigma)$, e.g., if $\psi_\sigma = \psi$ is constant, i.e., nonrandom.

formalism. To see this note that for any value $\sigma_0 = (\pi_0, t)$ of σ, we have that on $\{\sigma = \sigma_0\}$

$$\begin{aligned}\mathbf{P}(X_\sigma \in dx | Y_\sigma, \sigma) &= \mathbf{P}(X_\sigma \in dx | Y_\sigma, \sigma = \sigma_0) \\ &= \mathbf{P}(X_{\sigma_0} \in dx | Y_{\sigma_0}, \sigma = \sigma_0) \\ &= \mathbf{P}(X_{\sigma_0} \in dx | Y_{\sigma_0}) \equiv \mathbf{P}(X_t \in dx | Y_t) \\ &= |\psi_t(x)|^2 \, dx \equiv |\psi_{\sigma_0}(x)|^2 \, dx \\ &= |\psi_\sigma(x)|^2 \, dx,\end{aligned} \tag{9.10}$$

where we have used (6.1) and (9.6), as well as the obvious fact that X_σ, Y_σ, and ψ_σ agree respectively with $X_{\sigma_0}(\equiv X_t)$, $Y_{\sigma_0}(\equiv Y_t)$, and $\psi_{\sigma_0}(\equiv \psi_t)$ on $\{\sigma = \sigma_0\}$.[34]

10. Multitime distributions

> ...every atomic phenomenon is closed in the sense that its observation is based on registrations obtained by means of suitable amplification devices with irreversible functioning such as, for example, permanent marks on the photographic plate...the quantum-mechanical formalism permits well-defined applications only to such closed phenomena... (Bohr, ref. 22, pp. 73 and 90)

Now consider a sequence $\sigma_i = (\pi_i, T_i)$, $i = 1, \ldots, M$, of random systems, ordered so that (with probability 1)

$$T_1 \leq T_2 \leq \cdots \leq T_M. \tag{10.1}$$

We write X_i for X_{σ_i}, Y_i for Y_{σ_i}, and let

$$\mathcal{F}_i = \mathcal{F}(Y_{\sigma_i}, \sigma_i). \tag{10.2}$$

Suppose that for the wave function of the i-th system we have

$$\psi_{\sigma_i} = \psi_i \tag{10.3}$$

[34] The reader familiar with stochastic processes should note the similarity between (9.6) and (9.9) on the one hand, and the notions of stopping time and the strong Markov property from Markov process theory. Indeed, (6.1) can be regarded as a kind of Markov property, in relation to which (9.9) then becomes a strong Markov property.

where ψ_i is *nonrandom*, i.e., (with probability 1) the random wave function ψ_{σ_i} is the specific wave function ψ_i. This will be the case if the requirement that the i-th system have wave function ψ_i forms part of the basis of selection for this system, i.e., for σ_i—for example, if the i-th experiment, by prior decision, must be preceded by a successful preparation of the state ψ_i.

Finally, suppose that

$$X_i \in \mathcal{F}_j \quad \text{for all } i < j, \tag{10.4}$$

i.e., for all $i < j$ X_i is a function of Y_j and σ_j. This will hold, for example, if, with probability 1, each X_i is measured—if the i-th measurement has not been completed, and the result "recorded," prior to time T_j, then the i-th system, together with the apparatus which measures it, must still be isolated at time T_j, from σ_j as well as from the rest of its environment, remaining so until the completion of this measurement.

Notice that since ψ_j is nonrandom, it follows from (10.4) and the fundamental conditional probability formula (9.9) that

$$\begin{aligned} \mathbf{P}(X_j \in dx_j | X_1, \ldots, X_{j-1}) &= \mathbf{P}(X_j \in dx_j | Y_j, \sigma_j) \\ &= |\psi_j(x_j)|^2 dx_j. \end{aligned} \tag{10.5}$$

Thus

$$\begin{aligned} \mathbf{P}\left(X_i \in dx_i,\, i \le j\right) &= \mathbf{P}\left(X_i \in dx_i,\, i \le j-1\right) \mathbf{P}\left(X_j \in dx_j | X_1 = x_1, \ldots, X_{j-1} = x_{j-1}\right) \\ &= \mathbf{P}\left(X_i \in dx_i,\, i \le j-1\right) |\psi_j(x_j)|^2 dx_j \\ &= |\psi_1(x_1)|^2 \cdots |\psi_j(x_j)|^2 dx_1 \cdots dx_j, \end{aligned} \tag{10.6}$$

and

$$X_1, \ldots, X_M \textit{ are independent, with each } X_i \textit{ having distribution given by } |\psi_i|^2. \tag{10.7}$$

As it stands (10.7) is mildly useless, since the probability distribution $\mathbf{P}$ with respect to which it is formulated does not take into account any "prior" information, some of which we might imagine to be relevant to the outcomes of our sequence of experiments. Therefore, it is significant that our entire random system analysis (including (10.1), (10.3), and (10.4)) can be relativized to any set $\mathcal{M} \subset \mathcal{Q}$—i.e., we may replace $(\mathcal{Q}, \mathbf{P})$ by $(\mathcal{M}, \mathbf{P}^{\mathcal{M}})$ where $\mathbf{P}^{\mathcal{M}}(dQ) = \mathbf{P}(dQ|\mathcal{M})$—without essential modification, *provided* the random systems σ under consideration satisfy

$$\mathcal{M} \in \mathcal{F}(Y_\sigma, \sigma). \tag{10.8}$$

In particular, (10.7) is valid even with respect to $\mathbf{P}^{\mathcal{M}}$ provided that for all i

$$\mathcal{M} \in \mathcal{F}_i. \tag{10.9}$$

We might think of $\mathcal{M}$ as reflecting the "macroscopic state" at a time prior to all of our experiments, though one might argue about whether (10.9) would then be satisfied. Be that as it may, any event $\mathcal{M}$ describing any sort of prior information to which we could conceivably have access would be expected to satisfy (10.9), particularly if this information were recorded.

Now suppose that $\psi_i = \psi$ for all i. Then the joint distribution of $X_1, \ldots, X_M$ with respect to $\mathbf{P}^{\mathcal{M}}$ is precisely the same as in the equal time situation of Section 7.[35] Since the analysis there depended only upon this joint distribution, we may draw the same conclusions concerning empirical distributions and tests as before. We thus find for our sequence of experiments that *typical* initial configurations—typical with respect to $\mathbf{P}$ or $\mathbf{P}^{\mathcal{M}}$—yield empirical statistics governed by the quantum formalism.

Perhaps this claimed agreement with the quantum formalism requires elaboration. We have been explicitly concerned here only with the statistics governing the outcomes of *position* measurements. Now we were also concerned only with configurations in our equal-time analysis of Section 7. But our results there directly implied agreement with the quantum formalism for the results of measurements of any observable:

Our statistical conclusions there were valid regardless of whether or not the configurations—the X_i—were "measured." Thus, for the equal time case the joint distribution of any functions $Z_i = f_i(X_i)$ of the configurations must be inherited from the distribution of

[35] Notice that equal-time experiments are covered by our multitime analysis—all the T_i can be identical—and in this case (10.4) is automatically satisfied. However, for our earlier equal-time results it was necessary that ψ be the effective wave function, while here conditional is sufficient.

the X_i themselves. In particular, by considering subsystems of the form (5.4), where the apparatus "measures the observable"—i.e., self-adjoint operator—$\hat{Z}_i$, with wave functions $\hat{\psi}_i = \psi_i \otimes \phi_i$ where ϕ_i is the initial(ized) wave function of the i-th apparatus, letting Z_i be the outcome of this "measurement of $\hat{Z}_i$" and using what we know about the joint distribution of the X_i, it follows that the Z_i are independent, and, as in the last paragraph of Section 3, that each Z_i must have the distribution provided by the quantum formalism, namely, that given by the spectral measure $\rho^{\psi_i}_{\hat{Z}_i}(dz)$ for $\hat{Z}_i$ in the state ψ_i. (For a detailed account of how this comes about see [14,18,29].)

The corresponding result for the multitime case does not, in fact, follow from (10.7). The latter does require that the configurations be "measured," and a "measurement of $\hat{Z}_i$" need not involve, and indeed may be incompatible with, a "measurement" of X_i.

But, while it does not follow from the *result* for the X_i, the corresponding result for "general measurements" does, in fact, follow from the *analysis* for the X_i. We need merely suppose for the Z_i what we did for the X_i, namely, that

$$Z_i \in \mathcal{F}_j \quad \text{for all } i < j, \tag{10.10}$$

to conclude, for the sequence of outcomes Z_i of "measurements of observables" $\hat{Z}_i$ in states ψ_i, that (with respect to $\mathbf{P}^{\mathcal{M}}$ for $\mathcal{M}$ satisfying (10.8))

$$Z_1, \ldots, Z_M \textit{ are independent, with each } Z_i \textit{ having distribution given by } \rho^{\psi_i}_{\hat{Z}_i}, \tag{10.11}$$

from which the usual conclusions concerning empirical distributions and tests follow immediately.[36]

We emphasize that the assumptions (10.4), (10.10), and (10.9) are minimal. They demand merely that facts about results and initial experimental conditions not be "forgotten." Thus they are hardly assumptions at all, but almost the very conditions essential to

[36] That $Z_i = f_i(X_i)$ will in fact *be* the outcome of what would normally be considered a measurement of $\hat{Z}_i$ can be expected only if ψ_i is the effective wave function of the i-th system, and not merely the conditional wave function: The functional form of Z_i is based upon the evolution of a system initially with effective wave function ψ_i interacting with a suitable apparatus but otherwise isolated. However, the conclusion (10.11) for $Z_i = f_i(X_i)$ is valid even for ψ_i merely the conditional wave function, though in this case Z_i may have little connection with what is actually observed.

enable us, at the conclusion of our sequence of experiments, to talk in an informed manner about the experimental conditions and results and compare these with theory.

Moreover, it is not hard to see that if these conditions are relaxed, the "predictions" should not be expected to agree with those of the quantum formalism.[37] This is a striking illustration of the way in which Bohmian mechanics does not *merely* agree with the quantum formalism, but, eliminating ambiguities, illuminates, clarifies, and sharpens it.[38]

11. Absolute uncertainty

That the quantum equilibrium hypothesis $\rho = |\psi|^2$ conveys the *most detailed* knowledge *possible* concerning the present configuration of a subsystem (of which the "observer" or "knower" is not a part—see Point 23 of the Appendix), what we have called **absolute uncertainty**, is implicit in the results of Sections 7 and 10.[39] The key observation relevant to this conclusion is this: Whatever we may reasonably mean by knowledge, information, or certainty—and what precisely these do mean is not at all an easy question—it simply must be the case that the experimenters, their measuring devices, their records, and whatever other factors may form the basis for, or representation of, what could conceivably be regarded as knowledge of, or information concerning, the systems under investigation,

[37] Note that by selectively "forgetting" results we can dramatically alter the statistics of those that we have not "forgotten."

[38] The analysis we have presented does not allow for the possibility that with nonvanishing probability $T_i = \infty$, i.e., the conditions for the selection of σ_i are never satisfied. Our results extend to this case provided that $(X_1, \ldots, X_i)$ and $\{T_{i+1} < \infty\}$ are conditionally independent given $\{T_i < \infty\}$ for all $i = 1, \ldots, M-1$, in which case our results are valid given $\{T_M < \infty\}$. Note that without the aforementioned conditional independence our results would not be expected to hold: Suppose, for example, that if the initial results are "unfavorable," the depressed experimenter destroys humankind, and systems no longer get prepared properly. Thus, conditioning on $\{T_M < \infty\}$ yields a "biased" sample. The preceding points to perhaps a different, albeit rather minor, ambiguity in the quantum formalism, of which Bohmian mechanics again forces one to take note, and in so doing to rectify.

[39] Note, however, that as far as knowledge of the past is concerned, it is possible to do a good deal better than what would be permitted by absolute uncertainty for knowledge of the present: Having prepared our subsystem in a specific (not-too-localized) quantum state, with known wave function ψ, we may proceed to measure the configuration X of this system, thereby obtaining detailed knowledge of both its wave function and its configuration for some *past* time. But note well that the determination of the configuration may—indeed, as we show, must—lead to an appropriate "collapse" of ψ, and hence our knowledge of the (present) configuration will be compatible with $\rho = |\psi|^2$ for the *present* wave function. (Note also that for quantum orthodoxy as well it is sometimes argued that knowledge of the past need not be constrained by the uncertainty principle.)

must be a part of or grounded in the environment of these systems.

The possession by experimenters of such information must thus be reflected in *correlations* between the system properties to which this information refers and the features of the environment which express or represent this information. We have shown, however, that *given* its wave function there can be no correlation between (the configuration of) a system and (that of) its environment, even if the full microscopic environment Y—itself grossly more than what we could conceivably have access to—is taken into account.

Because we consider absolute uncertainty to be a very important conclusion, with significance extending beyond the conceptual foundations of quantum theory, we shall elaborate on how our results, for both the equal-time and the general multitime cases, entail this conclusion. The crucial point is that the possession of knowledge or information implies the existence of certain features of the environment, an environmentally based selection criterion, such that systems selected on the basis of this criterion satisfy the conditions expressed by this information. (For example, when a measuring device registers, or the associated computer printout records, that "$|X| < 1$", it should in fact be more or less the case that $|X| < 1$.)

Suppose that our M systems of Section 7 have been chosen on the basis of some features of the environment, say by selection from an ensemble of M' systems, also of the form considered there. The selection criterion can be based upon any property of the environment $Y_t = Y$ of the original (preselection) ensemble. (We allow for a rather arbitrary selection criterion, though in practice selection would of course be quite constrained. In particular, a realistic selection criterion should, perhaps, be the "same" for each system; i.e., whether or not the i-th system is selected should depend, for all i, upon the same property of Y relative to this system. However, we need here no such constraints.)

Since, with respect to $\mathbf{P}_t^Y$, the configurations of the systems of our original ensemble were independent, with each having distribution given by $|\psi|^2$, and since our selection criterion is based solely upon the environment Y of the original ensemble and in no way directly on the values of the configurations themselves, it follows that the configurations $X_1, \ldots, X_M$ of our selected subsystems have precisely the same distribution (also relative to $\mathbf{P}_t^Y$) as the original ensemble. Thus, for typical initial universal configurations, the empirical distribution of configurations across our selected ensemble will be given (approximately) by $|\psi|^2$, just as for the original ensemble. It follows that, whatever else it may be, our selection criterion cannot be based upon what we could plausibly regard as *information* concerning system configurations (more detailed than what is already expressed by $|\psi|^2$).

For the general case, of multitime experiments as described in Section 10, the analysis is perhaps even simpler. In fact, for this case there is really nothing to do, beyond observing that any (environmentally based) selection criterion, whatever it may be, can be incorporated into the definition of our random systems, as part of the basis for their selection. It thus follows from the results of Section 10 that no such criterion can be regarded as reflecting any information, beyond $|\psi|^2$, about the configurations of these systems. Therefore, no devices whatsoever, based on any present or future technology, will provide us with the corresponding knowledge. *In a Bohmian universe such knowledge is absolutely unattainable!*[40]

We emphasize that we do not claim that knowledge of the detailed configuration of a system is impossible, a claim that would be manifestly false. We maintain only that—as a consequence of the fact that the configuration X of a system and the configuration Y of its environment are conditionally independent given its wave function ψ—*all such knowledge must be mediated by* ψ. And we emphasize that a major reason for the not insignificant length of our argument, as presented in Sections 6-11, was the necessity to extract from the aforementioned conditional independence analogous conclusions concerning empirical

[40] The reader concerned that we have overlooked the possibility that information may sometimes be grounded in non-configurational features of the environment, for example in velocity patterns, should consider the following (recall as well footnote 12):

(1) Knowledge and information are, in fact, almost always, if not always, configurationally grounded. Examples are hardly necessary here, but we mention one—synaptic connections in the brain.
(2) Dynamically relevant differences between environments, e.g., velocity differences, which are not instantaneously correlated with configurational differences quickly generate them anyway. And we need not be concerned with differences which are not dynamically relevant!
(3) Knowledge and information must be communicable if they are to be of any social relevance; their content must be stable under communication. But communication typically produces configurational representations, e.g., pressure patterns in sound waves.
(4) In any case, in view of the effective product form (5.17), when a system has an effective wave function, the configuration Y provides an exhaustive description of the state of its environment (aside from the universal wave function Ψ—and through it Φ—which for convenience of exposition we are regarding as given—see also footnotes 27 and 31).

correlations.

From our conclusion that when a system has wave function ψ we cannot know more about its configuration X than what is expressed by $|\psi|^2$, it follows trivially that *knowledge* that its wave function is ψ similarly constrains our knowledge of the configuration. It also trivially follows that detailed knowledge of X, for example that $X \in I$ for a given set of values I, entails detailed conclusions concerning the wave function, for example that the (conditional) wave function of the system is supported by I.[41]

Finally, in order to further sharpen the character of our absolute uncertainty, one more point must be made. We have focused here primarily on the *statistical* aspect of the wave function of a system. But any "absolute uncertainty" based solely upon the fact that knowledge of the configuration X of a system must be mediated by (knowledge of) some "object," in the sense that the distribution of X can be expressed simply in terms of that "object," may be sorely lacking in substance if the "object" is *merely* statistical. In such a case, knowledge of the "object" need amount to nothing more than knowledge that X has the distribution so expressed.

What lends substance to the "absolute uncertainty" in Bohmian mechanics—and justifies our use of that phrase—is the fact that the relevant "object," the wave function ψ, plays a dual role: it has, in addition to its statistical aspect, also a dynamical one, as expressed, e.g., in equations (5.8) and (5.11). Thus, knowledge of the wave function of a system, which sharply constrains our knowledge of its configuration, is knowledge of something in its own right, something "real," and not merely knowledge that the configuration has distribution $|\psi|^2$.

Moreover, the *detailed* character of this dynamical aspect is such that a wave function with narrow support quickly spreads, owing to the dispersion in Schrödinger's equation, to one with broad support, a change which generates a similar change in the distribution of the configuration. It follows that the unavoidable price we must pay for sharp knowledge of the present configuration of a system is at best hazy knowledge of its future configuration, i.e., of its "effective velocity." In particular, our absolute uncertainty embodies absolute

[41]And even if the system does not have an effective wave function, we have that any density matrix describing the system must also be "supported" by I.

unpredictability. More generally, the usual uncertainty relations for noncommuting "observables" become a corollary of the quantum equilibrium hypothesis $\rho = |\psi|^2$ as soon as the dynamical role of the wave function is taken into account; a detailed analysis can be found in [14,18,29].

12. Knowledge and nonequilibrium

The alert reader may be troubled that we have established results about randomness and uncertainty, results of a flavor often associated with "chaos" and "strong ergodic properties," without having to invoke any of the hard estimates and delicate analysis usually required to establish such properties. Indeed, our analysis neither used nor referred to any such properties. How can this be?

The short answer is quantum equilibrium, with all that the notion of equilibrium entails and conveys, an answer upon which we shall elaborate in the next section. Here we would like merely to observe that what is truly remarkable is not absolute uncertainty, irreducible limitations on what we *can* know, but rather that it is possible to know anything at all!

We take (the possibility of) knowledge, our information gathering and storing abilities, too much for granted. (And we conclude all too readily that the unknowable is unreal.) Of course, it is not at all surprising that we should do so, in view of the essential role such abilities play in our existence and survival. But that there should arise stable systems embodying (what can reasonably be regarded as) such abilities is a perhaps astonishing fact about the way our universe works, about the laws of nature!

The point is that we, the knowers, are separate and distinct from the things about which we know, and know in marvelous detail. How can there be, between completely disjoint entities, sufficiently strong correlations to allow for a representation in one of these entities of detailed features of the other? Indeed, such correlations are absent in thermodynamic equilibrium. With respect to (any of the distributions describing) global thermodynamic equilibrium, disjoint systems are more or less independent, and systems are more or less independent of their environments, facts incompatible with the existence of knowledge or information.

What renders knowledge at all possible is nonequilibrium. In fact, rather trivially, the very existence of the devices and records, not to mention brains, yielding or embodying any sort of information is impossible under global equilibrium. And, according to Heisenberg, "every act of observation is by its very nature an irreversible process" [37], and thus fundamentally nonequilibrium.

Thus, the very notion of quantum equilibrium, of equilibrium of configurations relative to the wave function, already suggests the unknowability of these configurations beyond the wave function. Our results merely provide a firm foundation for this suggestion. What is, however, striking is the simplicity of the analysis and how absolute and clean are the conclusions.

Insofar as equilibrium is associated with the impossibility of knowledge, equilibrium alone does not provide an adequate perspective on our analysis. In particular, our results say perhaps little of physical relevance unless *some* knowledge is possible, e.g., of the wave function of a particular system, or of the results of observations. But for this *nonequilibrium* is essential.

13. Quantum equilibrium and thermodynamic (non)equilibrium

> [In] a complete physical description, the statistical quantum theory would...take an approximately analogous position to the statistical mechanics within the framework of classical mechanics. (Einstein, in ref. 50, p.672)

We would like now to place quantum equilibrium within a broader context by comparing it with classical thermodynamic equilibrium.

According to the quantum equilibrium hypothesis, when a system has wave function ψ, the distribution ρ of its configuration is given by

$$\rho = |\psi|^2. \tag{13.1}$$

Similarly, the Gibbs postulate of statistical mechanics asserts that for a system at temperature T, the distribution ρ of its phase space point is given by

$$\rho = \frac{e^{-H/kT}}{Z}, \tag{13.2}$$

where H is the classical Hamiltonian of the system (including, say, the "wall potential"), k is Boltzmann's constant, and Z, the partition function, is a normalization.

In addition, we found that (13.1) assumed sharp mathematical form when understood as expressing the conditional probability formula (6.1). (13.2) is perhaps also best regarded as a conditional probability formula, for the distribution of the phase point of the system given that of its environment—after all, the Hamiltonian H typically involves interactions with the environment, and the temperature T (like the wave function) can be regarded as a function of (the state of)the *environment.* (How otherwise would we know the temperature?) Furthermore, for a rigorous analysis of equilibrium distributions in the thermodynamic limit—i.e., of (the idealization given by) global thermodynamic equilibrium—the equations of Dobrushin and Lanford-Ruelle [27,41], stipulating that (13.2)—regarded as expressing such a conditional distribution—be satisfied for all subsystems, often play a defining role.[42]

Moreover, what we have just described is only a part of a deeper and broader analogy, between the scheme

$$\text{classical mechanics} \Longrightarrow \text{equilibrium statistical mechanics} \Longrightarrow \text{thermodynamics}, \qquad (13.3)$$

which outlines the (classical) connection between the microscopic level of description and a phenomenological formalism on the macroscopic level; and the scheme

$$\text{Bohmian mechanics} \Longrightarrow \begin{array}{l}\text{quantum equilibrium:}\\ \text{statistical mechanics rel-}\\ \text{ative to the wave function}\end{array} \Longrightarrow \text{the quantum formalism}, \qquad (13.4)$$

which outlines the (quantum) connection between the microscopic level and another phenomenological formalism—the quantum measurement formalism. We began this section by comparing only the middle components of (13.3) and (13.4), but it is in fact the full schemes which are roughly analogous.

[42] However, for a universe which, like ours, is not in global thermodynamic equilibrium, there is presumably no probability distribution on initial phase points with respect to which the probabilities (13.2), for all subsystems which happen to be "in thermodynamic equilibrium" and all times, are the conditional probabilities given the environments of the subsystems. In other words, roughly speaking, (13.2) is not equivariant. (See Krylov [39], as well as the discussion after (13.4).)

In particular, note that the middle of both schemes concerns the equilibrium distribution for the complete state description of the structure on the left with respect to the state for the structure on the right—the macrostate, as described by temperature (or energy) and, say, volume; or the quantum state, specified by the wave function. However, the quantum formalism does not live entirely on the macroscopic level, since the wave function for, say, an atom is best regarded as inhabiting (mainly) the microscopic level, at least for Bohmian mechanics.

The second arrow of (13.3) is, of course, associated primarily with the work of J. Willard Gibbs [34]; the corresponding arrow of (13.4), upon which we have not focused here, will be the subject of [29]. (See also [14,18].) We have here focused on the first arrow of (13.4), i.e., on deriving the quantum equilibrium hypothesis from Bohmian mechanics. The corresponding arrow of (13.3) remains an active area of research, though it does not appear likely that a comprehensive rigorous analysis will be forthcoming any time soon. Conventional wisdom to the contrary notwithstanding, the problem of the rigorous justification, from first principles, of the use of the "standard ensembles," i.e., of the derivation of randomness governed by detailed probabilities, is far more difficult for classical thermodynamic equilibrium than for quantum theory!

How can this be? How is it possible so easily to derive the quantum equilibrium hypothesis from first principles (i.e., from Bohmian mechanics), while the corresponding result for thermodynamics—the rigorous derivation of the Gibbs postulate from first principles—is so very difficult? The answer, we believe, is that "pure equilibrium" is easy, while nonequilibrium, even a little bit, is hard. In our nonequilibrium universe, systems which happen to be in thermodynamic equilibrium are surrounded by, and arose from, (thermodynamic) nonequilibrium. Thus with thermodynamic equilibrium we are dealing with *islands of equilibrium in a sea of nonequilibrium*. But with quantum equilibrium we are in effect dealing with a *global* equilibrium, albeit relative to the wave function.

What makes nonequilibrium so very difficult is the fact that for nontrivial dynamics it is extremely hard to get a handle on the evolution of nonequilibrium ensembles adequate to permit us rigorously to conclude much of anything concerning the present distribution that would arise from a given nonequilibrium distribution in the (distant) past. To establish "convergence to equilibrium" for times $t \to \infty$ (mixing) is itself extremely difficult, but even this would be of little physical relevance, since we generally deal with, and can survive only during, times much earlier than the epoch of global thermodynamic equilibrium.

We should perhaps elaborate on why global equilibrium is so easy. A key aspect of equilibrium is, of course, stationarity—or equivariance. But how can this be sufficient for our purposes? Mere stationarity is not normally sufficient in a dynamical system analysis to conclude that typical behavior embodies randomness governed by the stationary distribution. Such "almost everywhere"-type assertions usually require the ergodicity of the dynamics. Why did we not find it necessary to establish some sort of ergodicity?

The answer, we believe, lies in another critical aspect of the notion of equilibrium, shared by the schemes (13.3) and (13.4), and arising from the fact that both schemes are concerned with large "systems," with the thermodynamic limit as it were. In equilibrium, whether quantum or thermodynamic, most configurations or phase points are "macroscopically similar": quantities given by suitable spatial averages—e.g., density, energy density, or velocity fluctuations for thermodynamic equilibrium, and empirical correlations for quantum equilibrium—are more or less constant over the state space, in a sense defined by the equilibrium distribution. To say that a system is in equilibrium is then to say that its configuration or phase point is typical, in the sense that the values of these spatial averages are typical.

Now while the individual subsystems with which we have been concerned may be microscopic, our analysis, in fact, is effectively a "large system analysis." This is manifest in the equal-time analysis of Section 7, and for the general, multitime analysis it is implicit in our measurability conditions (10.4) and (10.8), which are plausible only for a universe having a large number of degrees of freedom. Thus, just as for a system *already in thermodynamic equilibrium,* we have no need for the ergodicity of the dynamics—just "stationarity"—since the kind of behavior we wish to establish occurs for a huge set of initial configurations, the "overwhelming majority."

(It might also be argued that we have, in fact, established for Bohmian mechanics a kind of effective Bernoulliness, and hence an effective ergodicity. And, again, the fact that we can do this with little work comes from the "thermodynamic limit" aspect of our analysis.)

The reader should compare the impossibility of perpetual motion machines, which is associated with the scheme (13.3), with that of "knowledge machines," as expressed by absolute uncertainty, associated with the scheme (13.4). In both cases the existence of devices of a certain character is precluded by general theoretical considerations—more or less equilibrium considerations for both—rather than by a detailed analysis of the workings of the various possible devices.

14. Global equilibrium beneath nonequilibrium

> But to admit things not visible to the gross creatures that we are is, in my opinion, to show a decent humility, and not just a lamentable addiction to metaphysics. (Bell [9])

The schemes (13.3) and (13.4) refer to different universes, a classical universe and a quantum (Bohmian) universe. Since our universe happens to be a quantum one, it would, perhaps, be better to consider, instead of (13.3), the analogous quantum scheme[43]

$$\text{Bohmian mechanics} \Longrightarrow \text{quantum statistical mechanics} \Longrightarrow \text{thermodynamics.} \tag{14.1}$$

While the second arrow of (14.1) is standard, and presumably nonproblematical, research on the first arrow has not yet reached its infancy.

Note that it would make little sense to ask for a derivation of quantum statistical mechanics from the first principles provided by *orthodox* quantum theory. The very meaning of orthodox quantum theory is so entwined with processes, such as measurements, in which thermodynamic considerations play a crucial role that it is difficult to imagine where such a derivation might begin, or, for that matter, what such a derivation could possibly mean! (And insofar as Bohmian mechanics clarifies the meaning and significance of the wave function of a system, and permits a coherent analysis of the microscopic and macroscopic

[43] While it can be shown that in the "macroscopic limit"

$$\text{Bohmian mechanics} \Longrightarrow \text{classical mechanics,}$$

a proper understanding of thermodynamics must be in terms of the *actual* behavior of the constituents of equilibrium systems, i.e., quantum behavior.

domains within a common theoretical framework, it may well be that the last word has not yet been written concerning the connection represented by the second arrow.)

If nonequilibrium is an essential aspect of our universe, and if configurations are in quantum equilibrium, i.e., pure equilibrium relative to the wave function, what then is the source, in our universe, of nonequilibrium? What is it that is *not in equilibrium*? The wave function, of course—both the universal wave function Ψ and, as a consequence, subsystem wave functions ψ. At the same time, the middle of the scheme (14.1) can be regarded as concerned with the distribution of the subsystem wave function ψ for subsystems which happen to be in thermodynamic equilibrium. But by exploiting global thermodynamic *nonequilibrium* we are able to see beneath the thermodynamic-macroscopic level of description, while with global quantum equilibrium there is no quantum nonequilibrium to reveal the system configuration X beneath the system wave function ψ.

It is important, however, not to succumb to the temptation to conclude, as does Heisenberg [37], that configurations therefore provide merely an "ideological superstructure" best left out of quantum theory; for, as we have seen, the very meaning of the wave function ψ of a subsystem requires the existence of configurations, i.e., those of its environment. And when we determine the wave function of a system we do so on the basis of the configuration of the environment. Recall also that both aspects of the wave function of a subsystem, the statistical and the dynamical, cannot coherently be formulated without reference to configurations. It is therefore not at all astonishing that orthodox quantum theory, by refusing to accept configurations as part of the description of the state of a system, has led to so much conceptual confusion.

Note that the fact that thermodynamics seems to depend only upon ψ, and not on any contribution to the total thermodynamic entropy from the actual configuration X, is an immediate consequence of quantum equilibrium: For a universe in quantum equilibrium the entropy associated with configurations is maximal, i.e., constant as a functional of ψ, and thus plays no thermodynamic role.

A crucial feature of our quantum universe is the peaceful coexistence between global equilibrium (quantum) and nonequilibrium (thermodynamic), providing us with what we may regard as an "equilibrium laboratory," a glimpse, as it were, of pure equilibrium, with all the surprising consequences it entails. Our analysis has shown how the interplay between the corresponding levels of structure—the nonequilibrium level given by the wave function, and, beneath the level of the wave function, that of the particles, described by their positions, in equilibrium relative to the wave function—leads to the randomness and uncertainty so characteristic of quantum theory. We shall explore elsewhere [30] how this (hierarchical) structure itself naturally arises, and what its deeper significance might be. (See also Bohm [17].)

We have argued, and believe our analysis demonstrates, that quantum randomness can best be understood as arising from ordinary "classical" uncertainty—about what is *there* but *unknown*. The denial of the existence of this unknowable—or only partially knowable—reality leads to ambiguity, incoherence, confusion, and endless controversy. What does it gain us?

Appendix: Random points

In the following remarks we expand upon concepts introduced in this paper, placing our conclusions within a broader perspective and comparing ours with related approaches.

1. Bohmian mechanics is what emerges from Schrödinger's equation, which is said to describe the evolution of the wave function of a system of *particles*, when we take this language seriously, i.e., when we insist that *"particles" means particles*. Thus Bohmian mechanics is the minimal interpretation of nonrelativistic quantum theory, arising as it does from the assertion that a familiar word has its familiar meaning.

In particular, if Bohmian mechanics is somehow strange or unacceptable, it must be because either Schrödinger's equation, or the assertion that "particles" means particles, or their combination is strange or unacceptable. Now the assertion that "particles" means particles can hardly be regarded as in any way problematical. On the other hand, Schrödinger's equation, for a field on *configuration* space, is a genuine innovation, though one that physicists by now, of course, take quite for granted. However, as we have seen in Section 2, when it is appropriately combined with the assertion that "particles" means particles, its strangeness is, in fact, very much diminished.

2. Quantum mechanics is notoriously nonlocal [52], a novelty which is in no way ameliorated by Bohmian mechanics. In fact, "in this theory an explicit causal mechanism exists whereby the disposition of one piece of apparatus affects the results obtained with a distant piece" [2]. We wish to emphasize, however, that *relative to the wave function*, Bohmian mechanics is completely *local*: the nonlocality in Bohmian mechanics derives solely from the nonlocality built into the structure of standard quantum theory, as provided by a wave function on configuration space.

> That the guiding wave, in the general case, propagates not in ordinary three-space but in a multidimensional-configuration space is the origin of the notorious 'nonlocality' of quantum mechanics. It is a merit of the de Broglie-Bohm version to bring this out so explicitly that it cannot be ignored. (Bell [5])

3. A rather fortunate property of Bohmian mechanics is that the behavior of the parts—of subsystems—reflects that of the whole. Indeed, if this were not the case it would have been difficult, if not impossible, to have ever discovered the full theory. We believe that a major reason nonlocality is so often regarded as problematical is not nonlocality per se but rather that it *suggests* the breakdown of precisely this feature.

4. Notice that the effective wave function ψ is, in effect, a "collapsed" wave function. Thus our analysis implicitly explains the status and role of "collapse of the wave packet" in the quantum formalism. (See also Point 21, recalling that the Wigner formula [60] for the joint distribution of the outcomes of a sequence of quantum measurements, to which we there refer, is usually based upon collapse.)

In particular, note that the effective wave function of a subsystem evolves according to Schrödinger's equation only when this system is suitably isolated. More generally, the evolution $\psi(t)$ of the effective wave function defines a stochastic process, one which embodies collapse in just the right way—with respect to the conditional probability distribution given the (initial) configuration of the environment of the composite system which includes the apparatus, with ψ the effective wave function of the system alone, i.e., not including the apparatus. For details see [29].

Note also that the very notion of effective wave function, as well as its behavior, depends upon the location of the split between the "observed" and the "observer," i.e., between the system of interest and the rest of the world, a dependence whose importance has been emphasized by Bohr [22], by von Neumann [56], and by a great many others, see for example [12,40,43]. In particular, while the effective wave function will "collapse" during measurement if the apparatus is *not* included in the system, it need not, in principle, collapse if the apparatus *is* included, precisely as emphasized by von Neumann [56]. But von Neumann was left with the "measurement paradox," while with Bohmian mechanics no hint of paradox remains.

5. The fact that knowledge of the configuration of a system must be mediated by its wave function may partially account, from a Bohmian perspective, for how the physics community could identify the state of a quantum system—its complete description—with its wave function without encountering any *practical* difficulties. Indeed, the conclusion of our analysis can be partially summarized with the assertion that the wave function ψ of a subsystem represents maximal information about its configuration X. This is primarily because of the wave function's statistical role, but its dynamical role is also relevant here. Thus it is natural, even in Bohmian mechanics, to regard the wave function as the "state" of the system.

6. It has been clear, at least since von Neumann [56], that for all practical purposes the quantum formalism, regarded in strictly operational terms, is consistent. However, it has not, at least for many (e.g., Einstein), been clear that the "full" quantum theory, regarded as including the assertion of "completeness" based upon Heisenberg's uncertainty principle—which has itself traditionally been regarded as arising from the apparent impossibility of certain measurements described in more or less *classical* terms—is also consistent. (See [54] for a recent expression of related concerns.) If nothing else, Bohmian mechanics establishes and makes clear this consistency—even including absolute uncertainty.

Indeed, as is well known, Einstein tried for many years to devise thought experiments in which the limitations expressed by the uncertainty principle could be evaded. The reason Einstein persisted in this endeavor is presumably connected with the fact that the arguments presented by Heisenberg and Bohr against such a possibility were, to say the least, not entirely convincing, relying, as they did, on a peculiar, nearly contradictory, combination of quantum and classical "reasoning." In this regard, recall that in order to rescue (a version of) the uncertainty principle from one of Einstein's final onslaughts (see [21]), Bohr felt compelled to exploit certain effects arising from Einstein's general theory of relativity [21].

However, from the perspective of a Bohmian universe the uncertainty principle is sharp and clear. In particular, from such a perspective it makes no sense to try to devise *thought* experiments by means of which the uncertainty principle can be evaded, since this principle is a mathematical consequence of Bohmian mechanics itself. One could, of course, imagine a universe governed by different laws, in which the uncertainty principle, and a great deal else, *would* be violated, but there can be no universe governed by Bohmian mechanics—and in quantum equilibrium— which fails to embody absolute uncertainty and the uncertainty principle which it entails.

7. The notion of effective wave function developed in Section 5 should perhaps be compared with a related notion of Bohm, namely, the "active" piece of the wave function [19,20] (see also Bohm [13]): If Ψ is of the form (5.12) with the supports of $\Psi^{(1)}$ and $\Psi^{(2)}$ "sufficiently disjoint," then $\Psi^{(i)}$ is "active" if the actual configuration Q is in the support of $\Psi^{(i)}$. (See (5.13) and the surrounding discussion.) When this active wave function appropriately factorizes—see (5.6)—the (active) wave function of a subsystem could be defined in terms of the obvious factor.

This notion of subsystem wave function will agree with ours if, as is likely to be the case, the active and inactive pieces have suitably disjoint y-supports, and it will otherwise disagree. (In this regard see also Point 20.) For example, if

$$\Psi^{(i)}(x,y) = \psi^{(i)}(x)\Phi(y) \tag{A.1}$$

with $\psi^{(1)}$ and $\psi^{(2)}$ suitably disjoint (e.g., because the x-system is macroscopic and ...) then the "active" wave function of the x-system is the appropriate $\psi^{(i)}$, while using our notion the x-system has effective wave function $\psi^{(1)} + \psi^{(2)}$. Note, in particular, that with our notion the effective wave function of the universe is the universal wave function Ψ, not the active piece of Ψ.

Our notion of effective wave function—and not the notion based upon the active piece—has a distinctly epistemological aspect: While for both choices we have that "$\rho = |\psi|^2$", the latter will be the conditional distribution given the configuration of the environment only if ψ agrees with our effective (or conditional) wave function. Moreover, whenever we can be said to *"know* that the x-system has wave function ψ," then the x-system indeed has effective wave function ψ in our sense.

Note that while both of these choices are somewhat vague, in that they appeal to the notion of the "macroscopic"—or to some such notion—our effective wave function, when it exists, is, as we have seen, completely unambiguous. Moreover, as we have also seen, with our notion reference to something like the macroscopic is not critical. Removing such a reference—as we did in defining the notion of the conditional wave function—leads to a precise formulation which remains entirely adequate (in fact, perfect) for our purposes. But for the choice based on the active piece, removing such a reference would lead to utter vagueness.

There is, of course, no real physics contingent upon a particular choice of (notion of) "effective wave function"; rather this choice is simply a matter of convenience of expression, of how we talk most efficiently about the physics. But such considerations can be quite important!

8. Sometimes it is helpful to try to imagine how things appear to God. This is of course audacious, but, in fact, the very activity of a physicist, his attempting to find the deepest laws of nature, is nothing if not audacious. Indeed, one might even argue that the defining activity of the physicist is the search for the divine perspective.

Be that as it may, to create a universe God must first decide upon the ontology—on what there is—and then on the dynamical laws—on how what is behaves. But this alone would not be sufficient. What is missing is a particular realization, out of all possible solutions, of the dynamics—the one corresponding to the actual universe. In other words, at least for a deterministic theory, what is further required is a choice of initial conditions. And unless there is somehow a natural special choice, the simplest possibility would appear to be a completely random initial condition, with an appropriate natural measure for the description of this randomness (whatever this might mean, even given the measure). The notion of typicality so defined would, in a sense, be an essential ingredient of the theory governing this hypothetical universe.

For Bohmian mechanics, *with somehow given initial wave function* Ψ_0, this measure of typicality is given by the quantum equilibrium distribution $|\Psi_0|^2$. Moreover, the dynamics itself is also generated by Ψ_0. It seems most fitting that God should design the universe in so efficient a manner, that a single object, the wave function Ψ_0, should generate all the necessary (extra-ontological) ingredients.

9. Regarding the question of universal initial conditions, we should perhaps contrast the issue of the initial configuration with that of the initial wave function. Insofar as the latter is a nonequilibrium wave function, the initial wave function must correspond to low entropy—it must be very atypical, i.e., of a highly improbable character. As has been much emphasized by R. Penrose [49], in order to understand our nonequilibriuim world we must face the problem of why God should have chosen such improbable initial conditions as demanded by nonequilibrium. On the other hand, for the universal initial configuration—in quantum equilibrium—we of course have no such problem. On the contrary, quantum randomness itself, including even absolute uncertainty, arising as it does from quantum equilibrium, in effect requires no explanation. (Concerning the choice of initial universal wave function, see also Point 13.)

10. Naive agreement with the quantum formalism *demands* the existence of a small set of bad initial configurations, corresponding to outcomes which are very unlikely but *not* impossible. It is thus hard to see how our results could be improved upon or significantly strengthened.

More generally, for any theory with probablistic content, particularly one describing a relativistic universe, we arrive at a similar conclusion: Once we recognize that there is but one world (of relevance to us), only one actual space-time history, we must also recognize that the ultimate meaning of probability, insofar as it employed in the formulation of the predictions of the theory, must be in terms of a specification of typicality—one such that theoretically predicted empirical distributions are typical. When all is said and done, the physical import of the theory must arise from its provision of such a notion of typical space-time histories (at the very least of "macroscopic" events), presumably specified via a probability distribution on the set of all (kinematically) possible histories. And given a theory, i.e., such a probability distribution, describing a large but finite universe, atypical space-time histories, with empirical distributions disagreeing with the theoretical predictions, are, though extremely unlikely, not impossible.

11. It is quite likely that the fiber $\mathcal{Q}_t^Y \equiv \{Q \in \mathcal{Q} \mid Y_t = Y\}$ of $\mathcal{Q}$ for which $Y_t = Y$, discussed in Section 7, is extremely small, owing to the expansive and dispersive effects of the Laplacian $\mathbf{\Delta}$ in Schrödinger's equation. If so, it follows that any regular (continuous) Ψ_0 (or $|\Psi_0|^2$) should be approximately constant on $\mathcal{Q}_t^Y$ (as on any sufficiently small set of initial conditions). This would imply that $\mathbf{P}_t^Y$, the conditional measure given $\mathcal{Q}_t^Y$, should be approximately the same as the uniform distribution—Lebesgue measure—on $\mathcal{Q}_t^Y$, so that typicality defined in terms of quantum equilibrium agrees with typicality in terms of Lebesgue measure.

Now, as we have already indicated in Section 4, under more careful scrutiny this argument does not sustain its appearance of relevance. However, it may nonetheless have some heuristic value.

12. We wish to emphasize that a byproduct of our analysis, quite aside from the relevance of this analysis to the interpretation of quantum theory, is the clarification and illumination of the meaning and role of probability in a deterministic (or even nondeterministic) universe. Moreover, our analysis of statistical tests in Section 7—the very triviality of this analyis, see equations (7.8) and (7.10)—sharply underlines the centrality of typicality in the elucidation of the concept of probability.

13. We should mention some examples of nonequilibrium (initial) universal wave functions:

(1) Suppose that physical space is finite, say the 3-torus $\mathbb{T}^3$ rather than $\mathbb{R}^3$, and suppose, say, that the potential energy $V = 0$. Let $\Psi_0(\mathbf{q}_1, \ldots, \mathbf{q}_N) = 1$ if all $\mathbf{q}_i \in B$, where $B \subset \mathbb{T}^3$ is a "small" region in physical space, and be otherwise 0. Then Ψ_0 is a nonequilibrium wave function, since an equilibrium wave function should be "spread out" over $\mathbb{T}^3$. Moreover the

initial quantum equilibrium distribution on configurations is uniform over configurations of N particles in B.

More generally, any well localized Ψ_0 is a nonequilibrium wave function. And if physical space is $\mathbb{R}^3$, any localized or square-integrable wave function is a nonequilibrium wave function.

(2) For a nonequilibrium wave function of a rather different character, consider the following: Take $\mathbb{T}^3$ again for physical space, but instead of considering free particles, suppose that V arises from Coulomb interactions, with half of the particles having charge $+e$ and half $-e$. Now suppose that Ψ_0 is constant, $\Psi_0 = 1$ on $\mathbb{T}^3$. (Thus, quantum equilibrium now initially corresponds to a uniform distribution on configurations.) That this Ψ_0, though "spread out," is nevertheless a nonequilibrium wave function can be seen in various ways. Dynamically, the Schrödinger evolution should presumably lead to the formation of "atoms," of suitable pairing in the (support properties of the) wave function. Entropically, Ψ_0 is very special. An equilibrium ensemble of initial wave functions is determined by the values of the infinite set of constants of the motion given by the absolute squares of the amplitudes with respect to a basis of energy eigenfunctions. Wave functions in this ensemble are then specified by the phases of these amplitudes. A random choice of phases leads to an equilibrium wave function, which should reflect the existence of "atoms." On the other hand, the wave function $\Psi_0 = 1$ corresponds to a particular, very special choice of phases, so that "atoms cancel out."

Note also that this example is relevant to the Penrose problem mentioned in Point 9. What choice of initial wave function could be simpler—and thus in a sense more natural—than the one which is everywhere constant? And, again, while it might at first glance seem that this choice corresponds to equilibrium, the attractive (in both senses) effects of the Coulomb interaction presumably imply that this is not so!

From a classical perspective the situation is similar: The initial state in which the particles are uniformly distributed in space with velocities all 0 (or with independent Maxwellian velocities) is a nonequilibrium state. In fact, an infinite amount of entropy can be extracted from suitable clustering of the particles, arising from the great volume in momentum space liberated when pairs of oppositely charged particles get close. (Of course, for Newtonian gravitation—as well as for general relativity—this tendency to cluster is, in a sense, far stronger still.)

14. To account for (the) most (familiar) applications of the quantum formalism one rarely needs to apply (the conclusions of) our quantum equilibrium analysis to systems of the form (5.4): Randomness in the result of even a quantum measurement usually arises solely from randomness in the system, randomness in the apparatus making essentially no contribution. This is because most real-world measurements are of the scattering-detection type—and a particle (or atom ...) will be detected more or less where it's at. Think, for example, of a two-slit-type experiment, or of the purpose of a cloud chamber, or of a Stern-Gerlach measurement of spin.

15. When all is said and done, what does the incorporation of actual configurations buy us? A great deal! It accounts for:

(1) randomness

(2) absolute uncertainty

(3) the meaning of the wave function of a (sub)system

(4) collapse of the wave packet

(5) coherent—indeed, familiar—(macroscopic) reality

Moreover, it makes possible an appreciation of the basic significance of the universal wave function Ψ, as an embodiment of *law*, which cannot be clearly discerned without a coherent ontology to be governed by some law.

16. Recall that in principle the wave function ψ of a (sub)system could depend upon the universal wave function Ψ and on the choice of system $\sigma = (\pi, T)$, as well as on the configuration Y of the environment of this system. In practice, however, in situations in which we in fact know what ψ is, it must be given by a function of Y alone, not depending upon σ, nor even on Ψ (for "reasonable" nonequilibrium Ψ). After all, what else, beyond Y, do we have at our disposal to take into account when we conclude that a particular system has wave function ψ? In particular, Ψ is unknown, apart from what we can conclude about it on the basis of Y (and perhaps some a priori assumptions about reasonable initial Ψ_0's. But even if Ψ_0 were known precisely, this information would be of little use here, since

solving Schrödinger's equation to obtain Ψ would be out of the question!)

Thus, whatever we can in practice conclude about ψ must be based upon a *universal* function—of Y. It would be worthwhile to explore and elucidate the details of this function, analyzing the rules we follow in obtaining knowledge and trying to understand the validity of these rules. However, such considerations are not directly relevant to our purposes in this paper, where our goal has been primarily to establish sharp *limitations* on the possibility of knowledge rather than to analyze what renders it at all possible. We have argued that the latter problem is perhaps far more difficult than the former, and, indeed, that this is not terribly astonishing.

17. In view of the similarity between Bohmian mechanics and stochastic mechanics [44,45,46], for which similarity see [35,28], all of our arguments and results can be transferred to stochastic mechanics without significant modification. More important, the motivation for stochastic mechanics is the rather plausible suggestion that quantum randomness might originate from the merging of classical dynamics with intrinsic randomness, as described by a diffusion process, and with "noise" determined by $\hbar$. Insofar as our results demonstrate how quantum randomness naturally emerges without recourse to any such "noise," they rather drastically erode the evidential basis of stochastic mechanics.

18. The analyis of Bohmian mechanics presented here is relevant to the problem of the interpretation and application of quantum theory in cosmology, specifically, to the problem of the significance of $\rho = |\psi|^2$ on the cosmological level—where there is nothing outside of the system to perform the measurements from which $\rho = |\psi|^2$ derives its very meaning in orthodox quantum theory.

19. Our random system analysis illuminates the flexibility of Bohmian mechanics: It illustrates how joint probabilities as predicted by the quantum formalism, even for configurations, may arise from measurement and bear little resemblance to the probabilities for unmeasured quantities. And our analysis highlights the mathematical features which make this possible. This flexibility could be quite important for achieving an understanding of the relativistic domain, where it may happen that quantum equilibrium prevails only on special space-time surfaces (see [28]). Our (random system) multitime analysis illustrates

how this need entail no genuine obstacle to obtaining the quantum formalism. (Our argument here of course involved the natural hypersurfaces given by $\{t = \text{const.}\}$, but the only feature of these surfaces critical to our analysis was the validity of quantum equilibrium, or, more precisely, of the fundamental conditional probability formula (6.1).)

20. A notion intermediate between that of the effective wave function and that of the conditional wave function of a subsystem, a *more-general-effective wave function* which like the effective wave function is "stable," may be obtained by replacing, in the definition (5.17)–(5.18) of effective wave function, the reference to macroscopically disjoint y-supports by "sufficiently disjoint" y-supports. This notion of more-general-effective wave function is, of course, rather vague. But we wish to emphasize that the y-supports of Φ and $\Psi^{\perp}$ may well be sufficiently disjoint to render negligible the (effects of) future interference between the terms of (5.17)—so that if (5.18) is satisfied, ψ will indeed fully function dynamically as the wave function of the x-system—without their having to be actually macroscopically disjoint.

In fact, owing to the interactions—expressed in Schrödinger's equation—among the many degrees of freedom, the amount of y-disjointness in the supports of Φ and $\Psi^{\perp}$ will typically tend to increase dramatically as time goes on, with, as in a chain reaction, more and more degrees of freedom participating in this disjointness (see [13,42,64,38]; see also [12])). When the effects of this dissipation or "decoherence" are taken into account, one finds that a small amount of y-disjointness will often tend quickly to become "sufficient," indeed becoming "much more sufficient" as time goes on, and very often indeed becoming macroscopic. Moreover, if ever we are in the position of knowing that a system has more-general-effective wave function ψ, then ψ must be its effective wave function, since our knowledge must be based on or grounded in macroscopic distinctions (if only in the eye or brain).

Concerning dissipation, we wish also to emphasize that in practice the problem is not how to arrange for it to occur but how to keep it under control, so that superpositions of (sub)system wave functions retain their coherence and thus may interfere.

21. If we relax the condition (10.3), requiring that ψ_{σ_i} be nonrandom, and stipulate instead

merely that

$$\psi_{\sigma_i} \in \mathcal{F}(Z_1, \ldots, Z_{i-1}), \tag{A.2}$$

we find that $Z_1, \ldots, Z_M$ have joint distribution given by the familiar (Wigner) formula [60] (see also [56] and [1]).

22. We wish to compare (what we take to be the lessons of) Bohmian mechanics with the approach of Gell-Mann and Hartle (GMH) [31,32]. Unhappy about the irreducible reference to the observer in the orthodox formulation of quantum theory, particularly insofar as cosmology is concerned, they propose a program to extract from the quantum formalism a "quasiclassical domain of familiar experience," which, if we understand them correctly, defines for them the basic ontology of quantum theory. This they propose to do by regarding the Wigner formula (referred to in Points 4 and 21), for the joint probabilities of the results of a sequence of measurements of quantum observables, as describing the probabilities of objective, i.e., not-necessarily-measured, events—what they call alternative histories. Of course, owing to interference effects one quickly gets into trouble here unless one restricts *this* use of the Wigner formula to what they call alternative (approximately) *decohering* histories, for which the Wigner formula can indeed be regarded as defining (approximate) probabilities, which are additive under coarse-graining. Thus far GMH in essence reproduce the work of Griffiths [36] and Omnes [47]. But, as GMH further note, the condition of (approximate) decoherence by itself allows for far too many possibilities. They thus introduce additional conditions, such as "fullness" and "maximality," as well as propose certain (as yet tentative) measures of "classicity" to define an optimization procedure they hope will yield a more or less unique quasiclassical domain. (They also consider the possibility that there may be many quasiclassical domains, each of which would presumably define a different physical theory.)

As in our analysis of Bohmian mechanics, universal initial conditions—for GMH the initial universal wave function(or density matrix)—play a critical role. And just as in Bohmian mechanics, the wave function does not provide a complete description of the universe, but rather attains physical significance from the role it plays in generating the behavior of something else, something *physically* primitive—for GMH the quasiclassical domain.

Insofar as nonrelativistic quantum theory is concerned, a significant difference between Bohmian mechanics and the proposal of GMH is that the latter defines a research program while the former is an already existing, and sharply formulated, physical theory. And as far as relativistic quantum theory is concerned, we believe that, appearances to the contrary notwithstanding, the lesson of Bohmian mechanics is one of flexibility (see also Point 19) while the approach of GMH is rigid. In saying this we have in mind, on the one hand, that GMH insist (1) that the possible ontologies be limited by the usual quantum description, i.e., correspond to a suitable (possibly time-dependent) choice of self-adjoint operators on Hilbert space; and (2) that this ontology be constrained further by the quantum formalism, demanding that its evolution be governed by the Wigner formula—so that for them, but not for Bohmian mechanics, the consideration of decoherence indeed becomes essential, bound up with questions of ontology.

On the other hand, one lesson of Bohmian mechanics is that ontology need not be so constrained. While the quantum formalism must—and for Bohmian mechanics does—emerge in measurement-type situations, the behavior of the basic variables, describing the fundamental ontology, outside of these situations need bear no resemblance to anything suggested by the quantum formalism. (Recall, in fact, that it quite frequently happens that simple, symmetric laws on a deeper level of description lead to a less symmetric phenomenological description on a higher level.) Indeed, these basic variables, whether they describe positions, or field configurations, or what have you, need not even correspond to self-adjoint operators. That they rather trivially do in Bohmian mechanics is, in part, merely an artifact of the equivariant measure's being a strictly local functional of the wave function, which was in no way crucial to our analysis.

In particular, while dissipation or decoherence are relevant both to Bohmian mechanics and to GMH, for GMH they are crucial to the *formulation* of the theory, to the specification of an *ontology*, while for Bohmian mechanics they are relevant only on the level of *phenomenology*. And insofar as the formation of new theories is concerned, the lesson of Bohmian mechanics is to look for fundamental microscopic laws appropriate to the (or a) natural choice of ontology, rather than to let the ontology itself be dictated by some law, let alone by what is usually regarded as a macroscopic measurement formalism.

It is perhaps worth considering briefly the two-slit experiment. In Bohmian mechanics the electron, indeed, goes through one or the other of the two slits, the interference pattern arising because the arrival of the electron at the "photographic" plate reflects the interference profile of the wave function governing the motion of the electron. In particular, and this is what we wish to emphasize here, in Bohmian mechanics a spot appears somewhere on the plate because the electron arrives there; while for GMH "the electron arrives somewhere" because the spot appears there.

23. There is one situation where we may, in fact, know more about configurations than what is conveyed by the quantum equilibrium hypothesis $\rho = |\psi|^2$: when we ourselves are part of the system! See, for example, the paradox of Wigner's friend [59]. In thinking about this situation it is important to note well that, while it may be merely a matter of convention whether or not we choose to include say ourselves in the subsystem of interest, the wave function to which the quantum equilibrium hypothesis refers—that of the subsystem—depends crucially on this choice.

24. We have shown, in part here and in part in [29], how the quantum formalism emerges within a Bohmian universe in quantum equilibrium. Thus, evidence for the quantum formalism is evidence for quantum equilibrium—global quantum equilibrium. This should be contrasted with the thermodynamic situation, in which the evidence points towards pockets of thermodynamic equilibrium within global thermodynamic nonequilibrium.

The reader may wish to explore quantum nonequilibrium. What sort of behavior would emerge in a universe which is initially in quantum nonequilibrium? What phenomenological formalism or laws would govern such behavior? We happen to have no idea! We know only that such a world is not our world! Or do we?

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References

1. Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, *Time symmetry in the quantum process of measurement*, Physical Review B **134** (1964), 1410–1416, reprinted in [58].
2. J. S. Bell, *On the problem of hidden variables in quantum mechanics*, Reviews of Modern Physics **38** (1966), 447–452, reprinted in [58] and in [10].
3. J. S. Bell, *On the Einstein Podolsky Rosen paradox*, Physics **1** (1964), 195–200, reprinted in [58] and in [10].
4. J. S. Bell, *The measurement theory of Everett and de Broglie's pilot wave*, Quantum Mechanics, Determinism, Causality, and Particles (L. de Broglie and M. Flato, eds.), Dordrecht-Holland, D. Reidel, Boston, 1976, pp. 11–17, reprinted in [10].
5. J. S. Bell, *De Broglie-Bohm, delayed-choice double-slit experiment, and density matrix*, International Journal of Quantum Chemistry: A Symposium **14** (1980), 155–159, reprinted in [10].
6. J. S. Bell, *Bertlmann's socks and the nature of reality*, Journal de Physique, C2 **42** (1981), 41–61, reprinted in [10].
7. J. S. Bell, *Quantum mechanics for cosmologists*, Quantum Gravity 2 (C. Isham, R. Penrose, and D. Sciama, eds.), Oxford University Press, New York, 1981, pp. 611–637, reprinted in [10].
8. J. S. Bell, *On the impossible pilot wave*, Foundations of Physics **12** (1982), 989–999, reprinted in [10].
9. J. S. Bell, *Are there quantum jumps?*, Schrödinger. Centenary celebration of a polymath (C. W. Kilmister, ed.), Cambridge University Press, Cambridge, 1987, reprinted in [10].
10. J. S. Bell, *Speakable and unspeakable in quantum mechanics*, Cambridge University Press, Cambridge, 1987.
11. J. S. Bell, *Against "measurement"*, Physics World **3** (1990), 33–40, also in A. I. Miller (ed.), *Sixty-two Years of Uncertainty: Historical, Philosophical, and Physical Inquiries into the Foundations of Quantum Mechanics*, Plenum Press, New York, 1990, pp. 17–31.
12. D. Bohm, *Quantum Theory*, Prentice-Hall, Englewood Cliffs, N.J., 1951.
13. D. Bohm, *A suggested interpretation of the quantum theory in terms of "hidden variables": Part I*, Physical Review **85** (1952), 166–179, reprinted in [58].
14. D. Bohm, *A suggested interpretation of the quantum theory in terms of "hidden variables": Part II*, Physical Review **85** (1952), 180–193, reprinted in [58].

15. D. Bohm, *Proof that probability density approaches $|\psi|^2$ in causal interpretation of quantum theory*, Physical Review **89** (1953), 458–466.

16. D. Bohm and B. J. Hiley, *On the intuitive understanding of non-locality as implied by quantum theory*, Foundations of Physics **5** (1975), 93–109.

17. D. Bohm, *Wholeness and the Implicate Order*, Routledge & Kegan Paul, London, 1980.

18. D. Bohm and B. J. Hiley, *Measurement understood through the quantum potential approach*, Foundations of Physics **14** (1984), 255–274.

19. D. Bohm and B. J. Hiley, *An ontological basis for the quantum theory I: Non-relativistic particle systems*, Physics Reports **144** (1987), 323–348.

20. D. Bohm and B. Hiley, *The Undivided Universe: an Ontological Interpretation of Quantum Theory*, 1991, preprint.

21. N. Bohr, *Discussion with Einstein on epistemological problems in atomic physics*, in [50], pp. 199–244, reprinted in [22] and in [58].

22. N. Bohr, *Atomic Physics and Human Knowledge*, Wiley, New York, 1958.

23. M. Born, *Quantenmechanik der Stoßvorgänge*, Zeitschrift für Physik **38** (1926), 803–827; English translation, *Quantum mechanics of collision processes,* G. Ludwig (ed.), *Wave Mechanics*, Pergamon Press, Oxford and New York, 1968.

24. L. de Broglie, *A tentative theory of light quanta*, Philosophical Magazine **47** (1924), 446–458.

25. L. de Broglie, *La nouvelle dynamique des quanta*, Electrons et Photons: Rapports et Discussions du Cinquième Conseil de Physique tenu à Bruxelles du 24 au 29 Octobre 1927 sous les Auspices de l'Institut International de Physique Solvay, Gauthier-Villars, Paris, 1928, pp. 105–132.

26. B. S. DeWitt and N. Graham, Eds., *The Many-Worlds Interpretation of Quantum Mechanics*, Princeton University Press, Princeton, N.J., 1973.

27. R. L. Dobrushin, *The description of a random field by means of conditional probabilities and conditions of its regularity*, Theory of Probability and its Applications **13** (1968), 197–224.

28. D. Dürr, S. Goldstein, and N. Zanghí, *On a realistic theory for quantum physics*, Stochastic Processes, Geometry and Physics (S. Albeverio, G. Casati, U. Cattaneo, D. Merlini, R. Mortesi, eds.), World Scientific, Singapore, 1990, pp. 374–391.

29. D. Dürr, S. Goldstein, and N. Zanghí, *On the role of operators in quantum theory*, in preparation.

30. D. Dürr, S. Goldstein, and N. Zanghí, *The mystery of quantization*, in preparation.

31. M. Gell-Mann and J. B. Hartle, *Quantum mechanics in the light of quantum cosmology*, Complexity, Entropy, and the Physics of Information (W. Zurek, ed.), Addison-Wesley, Reading, 1990, pp. 425–458; also in S. Kobayashi, H. Ezawa, Y. Murayama, and S. Nomura (eds.), *Proceedings of the 3rd International Symposium on Quantum Mechanics in the Light of New Technology*, Physical Society of Japan, 1990.

32. M. Gell-Mann and J. B. Hartle, *Alternative decohering histories in quantum mechanics* (1991), preprint.

33. G. C. Ghirardi, A. Rimini, and T. Weber, *Unified dynamics for microscopic and macroscopic systems*, Physical Review D **34** (1986), 470–491.

34. J. W. Gibbs, *Elementary Principles in Statistical Mechanics*, Yale University Press, 1902; Dover, New York, 1960.

35. S. Goldstein, *Stochastic mechanics and quantum theory*, Journal of Statistical Physics **47** (1987), 645–667.

36. R. B. Griffiths, *Consistent histories and the interpretation of quantum mechanics*, Journal of Statistical Physics **36** (1984), 219–272.

37. W. Heisenberg, *Physics and Philosophy*, Harper and Row, New York, 1958, p. 138.

38. E. Joos and H. D. Zeh, *The emergence of classical properties through interaction with the environment*, Zeitschrift für Physik B **59** (1985), 223–243.

39. N. S. Krylov, *Works on the Foundations of Statistical Mechanics*, Princeton University Press, Princeton, N.J., 1979.

40. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory*, translated from the Russian by J. B. Sykes and J. S. Bell, Pergamon Press, Oxford and New York, 1958.

41. O. E. Lanford, III, and D. Ruelle, Communications in Mathematical Physics **13** (1969), 194–215.

42. A. J. Leggett, *Macroscopic quantum systems and the quantum theory of measurement*, Supplement of the Progress of Theoretical Physics **69** (1980), 80–100.

43. F. W. London and E. Bauer, *La Théorie de l'Observation en Mécanique Quantique*, Hermann, Paris, 1939; English translation by A. Shimony, J. A. Wheeler, W. H. Zurek, J. McGrath, and S. McLean McGrath in [58].

44. E. Nelson, *Derivation of the Schrödinger equation from Newtonian mechanics*, Physical Review **150** (1966), 1079–1085.

45. E. Nelson, *Dynamical Theories of Brownian Motion*, Princeton University Press, Princeton, N.J., 1967.

46. E. Nelson, *Quantum Fluctuations*, Princeton University Press, Princeton, N.J., 1985.

47. R. Omnes, *Logical reformulation of quantum mechanics I*, Journal of Statistical Physics **53** (1988), 893–932.

48. R. Penrose, *Quantum gravity and state-vector reduction*, Quantum Concepts in Space and Time (R. Penrose and C. J. Isham, eds.), Oxford University Press, Oxford, 1985, see also [49].

49. R. Penrose, *The Emperor's New Mind*, Oxford University Press, New York and Oxford, 1989.

50. P. A. Schilpp, Ed., *Albert Einstein, Philosopher-Scientist*, Library of Living Philosophers, Evanston, Ill., 1949.

51. E. Schrödinger, *Die gegenwärtige Situation in der Quantenmechanik*, Naturwissenschaften **23** (1935), 807–812, 823–828, 844–849; English translation by J. D. Trimmer, *The present situation in quantum mechanics: a translation of Schrödinger's "cat paradox" paper*, Proceedings of the American Philosophical Society **124** (1980), 323–338, reprinted in [58].

52. E. Schrödinger, *Discussion of probability relations between separated systems*, Proceedings of the Cambridge Philosophical Society **31** (1935), 555–563, **32** (1936), 446–452.

53. J. T. Schwartz, *The pernicious influence of mathematics on science*, Discrete Thoughts: Essays on Mathematics, Science, and Philosophy, by M. Kac, G. Rota, and J. T. Schwartz, Birkhauser, Boston, 1986, p. 23.

54. M. O. Scully and H. Walther, *Quantum optical test of observation and complementarity in quantum mechanics*, Physical Review A **39** (1989), 5229–5236.

55. H. P. Stapp, *Light as foundation of being*, Quantum Implications: Essays in Honor of David Bohm (B. J. Hiley and F. D. Peat, eds.), Routledge & Kegan Paul, London and New York, 1987.

56. J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Springer Verlag, Berlin, 1932; English translation by R. T. Beyer, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, N.J., 1955.

57. S. Weinberg, *Precision tests of quantum mechanics*, Physical Review Letters **62** (1989), 485—488.
58. J. A. Wheeler and W. H. Zurek (eds.), *Quantum Theory and Measurement*, Princeton University Press, Princeton, N.J., 1983.
59. E. P. Wigner, *Remarks on the mind-body question*, The Scientist Speculates (I. J. Good, ed.), Basic Books, New York, 1961, reprinted in [61] and in [58].
60. E. P. Wigner, *The problem of measurement*, American Journal of Physics **31** (1963), 6–15, reprinted in [61] and in [58].
61. E. P. Wigner, *Symmetries and Reflections*, Indiana University Press, Bloomington, 1967.
62. E. P. Wigner, *Interpretation of quantum mechanics*, in [58] (1976).
63. E. P. Wigner, *Review of the quantum mechanical measurement problem*, Quantum Optics, Experimental Gravity and Measurement Theory (P. Meystre and M. O. Scully, eds.), Plenum, New York, 1983, pp. 43–63.
64. W. H. Zurek, *Environment-induced superselection rules*, Physical Review D **26** (1982), 1862–1880.