Nonlocality without Inequalities for Almost All Entangled States for Two Particles

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We provide a streamlined proof of Hardy's Theorem, that almost every entangled state for a pair of quantum particles admits a "proof of nonlocality" without inequalities. Moreover, our analysis covers a larger class of observables. Thus we also strengthen Hardy's assertion that the argument fails for maximally entangled states, such as the singlet state. At the same time, we formulate the argument in such a manner that the relations which must be satisfied for local hidden variables are entirely deterministic, making no reference whatsoever to probability, let alone probablistic inequalities.

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In a recent very interesting Letter, Hardy [1] shows that almost all entangled states for a pair of spin-$\frac{1}{2}$ particles admit a "proof of nonlocality" which does not involve inequalities. While Hardy's proof is very simple, we believe that Hardy's Theorem is so nice that an even simpler argument would be worthwhile. Moreover, our proof generalizes Hardy's result in the following sense: while Hardy's analysis concerns four observables the choice of each of which is very much constrained by the quantum state, for our argument the choice of one of the observables is almost arbitrary.

I. Let $|u_i\rangle$ and $|v_i\rangle$ be a basis for particle i, $i = 1, 2$. ($|u_1\rangle$ and $|u_2\rangle$ need bear no relation to each other.) Then a state $|\Psi\rangle$ of the form

$$|\Psi\rangle = a|v_1\rangle|v_2\rangle + b|u_1\rangle|v_2\rangle + c|v_1\rangle|u_2\rangle, \quad (abc \neq 0) \quad (1)$$

i.e., a state for which only one term—here $|u_1\rangle|u_2\rangle$—of the product basis is missing, admits a nonlocality proof of the desired form.

This is rather immediate, as Hardy has shown, but we include the argument because the form (1) is more general than the one analyzed by Hardy, as well as for completeness. We also wish to rearrange Hardy's ingredients in order to facilitate comparison with Bell's inequality.

Letting $U_1 = |u_1\rangle\langle u_1|$ and so on, it follows from (1) that in the state $|\Psi\rangle$ we have: i) $U_1U_2 = 0$, ii) $U_1 = 0$ implies that $W_2 = 1$, where $W_2 = |w_2\rangle\langle w_2|$ with $|w_2\rangle = (a|v_2\rangle + c|u_2\rangle)/\sqrt{|a|^2+|c|^2}$, iii) $U_2 = 0$ implies that $W_1 = 1$, where $W_1 = |w_1\rangle\langle w_1|$ with $|w_1\rangle = (a|v_1\rangle + b|u_1\rangle)/\sqrt{|a|^2+|b|^2}$, and iv) with nonvanishing probability $W_1 = W_2 = 0$ (since $abc \neq 0$), as predictions of quantum mechanics for the measurement of these observables. But if we assume local hidden variables for these observables, it follows from i)–iii) that iv$'$) W_1 and W_2 cannot simultaneously be 0, contradicting iv).

Notice that neither iv$'$), which is analogous to Bell's inequality, nor i)–iii), involve probability, let alone a probablistic inequality.

II. Almost every spin state for the pair of spin-$\frac{1}{2}$ particles is of the form (1) for a suitable choice of bases. In fact, for *any* basis $|u_1\rangle, |v_1\rangle$ for particle 1, the general state $|\Psi\rangle$ assumes the form

$$|\Psi\rangle = |u_1\rangle|\phi_2^u\rangle + |v_1\rangle|\phi_2^v\rangle, \quad (2)$$

in which, for almost all states $|\Psi\rangle$, $|\phi_2^u\rangle \neq 0$, and $|\phi_2^v\rangle$ is neither proportional to nor orthogonal to $|\phi_2^u\rangle$, i.e., $|\phi_2^v\rangle = \alpha|\phi_2^u\rangle + |\phi_2^{u,\perp}\rangle$ ($\alpha \neq 0$) where $|\phi_2^{u,\perp}\rangle \neq 0$ and $\langle\phi_2^{u,\perp}|\phi_2^u\rangle = 0$. Thus such a $|\Psi\rangle$ is of the form (1) with $|u_2\rangle$ proportional to $|\phi_2^{u,\perp}\rangle$ and $|v_2\rangle$ proportional to $|\phi_2^u\rangle$.

III. The argument in II of course fails when $|\phi_2^v\rangle$ is proportional to $|\phi_2^u\rangle$, in which case $|\Psi\rangle$ is a product state. It also fails when $|\phi_2^v\rangle$ is orthogonal to $|\phi_2^u\rangle$, but this defect will typically be removed by a nontrivial change of basis for particle 1: $|\phi_2^u\rangle = \langle u_1|\Psi\rangle$ depends antilinearly on $|u_1\rangle$. Thus $|\phi_2^{u'}\rangle$ and $|\phi_2^{v'}\rangle$ will be related to $|\phi_2^u\rangle$ and $|\phi_2^v\rangle$ by the (complex conjugation of) the relations connecting the transformed basis $|u_1'\rangle, |v_1'\rangle$ to the original $|u_1\rangle, |v_1\rangle$ basis. Hence $|\phi_2^{u'}\rangle$ and $|\phi_2^{v'}\rangle$ will be nonorthogonal unless $\langle\phi_2^u|\phi_2^u\rangle = \langle\phi_2^v|\phi_2^v\rangle$, in which case $|\Psi\rangle$ is said to be maximally entangled.

IV. Suppose that the particles are associated with Hilbert spaces of dimension ≥ 2. Then by the Schmidt decomposition of a general state $|\Psi\rangle$ for the pair,

$$|\Psi\rangle = \sum_i c_i|e_1^i\rangle|f_2^i\rangle \quad (c_i > 0) \quad (3)$$

where the e-states form (part of) a basis for particle 1 and the f-states (part of) a basis for particle 2, we have that for any entangled state which is not maximally entangled there are—by the definition of maximally entangled—terms i and j in (3) for which $c_i \neq c_j$, i.e., to which the argument in III may be applied. It thus follows generally that any entangled state which is not maximally entangled admits a "proof of nonlocality" which does not involve inequalities.

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[1] L. Hardy, Phys. Rev. Lett. **71**, 1665 (1993)