Stationary States of the One-Dimensional Discrete-Time Facilitated Symmetric Exclusion Process[*]

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Abstract

We describe the extremal translation invariant stationary (ETIS) states of the facilitated exclusion process on $\mathbb{Z}$. In this model all particles on sites with one occupied and one empty neighbor jump at each integer time to the empty neighbor site, and if two particles attempt to jump into the same empty site we choose one randomly to succeed. The ETIS states are qualitatively different for densities $\rho < 1/2$, $\rho = 1/2$, and $1/2 < \rho < 1$, but in each density region we find states which may be grouped into families, each of which is in natural correspondence with the set of all ergodic measures on $\{0,1\}^{\mathbb{Z}}$. For $\rho < 1/2$ there is one such family, containing all the ergodic states in which the probability of two adjacent occupied sites is zero. For $\rho = 1/2$ there are two families, in which configurations translate to the left and right, respectively, with constant speed 2. For the high density case there is a continuum of families. We show that all ETIS

states at densities $\rho \leq 1/2$ belong to these families, and conjecture that also at high density there are no other ETIS states. We also study the possible ETIS states which might occur if the conjecture fails.

1   Introduction

The *facilitated symmetric simple exclusion process* (F-SSEP) is a model of particles moving on a lattice, in which a particle can jump to a neighboring (empty) site only if another of its neighboring sites is occupied (by a *facilitating* particle). In this paper we consider only the case of synchronous discrete-time dynamics on the one-dimensional lattice $\mathbb{Z}$, except that in Remark 1.1 below we discuss briefly the situation for one-dimensional continuous time dynamics. For further results on the one-dimensional case, see [1, 2, 3, 4, 5, 6, 12, 9, 10, 17, 19]; for results of simulations of the continuouis-time model in higher dimensions see [14, 16, 18].

The configuration space of the model is $X = \{0, 1\}^{\mathbb{Z}}$; if η is a configuration in X then we say that a site i with $\eta(i) = 1$ is *occupied* by a particle, and a site with $\eta(i) = 0$ is *unoccupied* or *empty*. The (stochastic) dynamics is defined as follows: if η_t is the configuration at time t, $t \in \mathbb{Z}$, then each particle in η_t with exactly one occupied neighboring site attempts to jump to its unoccupied neighboring site; the jump takes place unless two particles attempt to jump on the same site, in which case one of them is chosen at random to succeed, with each choice equally likely. η_{t+1} is the resulting configuration.

Our goal is to classify the states—probability measures on X—which are translation invariant (TI) and stationary for the F-SSEP dynamics, the *TIS states*. Every TIS state is a convex combination of the extremal TIS (ETIS) states, that is, of the TIS states which are not proper convex combinations of others, so it suffices to find the ETIS states. The ETIS states need not be extremal TI (ETI)—i.e., ergodic under translations—but since particles are neither created nor destroyed, extremality in the class of TIS states suffices to guarantee (see Lemma 2.4) that each ETIS state will be supported on the set X_ρ of configurations having particle density ρ for some ρ with $0 \leq \rho \leq 1$,

2

that is, satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \eta_i = \lim_{N\to\infty} \frac{1}{N} \sum_{i=-N}^{-1} \eta_i = \rho. \qquad (1.1)$$

We will say that a TI state *has density* ρ if it is supported on X_ρ. (Note that this condition implies that for each $i \in \mathbb{Z}$ the expected value of $\eta(i)$ is ρ, but is in fact a stronger statement.)

It is convenient to consider also a second particle system on $\mathbb{Z}$, again evolving in discrete time: the *symmetric stack model* (SSM). In this model there are no restrictions on the number of particles at any site, so that the configuration space is $\widehat{X} = \mathbb{Z}_+^{\mathbb{Z}}$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. We denote stack configurations by boldface letters, and to distinguish explicit stack configurations from F-SSEP particle configurations we will use italics for the former, so that for $\mathbf{n} \in \widehat{X}$ we might have $\mathbf{n}(0) = 2$. To specify the evolution of the SSM, let us say that the stack at site i is *short* if $\mathbf{n}(i) \le 1$ and *tall* otherwise. Then in the transition from $\mathbf{n}_t$ to $\mathbf{n}_{t+1}$ either zero or one particle moves along each bond $\langle k, k+1 \rangle$, either to the left or to the right: if the stacks at k and $k + 1$ are both short then no particle moves on the bond; if one is short and one tall then a particle moves from the tall to the short stack, and if both are tall then a particle moves from one to the other in a randomly chosen direction, with each direction equally likely.

For the SSM we again speak of TIS and ETIS states and, for $\hat{\rho} < \infty$, of states of density $\hat{\rho}$, where the latter are those supported on $\widehat{X}_{\hat{\rho}}$, the set of SSM configurations of density $\hat{\rho}$, defined in parallel with (1.1). If the expected value of $\mathbf{n}(0)$ is finite in a TI state on $\widehat{X}$ then we say that the state is *regular*. A TI state on X is called regular if it gives zero probability to the configuration η for which $\eta(i) = 1$ for all i.

The SSM is connected with the F-SSEP through a *substitution map* $\phi :$ $\widehat{X} \to X$: if $\mathbf{n} \in \widehat{X}$ then $\phi(\mathbf{n})$ is obtained by replacing each $\mathbf{n}(i)$ with a zero followed by $\mathbf{n}(i)$ ones. (Such a mapping has also been used to relate exclusion and zero range processes; see, e.g., [7, 8].) In Appendix A we show that this substitution, and members of a large class of similar substitutions, give rise to a bijection of the regular TI or ETI states of the models related by the substitution. Moreover, we show there that, for the particular substitution ϕ above, the bijection Φ_ϕ of the regular TI states is also a bijection from the regular TIS or ETIS states of the SSM to those of the F-SSEP. Thus for the question of interest here—the nature and classification of the TIS states—the

3

F-SSEP and SSM are essentially equivalent, and we can and will pass freely from one to the other. Note that if $\mathbf{n}$ has density $\hat{\rho}$ then $\phi(\mathbf{n})$ has density $\rho = \hat{\rho}/(1+\hat{\rho})$; correspondingly, Φ_ϕ carries regular SSM states of density $\hat{\rho}$ to regular F-SSEP states of density ρ.

Recall now that each regular ETIS state is associated with some density ρ in the F-SSEP or equivalently $\hat{\rho} = \rho/(1-\rho)$ in the SSM. The classification of the ETIS states of the models is qualitatively different in the three density regions $0 \le \rho \le 1/2$, $\rho = 1/2$, and $1/2 \le \rho < 1$, or equivalently $0 \le \hat{\rho} \le 1$, $\hat{\rho} = 1$, and $1 \le \hat{\rho} < \infty$. The states in the first two of these regions are of course all regular.

Consider first the low density region. The set of ETIS states of the SSM with $0 \le \hat{\rho} \le 1$ is precisely the set of ETI states supported on $\widehat{F} \subset \widehat{X}$, the set of *frozen* SSM configurations for which every stack has height zero or one (note that in fact $\widehat{F} = X$). For the F-SSEP the corresponding result is that the ETIS states are the ETI states supported on the set F of frozen F-SSEP configurations: those in which no two adjacent sites are occupied and hence no particle jumps are possible.

When $\hat{\rho} = 1$ in the SSM ($\rho = 1/2$ in the F-SSEP) there are two families of ETIS states in each model; these describe patterns moving to the left or to the right, respectively, with speed 1 in the SSM and speed 2 in the F-SSEP. In the SSM the left-moving family consists of all ETI states supported on $\widehat{X}_{\text{left}}$, the set of configurations in which no stack has height more than 2, a stack of height 2 can be followed only by one of height 0, and a stack of height 0 can be preceded only by one of height 2. Similarly, the right-moving family consists of the ETI states on $\widehat{X}_{\text{right}}$, the spatial reflection of $\widehat{X}_{\text{left}}$. Two states belong to both families: the state $\hat{\mu}^{(1)}$ (which is also one of the low-density states of the previous paragraph) supported on the single configuration in which all stacks have height *1*, and the state $\hat{\mu}^{(2)}$ supported with equal probability on the two configurations in which stacks of height *0* and *2* alternate. The left- and right-moving families in the F-SSEP are obtained from those of the SSM via the map Φ_ϕ.

All TIS states in the high density region of the SSM, $1 \le \hat{\rho}$, are supported on $\widehat{X}^* \subset \widehat{X}$, the set of configurations for which no two adjacent sites both have short stacks. On $\widehat{X}^*$ the dynamics preserves the parity of each stack height, so that if for $\sigma \in X$ we let $\widehat{X}_\sigma^* \subset \widehat{X}^*$ be the set of configurations $\mathbf{n}$ for which $\mathbf{n}(i)$ has parity $(-1)^{\sigma(i)}$ then each $\widehat{X}_\sigma^*$ is invariant for the dynamics. (In these circumstances we call σ a *parity sequence*.)

Let e be the parity sequence with $e(i) = 0$ for all i, so that $\widehat{X}_e^*$ is the set of configurations for which each stack height is even and there are no adjacent zeros. For each $\hat{\rho}_e \geq 1$ we find an ETIS state $\hat{\mu}_e^{(\hat{\rho}_e)}$ on $\widehat{X}_e^*$ of density $\hat{\rho}_e$, which we conjecture to be unique: if $\hat{\rho}_e = 1$ then $\hat{\mu}_e^{(\hat{\rho}_e)}$ is the state $\hat{\mu}^{(2)}$ described above, while if $\hat{\rho}_e > 1$ then $\hat{\mu}_e^{(\hat{\rho}_e)}$ is a Gibbs state for an interaction which is simply a one-body potential together with the constraints—hard-core and evenness—implicit in $\widehat{X}_e^*$. Further, for each such $\hat{\rho}_e$ we obtain from $\hat{\mu}_e^{(\hat{\rho}_e)}$ a family of regular ETIS states on $\widehat{X}^*$, and show that if the conjecture mentioned above holds then these are all such states. Specifically, for each $\hat{\rho}_e \geq 1$ and each ETI state λ on X there is an ETIS state $\hat{\mu}^{(\hat{\rho}_e, \lambda)}$ for the SSM; $\hat{\mu}^{(\hat{\rho}_e, \lambda)}$ has the distribution of $\eta + \sigma$ (pointwise addition), where η has distribution $\hat{\mu}_e^{(\hat{\rho}_e)}$, σ has distribution λ, and η and σ are independent. Note that if λ has density κ then $\hat{\mu}_e^{(\hat{\rho}_e, \lambda)}$ has density $\hat{\rho}_e + \kappa$. The corresponding families for the F-SSEP are obtained via the map Φ_ϕ.

Remark 1.1 A discussion of the TIS states of the continuous-time version of the model, generalized to include an asymmetry in the jumps, was given in [1]. The asymmetry is controlled by a parameter $p \in [0, 1]$: a particle at site $i \in \mathbb{Z}$ jumps to site $i + 1$ (respectively $i - 1$) with rate p (resp. $1 - p$), provided that site $i - 1$ (resp. $i + 1$) is occupied and site $i + 1$ (resp. $i - 1$) is empty. For $\rho < 1/2$ the TIS states are, as for the current model, just the TI states supported on F, but for the continuous-time model it was possible to determine the limiting state $\underline{\mu}$ when the initial state μ_0 is Bernoulli; rather surprisingly, $\underline{\mu}$ is independent of p. ($\underline{\mu}$ is also [11, 12] the limiting state, with initial state μ_0, under totally asymmetric discrete-time dynamics.) For $\rho = 1/2$, the unique TIS state is supported with equal probability on the two configurations in which occupied and empty sites alternate. For each $\rho > 1/2$ there is again a unique TIS state, the Gibbs state for a particle system in which the only interaction is an exclusion rule forbidding adjacent empty sites; the uniqueness was established via a coupling of the model with the usual asymmetric simple exclusion process.

2 Preliminary considerations

We here introduce some further notation and provide some simple results for the F-SSEP and SSM models, often speaking in terms of the F-SSEP with the understanding that parallel notation will be used, and similar results

hold, for the SSM. Let us mention several pieces of general notation: for any sets A and B, function $f : A \to B$, and measure λ on A we let $f_*\lambda$ be the measure on B with $(f_*\lambda)(C) = \lambda(f^{-1}(C))$; moreover, if $B = \mathbb{R}$ we let $\lambda(f) = \int_A f \, d\lambda$ denote the expected value of f under λ. When $C \subset B$ we let $\mathbf{1}_C : B \to \{0,1\}$ denote the indicator function of the set C. If S is a finite set then $|S|$ denotes the size of S.

Recall from Section 1 that the configuration spaces for these models are $X := \{0,1\}^{\mathbb{Z}}$ and $\widehat{X} := \mathbb{Z}_+^{\mathbb{Z}}$, respectively, with $\eta \in X$ and $\mathbf{n} \in \widehat{X}$ denoting configurations. For $\eta \in X$ and $j, k \in \mathbb{Z}$ with $j \leq k$ we let $\eta(j{:}k) = (\eta(i))_{j \leq i \leq k}$ denote the portion of the configuration η lying between sites j and k (inclusive). We will occasionally use string notation for configurations or partial configurations, writing for example $\eta(0{:}4) = \eta(0)\cdots\eta(4) = 01101 = 01^201$. $\tau : X \to X$ (or $\hat{\tau} : \widehat{X} \to \widehat{X}$) denotes the translation operator: if $\eta \in X$ then $(\tau\eta)(i) = \eta(i-1)$, if f is any function on X then $\tau f(\eta) = f(\tau^{-1}\eta)$, and if μ is a (Borel) measure on X then τ acts on μ via τ_*.

It will sometimes be convenient to associate to each F-SSEP configuration $\eta \in X$ a height profile $h_\eta : \mathbb{Z} \to \mathbb{Z}$, which, in the usual convention, rises by one unit when $\eta(i) = 0$ and sinks by one unit when $\eta(i) = 1$. Specifically,

$$h_\eta(k) = \begin{cases} 0, & \text{if } k = 0, \\ \sum_{i=1}^{k}(-1)^{\eta(i)}, & \text{if } k > 0, \\ -\sum_{i=k+1}^{0}(-1)^{\eta(i)}, & \text{if } k < 0. \end{cases}$$

We do not introduce height profiles for SSM configurations.

From the somewhat informal description of the dynamics of the models given in Section 1 it is straightforward but tedious to specify, for a configuration $\eta \in X$ and a measurable subset $A \subset X$, the transition kernel $Q(\eta, A)$ of the F-SSEP Markov process, or similarly the kernel $\widehat{Q}(\mathbf{n}, B)$ for the SSM model. We omit the details. A measure μ on X is stationary if $\mu = \mu Q$; here $(\mu Q)(A) = \int_X Q(\cdot, A) \, d\mu$ (we also write $\mu Q^n := (\mu Q^{n-1})Q$ for $n \geq 2$).

In the remainder of the paper we will consider primarily *regular* (see Section 1) states on X and $\widehat{X}$; the sets of regular TI, ETI, TIS, and ETIS states for the F-SSEP are denoted by $\mathcal{M}(X)$, $\overline{\mathcal{M}}(X)$, $\mathcal{M}_s(X)$, and $\overline{\mathcal{M}}_s(X)$, respectively. We write similarly $\mathcal{M}(\widehat{X})$, etc., as well as $\mathcal{M}(A)$, $\mathcal{M}(\widehat{A})$, etc., for $A \subset X$ or $\widehat{A} \subset \widehat{X}$ TI sets. As a consequence of the results of Appendix A (see also the discussion of Section 1) we have immediately:

Theorem 2.1 *There exists a bijection $\Phi_\phi : \mathcal{M}(\widehat{X}) \to \mathcal{M}(X)$, arising from the substitution map ϕ defined in Section 1, which satisfies $\Phi_\phi(\overline{\mathcal{M}}(\widehat{X})) = \overline{\mathcal{M}}(X)$, $\Phi_\phi(\mathcal{M}_s(\widehat{X})) = \mathcal{M}_s(X)$, and $\Phi_\phi(\overline{\mathcal{M}}_s(\widehat{X})) = \overline{\mathcal{M}}_s(X)$. Φ_ϕ carries states of density $\hat\rho$ to states of density $\hat\rho/(1+\hat\rho)$. Moreover, if $\widehat{A} \subset \widehat{X}$ is TI then there is a similarly defined bijection from $\mathcal{M}(\widehat{A})$ to $\mathcal{M}(A)$, etc., where A is the minimal TI subset of X containing $\phi(\widehat{A})$.*

Remark 2.2 It is clear that one may also define, in a straightforward way, models with a fixed number of particles moving on a finite ring under either the F-SSEP or SSM dynamics. These models will play a role in Sections 4 and 5.2.

Since we are studying stationary states it is natural to introduce the set of space-time F-SSEP configurations $X_2 = \{0,1\}^{\mathbb{Z}^2} = \{(\xi_t(i))_{(t,i)\in\mathbb{Z}^2}\}$; a state $\mu \in \mathcal{M}(X)$ which is stationary for the dynamics induces a "path measure" on X_2, invariant under vertical translation, which we denote P_μ. $\widehat{X}_2$ and $\widehat{P}_{\hat\mu}$ denote the corresponding SSM quantities.

In Sections 3–5 we will describe all ETIS states for the two models. As indicated in Section 1, these fall into certain natural groups, which we will call λ-families. In this context we write $\mathcal{L}$ for the set of ergodic TI measures on X (in fact, $\mathcal{L} = \overline{\mathcal{M}}(X)$, but the special role that this space plays here motivates a special symbol).

Definition 2.3 A λ-*family* is a collection of ETIS states, for either the SSM or the F-SSEP, which is bijectively equivalent (with a "natural" bijection) to $\mathcal{L}$. We think of $\mathcal{L}$ as indexing the λ-family and let $\lambda \in \mathcal{L}$ denote a typical index. We will typically write $\widehat{\mathcal{F}}_*$ and $\mathcal{F}_*$ for λ-families for the SSM and F-SSEP, respectively, with $*$ a subscript distinguishing the various families and with $\mathcal{F}_* = \Phi_\phi(\widehat{\mathcal{F}}_*)$. $\widehat{\Psi}_* : \mathcal{L} \to \widehat{\mathcal{F}}_*$ and $\Psi_* = \Phi_\phi \circ \widehat{\Psi}_* : \mathcal{L} \to \mathcal{F}_*$ are the corresponding indexing bijections.

Certain simple spatially-periodic configurations and related states, some already mentioned in Section 1, will play a special role in our discussions. Let $\mathbf{n}^{(1)}, \mathbf{n}^{(2)} \in \widehat{X}$ be the configurations with $\mathbf{n}^{(1)}(i) = 1$ and $\mathbf{n}^{(2)}(i) = (1+(-1)^i)$ for all i; in Section 1 we introduced the states $\hat\mu^{(1)}, \hat\mu^{(2)} \in \mathcal{M}(\widehat{X})$ defined by $\hat\mu^{(1)} = \delta_{\mathbf{n}^{(1)}}$, $\hat\mu^{(2)} = (\delta_{\mathbf{n}^{(2)}}+\delta_{\tau\mathbf{n}^{(2)}})/2$. The corresponding states $\mu^{(1)} = \Phi_\phi(\mu^{(1)})$,

$\mu^{(2)} = \Phi_\phi(\mu^{(2)})$ are given by

$$\mu^{(1)} = \frac{1}{2}\sum_{j=0}^{1}\delta_{\tau^j\eta^{(1)}} \qquad \text{and} \qquad \mu^{(2)} = \frac{1}{4}\sum_{j=0}^{3}\delta_{\tau^j\eta^{(2)}}, \qquad (2.1)$$

where $\eta^{(1)} \in X$ is the period-two configuration with $\eta^{(1)}(1{:}2) = 10$ and $\eta^{(2)} \in X$ is the period-four configuration with $\eta^{(2)}(1{:}4) = 1100$. It is easy to check directly that $\hat{\mu}^{(1)}$ and $\hat{\mu}^{(2)}$ are ETIS states for the SSM, as are $\mu^{(1)}$ and $\mu^{(2)}$ for the F-SSEP.

We conclude this section with three general results; we state these for the F-SSEP, but the obvious translations to the SSM also hold. Recall from Section 1 that we say that a state $\mu \in \mathcal{M}(X)$ has density ρ if it is supported on the space X_ρ (see (1.1)).

Lemma 2.4 *Every ETIS state $\mu \in \overline{\mathcal{M}}_s(X)$ has a definite density ρ and satisfies either $\mu(F) = 0$ or $\mu(F) = 1$.*

Proof: Take $\mu \in \mathcal{M}_s(X)$; it suffices to show that μ is a convex combination of states in $\mathcal{M}_s(X)$ with a definite density ρ and for which F has probability 0 or 1. Let $\nu = r_*\mu$; here $r : X \to \mathbb{R}$, $r(\eta) := \lim_{N\to\infty}(2N+1)^{-1}\sum_{i=-N}^{N}\eta(i)$, is defined μ-a.e. by the ergodic theorem. ν is just the distribution of the density with respect to μ. Then [15] there exists a unique regular conditional probability distribution $(\mu_\rho)_{\rho\in[0,1]}$ for μ such that μ_ρ has density ρ and for any measurable $A \subset X$,

$$\mu(A) = \int_{0\leq\rho\leq 1}\mu_\rho(A)d\nu(\rho). \qquad (2.2)$$

Since F is invariant under the dynamics, i.e., $Q(\eta, F) = 1$ for $\eta \in F$, we see that if we further write $\mu_\rho = \mu_\rho\big|_F + \mu_\rho\big|_{X\setminus F}$ we obtain, after normalization of $\mu_\rho\big|_F$ and $\mu_\rho\big|_{X\setminus F}$, the desired representation. More details are given in [1]. ∎

We next give a lemma which shows that a connected portion of an F-SSEP configuration, other than possibly a single 1 at some site, cannot be frozen unless the entire configuration is.

Lemma 2.5 *Let $\mu \in \mathcal{M}_s(X)$ satisfy $\mu(F) = 0$. Suppose that $I \subset \mathbb{Z}$ is an interval, that $\theta \in \{0,1\}^I$, and that either $|I| \geq 2$ or $I = \{i\}$ and $\theta(i) = 0$. Then $P_\mu(\{\xi \in X_2 \mid \text{for all } t, \xi_t\big|_I = \theta\}) = 0$.*

We remark that the possibility $I = \mathbb{Z}$, $\theta(i) = 1$ for all i, an apparent counterexample to Lemma 2.5, is forbidden by the regularity of μ.

Proof of Lemma 2.5: Let $A, B, C \subset X_2$ denote the sets of space-time histories ξ such that, respectively, $\xi_t(0) = 0$ for all t (we write $\xi_t(0) \equiv 0$), $\xi_t(-1{:}0) \equiv 01$, and $\xi_t(-1{:}0) \equiv 11$, and such that in each case $\xi_t(1)$ changes infinitely often as $t \nearrow \infty$. We show that $P_\mu(A) = P_\mu(B) = P_\mu(C) = 0$; using the translation invariance of μ, translation invariance in time of P_μ, and reflection invariance of the system (although the latter is not really needed), one sees easily that this implies the result.

First, we observe that if $\xi \in X_2$ satisfies $\xi_t(0) \equiv 0$ and s is such that $\xi_s(1) = 1$ then, P_μ-a.s., $\xi_t(1) = 1$ for all $t \geq s$ and so $\xi \notin A$. Thus $P_\mu(A) = 0$. Next, suppose that $\xi_t(-1{:}0) \equiv 01$; if $\xi_s(1) = 1$ for some s then, P_μ-a.s., $\xi_{s+1}(-1) = 1$, a contradiction. Thus $P_\mu(\xi \in B) = 0$. Finally, if $\xi \in C$ we define $(\tau_k)_{k \in \mathbb{Z}_+}$, with $0 \leq \tau_0 < \tau_1 < \cdots$, to be the nonnegative times satisfying $\xi_{\tau_k}(1) = 0$; the τ_k are well defined on C. Now $\xi \in C$ implies that $\xi_{\tau_k+1}(0) = 1$, and this is possible only if $\xi_{\tau_k}(-1{:}3) = 11011$ and, moreover, L_k occurs, where L_k is the event that when two particles attempt at time τ_k to jump to site 1, it is the leftward jump which succeeds. Given C, the L_k are independent events, each with probability $1/2$, so that $P_\mu(C) = 0$. ∎

Finally, we give a result showing that non-frozen TIS states cannot have too large a local density of 0's.

Lemma 2.6 *Suppose that $\mu \in \mathcal{M}_s(X)$ satisfies $\mu(F) = 0$. Then:*

(a) μ-a.s., no configuration contains three consecutive 0's.

(b) μ-a.s., the height profile h_η of the configuration η does not increase by more than two units over any interval.

Proof: (a) Elementary analysis of the dynamics shows that, P_μ-a.s., if $\xi \in X_2$ satisfies $\xi_t(i) = \xi_t(i+1) = \xi_t(i+2) = 0$ for some $t, i \in \mathbb{Z}$, then also $\xi_{t-1}(i) = \xi_{t-1}(i+1) = \xi_{t-1}(i+2) = 0$. But then $\xi_s(i) = \xi_s(i+1) = \xi_s(i+2) = 0$ for all $s \leq t$ and, by the invariance of P_μ under time translations, for all s. The conclusion follows from Lemma 2.5.

(b) Suppose that the conclusion is false. By translation invariance we may assume that for some (necessarily odd) positive integer j, $\mu(A_j) > 0$, where $A_j = \{\eta \mid h_\eta(j) = 3\}$. We take j to be the minimal value for which this holds. By (a), $j > 3$, and minimality of j implies that if $\eta \in A_j$ then a.s. $\eta(1{:}j) = 00(10)^{(j-3)/2}0$. Now by Lemma 2.5 (as in (a)), $P_\mu(B) > 0$,

where $B = \{\xi \in X_2 \mid \xi_0 \in A_j, \xi_{-1} \notin A_j\}$. On the other hand, if $\xi \in B$ then elementary analysis shows that, P_μ-a.s., some translate of ξ_{-1} belongs to $A_{j'}$ for some $j' < j$ (for example, if $j = 9$ then $\xi_{-1}(1{:}j)$ must be one of $1100(10)^2 0$, $00(10)^2 011$, 110010011, or 000111000). This contradicts the minimality of j. ∎

3 Low density

In this section we describe all TIS states of low density ($\rho < 1/2$ for the F-SSEP, $\hat\rho < 1$ for the SSM). We first show that all ETIS states for the F-SSEP are frozen.

Lemma 3.1 *Let $\mu \in \overline{\mathcal{M}}_s(X)$ have density $\rho < 1/2$. Then $\mu(F) = 1$.*

Proof: By Lemma 2.4 it suffices to show that $\mu(F) > 0$. But if $\mu(F) = 0$ then for any $k > 0$,

$$2\rho - 1 = \frac{1}{k}\mu\left(\sum_{i=1}^{k}(2\eta(i) - 1)\right) = -\frac{1}{k}\mu\big(h(k)\big) \geq -\frac{2}{k}, \qquad (3.1)$$

by Lemma 2.6(b). Since k is arbitrary, $2\rho - 1 \geq 0$. ∎

To describe all low-density states we introduce $\mathcal{F}_{\mathrm{low}} := \overline{\mathcal{M}}(F)$, the set of ETI states supported on F, and $\widehat{\mathcal{F}}_{\mathrm{low}} := \overline{\mathcal{M}}(\widehat{F})$. Since $\widehat{F} = X$, $\widehat{\mathcal{F}}_{\mathrm{low}} = \overline{\mathcal{M}}(X) = \mathcal{L}$, and thus $\widehat{\mathcal{F}}_{\mathrm{low}}$ is indeed a λ-family (see Definition 2.3), with indexing map $\widehat{\Psi}_{\mathrm{low}}$ the identity. One checks easily that $\Phi_\phi(\widehat{\mathcal{F}}_{\mathrm{low}}) = \mathcal{F}_{\mathrm{low}}$, so that $\mathcal{F}_{\mathrm{low}}$ is also a λ-family, with indexing map $\Psi_{\mathrm{low}} = \Phi_\phi$. Note that $\hat\mu^{(1)} \in \widehat{\mathcal{F}}_{\mathrm{low}}$ has density 1 and $\mu^{(1)} = \Phi_\phi(\hat\mu^{(1)}) \in \mathcal{F}_{\mathrm{low}}$ has density 1/2, but that otherwise the states in $\widehat{\mathcal{F}}_{\mathrm{low}}$ and $\mathcal{F}_{\mathrm{low}}$ have densities less than 1 or less than 1/2, respectively.

Theorem 3.2 *(a) The set of ETIS states for the F-SSEP with density $\rho < 1/2$ is $\mathcal{F}_{\mathrm{low}} \setminus \{\mu^{(1)}\}$, and (b) the set of ETIS states for the SSM with density $\hat\rho < 1$ is $\widehat{\mathcal{F}}_{\mathrm{low}} \setminus \{\hat\mu^{(1)}\}$.*

Proof: For (a), note that the inclusion $\mathcal{F}_{\mathrm{low}} \subset \overline{\mathcal{M}}_s(X)$ is trivial;, while conversely every state in $\overline{\mathcal{M}}_s(X)$ with density less than 1/2 belongs to $\mathcal{F}_{\mathrm{low}}$ by Lemma 3.1. By virtue of Theorem 2.1, (b) is an immediate consequence of (a). ∎

Finally we ask the question: if the F-SSEP is started in an initial state μ_0 which is a Bernoulli measure with density $\rho < 1/2$, what is the final distribution of frozen configurations? As discussed in Remark 1.1, in earlier work we answered this question for several other facilitated exclusion processes, finding a common limiting distribution $\underline{\mu}$. For the current model the limit is different (see below), but beyond that we have only a partial description.

Let $\mu_t = \mu_{t-1}Q$, $t = 1, 2, \ldots$, be the state at time t, with μ_0 as above, and for $\eta \in X$ let $S_\eta = \{i \in \mathbb{Z} \mid \eta(i-2{:}i) = 000\}$.

Theorem 3.3 *The limiting measure $\mu_\infty = \lim_{t \to \infty} \mu_t$ exists and satisfies $\mu_\infty(\{0 \in S_\eta\}) > 0$. Under the conditional measure $\mu_\infty(\cdot \mid 0 \in S_\eta)$, S is a renewal process.*

Proof: Let $\widetilde{P}$ be the path measure on $\widetilde{X}_2 = \{0,1\}^{\mathbb{Z}_+ \times \mathbb{Z}}$ obtained from the initial state μ_0 and the F-SSEP dynamics. Elementary analysis shows that if $\xi \in X_2$ satisfies $\xi_t(i-2{:}i) = 000$ then $\widetilde{P}$-a.s. also $\xi_s(i-2{:}i) = 000$ for $0 \le s \le t$. But then for all $i \in \mathbb{Z}$,

$$\begin{aligned}
B_i &:= \{\xi \mid \xi_t(i-2{:}i) = 000 \text{ for all } t \ge 0\} \\
&= \{\xi \mid \xi_t(i-2{:}i) = 000 \text{ for all sufficiently large } t\}. \quad (3.2)
\end{aligned}$$

Moreover, $\widetilde{P}(B_i) = q^2(1-\rho)^3$, with q the probability that, for an initial measure under which the configuration on $\mathbb{N} := \{i \in \mathbb{Z} \mid i > 0\}$ is distributed as a Bernoulli measure with density ρ but all other sites are empty, no particle crosses the $\langle 0, 1 \rangle$ bond at any time during the evolution.

Next we show that $q > 0$. Let A_t be the event that for all $L \ge 1$ there are in the configuration ξ_t at most $L/2$ particles on sites $1, 2, \ldots, L$. A standard gambler's ruin computation [13] shows that $\mu_0(A_0) = (1-2\rho)/(1-\rho)$. Now when A_t holds no particle can cross the bond $\langle 0, 1 \rangle$ at the next time step, from the $L = 1$ condition. Moreover, $A_t \subset A_{t+1}$. Thus $q \ge \mu_0(A_0) > 0$.

To establish the existence of μ_∞, and in fact a stronger result, the $\widetilde{P}$-almost sure existence of $\eta_\infty := \lim_{t \to \infty} \xi_t$, one shows from $\widetilde{P}(B_i) > 0$ that for any $L > 0$, $\widetilde{P}\left(\bigcup_{k,l \ge L} B_{-k} \cap B_l\right) = 1$. For $\xi \in B_{-k} \cap B_l$, simple considerations of the system in a finite region then imply that $\lim_{t \to \infty} \xi_t\big|_{[-k+1, l-3]}$ exists, $\widetilde{P}$-a.s. For more details see the proof of Lemma 3.6 of [1].

Since μ_∞ is the distribution of η_∞ under $\widetilde{P}$ we must show that S_{η_∞} is a renewal process under the conditional measure $\widetilde{P}(\cdot \mid 0 \in S_{\eta_\infty})$. Now from (3.2) we have that $i \in S_{\eta_\infty}$ iff $\xi \in B_i$, and if we condition on $0 \in S_{\eta_\infty}$, that

11

is, on B_0, then what happens to the left of site -2 is independent of what happens to the right of site 0. Thus if, under this conditioning, we label the points of S_{η_∞} sequentially as $(s_k)_{k\in\mathbb{Z}}$, with $s_0 = 0$, so that $\xi_t(s_k - 2\!:\!s_k) = 000$ for all t and k, the differences $s_k - s_{k-1}$ are independent. ∎

Let us condition on $0 \in S_{\eta_\infty}$ and adopt the notation of the previous proof. Then either $s_1 = 1$, an event with (conditional) probability $1 - \rho$, or $s_1 \geq 4$; in the latter case $\eta_\infty(-2\!:\!s_1) = 000\,\sigma\,000$, with σ any string which begins and ends with 1 and contains no substrings 11 or 000. To complete the description of μ_∞ one would need to find the distribution of these σ (and hence of $s_1 - s_0$); we have only partial results in this direction. However, one finds easily that, for example, $\mu_\infty(\eta(1\!:\!6) = 101000 \mid 0 \in S_\eta) = \rho^2(1-\rho)^4$; on the other hand, with $\underline{\mu}$ as introduced in Remark 1.1, $\underline{\mu}(\eta(1\!:\!6) = 101000 \mid 0 \in S_\eta) = 2\rho^2(1-\rho)^4$ [1, 11, 12], establishing the difference of μ_∞ and $\underline{\mu}$.

4 Densities $\rho = 1/2$ and $\hat{\rho} = 1$

In this section we describe all ETIS states of density $1/2$ for the F-SSEP or density 1 for the SSM. We first show that, for such a state in the F-SSEP, the height profile h_η (see (2)) is a.s. confined to a strip of height at most two. For $\eta \in X$ we let $\Delta(\eta) = \sup_{i<j} |h_\eta(j) - h_\eta(i)|$.

Lemma 4.1 *If $\mu \in \overline{\mathcal{M}}_s(X)$ has density $\rho = 1/2$ then, μ-a.s., $\Delta(\eta) \leq 2$.*

Proof: By Lemma 2.4 we may assume that either $\mu(F) = 0$ or $\mu(F) = 1$. Now $\mu^{(1)}$ is the only TI state of density $1/2$ supported on F, and it satisfies the conclusion of the lemma. Consider then the case $\mu(F) = 0$. By Lemma 2.6(b), $h_\eta(j) - h_\eta(i) \leq 2$ for $i < j$, so by translation invariance it suffices to show that for any $j > 0$, $h_\eta(j) \geq -2$ μ-a.s.; moreover, it suffices to verify the result for each ergodic component of μ.

Suppose then that μ is ergodic but that for some $j > 0$, which we take to be minimal, $\mu(\{\eta \mid h_\eta(j) \leq -3\}) > 0$. Then from Lemma 2.6(b) and the minimality of j, also $\mu(\{\eta \mid \eta(0) > \sup_{i>0} h_\eta(i)\}) > 0$, and with ergodicity this implies that, μ-a.s., h_η has a negative overall slope. But this is inconsistent with μ having density $1/2$. ∎

We can now classify the states in $\overline{\mathcal{M}}_s(X)$ with density $1/2$. Consider a general η with $\Delta(\eta) \leq 2$; a portion of a typical height profile (with $\Delta(\eta) = 2$) is shown in Figure 1. As indicated there, we may partition η into *left-moving*

12

(L), *right-moving (R)*, and *transition (T)* regions, where T denotes a maximal region in which pairs of 0's alternate with pairs of 1's, L a region between two T regions having the form $(10)^k$ for some $k \geq 1$, and R a region between two T regions having the form $(01)^k$, $k \geq 1$.

Figure 1: Portion of a typical height profile, showing Left-moving, Right-moving, and Transition regions.

The naming of these regions reflects the fact that a configuration in which only L and T (respectively R and T) regions appear translates to the left (respectively right) at velocity 2 under the dynamics: if η_t is such a configuration, then $\eta_{t+1} = \tau^{-2}\eta_t$ (respectively $\eta_{t+1} = \tau^2 \eta$). We will see below that all TIS states are supported on such configurations. The two simplest examples of such states, $\mu^{(1)}$ and $\mu^{(2)}$, were defined in (2.1); $\mu^{(1)}$ may be viewed as supported on configurations with a single L, or equivalently a single R, region, and $\mu^{(2)}$ is supported on configurations consisting of a single T region.

It is straightforward to work out further rules for the evolution of the configurations with $\Delta(\eta) \leq 2$. Such configurations cannot contain the pattern 11011, and hence the evolution is deterministic. Boundaries between regions usually move with velocity 2, L|T and T|L boundaries to the left and R|T and T|R boundaries to the right. However, if R and L regions are separated by a T region which is a single pair of 0's, R|00|L, then both boundaries are stationary. If in the resulting T|R|00|L|T situation the L region is shorter than the R region then the (left-moving) L|T boundary will eventually reach the 00|L boundary, at which time the L region will disappear and the 00 and T regions amalgamate into a single T region; the situation is similar when the R region is shorter, or the two regions are the same length.

Following these ideas one easily sees that, on a ring, any initial configuration evolves to one in which either no L region, or no R region, occurs. No such conclusion is possible when the ring is replaced by $\mathbb{Z}$, but similar considerations do imply the following: For any configuration η with $\Delta(\eta) \leq 2$, the minimum size $\delta = \delta(\eta)$ of a maximal uniform block—a maximal right

13

block, a region of the form $T|R|T|\cdots|T|R|T$ preceded and followed by L regions, or a maximal left block, defined in a parallel way—must increase, if $\delta < \infty$, within a time proportional to δ. From this it follows that any TI stationary state for $\rho = 1/2$ must be supported on configurations with $\delta = \infty$, i.e., on configurations consisting of a single maximal uniform block or of a maximal right block adjacent to a maximal left block. However, the set of configurations of the latter sort must have probability 0 in any TI state.

Rather than filling in the straightforward details of the argument just sketched, we now give a different and shorter (though perhaps less intuitive) argument. Let X_{left} (respectively X_{right}) denote the set of configurations $\eta \in X$ with $\Delta(\eta) = 2$ which contain no R (respectively no L) region. By convention we suppose that X_{left} and X_{right} contain $\eta^{(1)}$ and $\tau\eta^{(1)}$ (see Section 2); then $X_{\text{left}} \cap X_{\text{right}}$ consists of $\eta^{(1)}$, $\eta^{(2)}$, and their translates. Recall also the spaces X_ρ, $\widehat{X}_{\hat\rho}$, $\widehat{X}_{\text{left}}$ and $\widehat{X}_{\text{right}}$, defined in Section 1.

Theorem 4.2 *(a) When $\rho = 1/2$, $\overline{\mathcal{M}}_s(X_\rho) = \overline{\mathcal{M}}(X_{\text{left}}) \cup \overline{\mathcal{M}}(X_{\text{right}})$.*
(b) When $\hat\rho = 1$, $\overline{\mathcal{M}}_s(\widehat{X}_{\hat\rho}) = \overline{\mathcal{M}}(\widehat{X}_{\text{left}}) \cup \overline{\mathcal{M}}(\widehat{X}_{\text{right}})$.

Before giving the proof we explain how the ETIS states of the theorem are organized into the λ-families of Definition 2.3. Consider first $\overline{\mathcal{M}}_s(\widehat{X}) = \overline{\mathcal{M}}(\widehat{X}_{\text{left}}) \cup \overline{\mathcal{M}}(\widehat{X}_{\text{right}})$; here $\overline{\mathcal{M}}(\widehat{X}_{\text{left}})$ and $\overline{\mathcal{M}}(\widehat{X}_{\text{right}})$ are λ-families which we denote respectively $\widehat{\mathcal{F}}_{\text{left}}$ and $\widehat{\mathcal{F}}_{\text{right}}$. The substitution map $\phi_{\text{left}} : X \to \widehat{X}$ given by $1 \to 20$, $0 \to 1$ gives rise to an indexing bijection $\widehat{\Psi}_{\text{left}} = \Phi_{\phi_{\text{left}}} : \mathcal{L} \to \widehat{\mathcal{F}}_{\text{left}}$, and similarly we obtain $\widehat{\Psi}_{\text{right}} : \mathcal{L} \to \widehat{\mathcal{F}}_{\text{right}}$ from ϕ_{right}, the substitution map sending $1 \to 02$, $0 \to 1$. The indexing maps for $\mathcal{F}_{\text{left}} = \overline{\mathcal{M}}(X_{\text{left}})$ and $\mathcal{F}_{\text{right}} = \overline{\mathcal{M}}(X_{\text{right}})$ are $\Psi_{\text{left}} = \Phi_\phi \circ \widehat{\Psi}_{\text{left}}$ and $\Psi_{\text{right}} = \Phi_\phi \circ \widehat{\Psi}_{\text{right}}$, or can be obtained directly from the substitution maps $1 \to 1100$, $0 \to 10$ and $1 \to 0011$, $0 \to 01$, respectively. Note that $\widehat{\mathcal{F}}_{\text{left}} \cap \widehat{\mathcal{F}}_{\text{right}} = \{\hat\mu^{(1)}, \hat\mu^{(2)}\}$ and $\mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}} = \{\mu^{(1)}, \mu^{(2)}\}$.

Proof of Theorem 4.2: (a) Since the F-SSEP dynamics on X_{left} or X_{right} is simply left or right translation, respectively, clearly $\overline{\mathcal{M}}(X_{\text{left}}) \cup \overline{\mathcal{M}}(X_{\text{right}}) \subset \overline{\mathcal{M}}_s(X)$. Suppose conversely that $\mu \in \overline{\mathcal{M}}_s(X)$. Since μ is an ETIS state and X_{left} and X_{right} are invariant under the dynamics, it suffices to prove that $\mu(X_{\text{left}} \cup X_{\text{right}}) = 1$, i.e., that the set of configurations η which have $\Delta(\eta) = 2$, and contain both L and T regions, has measure zero.

We first show that, μ-a.s., no configuration contains the string 010010 (corresponding to regions R|T|L with T having just two sites). For let E be

14

the event that sites 1 and 2 both belong to an R region and that $\eta(1{:}2) = 01$. Then from the dynamical rules,

$$P_\mu(\xi_{t+1} \in E) = P_\mu(\xi_t \in E) - P_\mu(\xi_t(-1{:}2) = 1101) + P_\mu(\xi_t(-1{:}4) = 010011)$$
$$= P_\mu(\xi_t \in E) - P_\mu(\xi_t(-1{:}2) = 1101) + P_\mu(\xi_t(-1{:}2) = 0100)$$
$$- P_\mu(\xi_t(-1{:}4) = 010010). \tag{4.1}$$

But $\mu\big(\eta(-1{:}2) = 1101\big) = \mu\big(\eta(-1{:}2) = 0100\big)$, since the density of left and right ends of R regions must be the same, so that, with the stationarity of μ, (4.1) implies that $\mu\big(\eta(-1{:}4) = 010010\big) = 0$, as claimed.

Now if $\mu(X_{\text{left}} \cup X_{\text{right}}) < 1$ then there is a minimal k such that $\mu\big(\eta(1{:}4k + 6) = 0100(1100)^k 10\big) > 0$. From the claim above, $k > 0$. But if $\xi_t(1{:}4k+6) = 0100(1100)^k 10$ then $\xi_{t+1}(3{:}4k + 4) = 0100(1100)^{k-1}10)$, contradicting the minimality of k.

(b) By (a) and Theorem 2.1 (using $\widehat{A} = \widehat{X}_{\text{left}}$ and $\widehat{A} = \widehat{X}_{\text{right}}$) we have

$$\overline{\mathcal{M}}_s(\widehat{X}_1) = \Phi_\phi^{-1}\big(\overline{\mathcal{M}}_s(X_{1/2})\big) = \Phi_\phi^{-1}\big(\overline{\mathcal{M}}(X_{\text{left}})\big) \cup \Phi_\phi^{-1}\big(\overline{\mathcal{M}}(X_{\text{right}})\big)$$
$$= \overline{\mathcal{M}}(\widehat{X}_{\text{left}}) \cup \overline{\mathcal{M}}(\widehat{X}_{\text{right}}). \tag{4.2}$$

∎

5   High density

Most of our discussion of the high density region, $\rho > 1/2$ for the F-SSEP or $\hat\rho > 1$ for the SSM, will be carried our for the SSM. We first obtain a reduction of the configuration space of the model to the space $\widehat{X}^* \subset \widehat{X}$, the set of configurations for which no two adjacent sites both have short stacks (see Section 1). Note that any extremal TI state in $\widehat{X}_e^*$ must have density $\hat\rho_e$ satisfying $\hat\rho_e \geq 1$. For future reference we also note that $H := \phi(\widehat{X}^*)$ is the set of F-SSEP configurations which do not contain any of the substrings 000, 0100, 0010, and 01010.

Theorem 5.1 *Every TIS state for the stack dynamics with density $\hat\rho > 1$ is supported on $\widehat{X}^*$; equivalently, every TIS state for the F-SSEP dynamics with density $\rho > 1/2$ is supported on H.*

Proof: It is convenient to work primarily in the stack model. We observe first that, since *00* in the stack model corresponds to 000 in the F-SSEP model, Lemma 2.6 implies that every $\hat{\mu} \in \mathcal{M}_s(\widehat{X})$ with density $\hat{\rho} > 1$ assigns zero probability to the set of all configurations containing the substring *00*.

Now we suppose that $\hat{\mu} \in \mathcal{M}_s(\widehat{X})$ has density $\hat{\rho} > 1$ and is such that configurations containing the string *10* occur with nonzero probability under $\hat{\mu}$, and derive a contradiction; we may assume that $\hat{\mu}$ is ergodic. For $(t, i) \in \mathbb{Z}^2$ let $E_{t,i} \subset \widehat{X}_2$ be the event that $\mathbf{m} \in \widehat{X}_2$ satisfies $\mathbf{m}_t(i\!:\!i+1)) = 10$; then for $\mathbf{m} \in E_{t,i}$ we know from Lemma 2.5 that there must a.s. exist times t_0 and t_1, with $t_0 \leq t < t_1$, such that $\mathbf{m} \in E_{s,i}$ for $t_0 \leq s < t_1$ but $\mathbf{m} \notin E_{t_0-1,i}$ and $\mathbf{m} \notin E_{t_1,i}$. Then elementary consideration of the dynamics, using the fact the *00* does not occur, shows that necessarily $\mathbf{m}_{t_0-1}(i-1\!:\!i) = 10$, i.e., the *10* string cannot be "created" at sites $(i, i+1)$ but rather "moves" there from sites $(i-1, i)$. But then such a string cannot vanish, since $\hat{\mu}$ is stationary and hence the density of *10* strings is constant in time, and again simple considerations show that necessarily $m_{t_1-1}(i\!:\!i+2)) = 102$ and $m_{t_1}(i+1\!:\!i+2) = 10$. The conclusion is that, $\widehat{P}_{\hat{\mu}}$-a.s., *10* substrings persist throughout time, moving to the right in the sense that there exist times $t_0 < t_1 < t_2 < \cdots$ such that $m_t(i+k\!:\!i+k+1) = 10$ for $t_k \leq t < t_{k+1}$.

A similar analysis applies to *01* strings, except that these move to the left. But we claim that $\hat{\mu}$ cannot give positive density to both *10* and *01* strings. For if it did, both would occur with positive frequency in the same configuration, by the ergodicity of $\hat{\mu}$, and would then "collide", that is, the configuration *101* would occur with positive frequency (*1001* is ruled out because *00* is), and then neither could continue to move without the destruction of the other, a contradiction. Thus we may assume, without loss of generality, that $\hat{\mu}$ supports a positive density of *10* substrings but that, $\hat{\mu}$-a.s., *01* strings do not occur.

Now let k be the minimal positive integer such that $\hat{\mu}(A_k) > 0$, where $A_k := \{\mathbf{n} \in \widehat{X} \mid \mathbf{n}(1\!:\!2) = 10 \text{ and } \sum_{i=1}^{k} \mathbf{n}(i) > k\}$; such a k exists because $\hat{\rho} > 1$ and we have assumed that $\hat{\mu}(\mathbf{n}(1\!:\!2)) = 10) > 0$. From the fact that *00* and *01* do not occur it follows that if $\mathbf{n}_t \in A_k$ then $\mathbf{n}_t(1\!:\!k)$ has one of two forms: (i) $\mathbf{n}_t(1\!:\!k) = 102(02)^p 1^q x$, with $x \geq 2$, or (ii) $\mathbf{n}_t(1\!:\!k) = 10(20)^p y$, with $y \geq 3$; here p and q are nonnegative integers. We show that each of these cases leads to a contradiction.

If (i) holds then one checks, by considering various special cases, that with probability at least 1/2 either (i.a) neither $\mathbf{n}_{t+1}(1\!:\!2) = 10$ nor $\mathbf{n}_{t+1}(2\!:\!3) =$

10, contradicting the possible evolution of a *10* pair as discussed above, or (i.b) $\tau^{-1}\mathbf{n}_{t+1}$ lies in case (i) of A_{k-2}, contradicting the minimality of k. For example, if $p = q = 0$, so that $k = 4$ and $\mathbf{n}_t(1\!:\!4) = 102x$, then $\mathbf{n}_t(2:3) = 12$ with probability $1/2$ (case 1.a)), while if $q \geq 2$ then $n_{t+1}(2\!:\!k-2) = 102(02)^p1^{q-2}2$, so that $\tau^{-1}\mathbf{n}_{t+1} \in A_{k-2}$ (case (1.b)).

Suppose then that (ii) holds. If $p = 0$ then, as in case (i.a), the *10* disappears, and if $p \geq 1$ and $y \geq 4$ then $\tau^{-1}\mathbf{n}_{t+1} \in A_{k-1}$; each of these outcomes contradicts our assumptions. If $p \geq 1$ and $y = 3$ then we must consider an additional site, and so write $\mathbf{n}_t(k+1) = z$. If $z \geq 2$ then, with probability $1/2$, $\mathbf{n}_{t+1}(2\!:\!k) = 102(02)^{p-1}3$ and so $\tau^{-1}\mathbf{n}_{t+1} \in A_{k-1}$. Finally, if $z = 1$ or $z = 0$ then $\mathbf{n}_{t+1}(2\!:\!k) = 102(02)^{p-1}1z'$ with $z' \geq 2$, so that $\tau^{-1}\mathbf{n}_{t+1}$ lies in A_k but falls under case (i), and this is a contradiction by the analysis of the previous paragraph. For this last step we need to observe that if $z = 0$ then necessarily $\mathbf{n}_t(k+2) \geq 2$, since neither *00* nor *01* can occur in $\mathbf{n}_t$.

To complete the proof we must show that, $\hat{\mu}$-a.s., the string *11* cannot occur in $\mathbf{n}$. But, as for the initial analysis of *10* above, if $\mathbf{m} \in \widehat{X}_2$ satisfies $\mathbf{m}_t(i\!:\!i+1) = 11$ then, $\widehat{P}_{\hat{\mu}}$-a.s., there must be a time $t_0 \leq t$ with $\mathbf{m}_{t_0}(i\!:\!i+1) = 11$ but $\mathbf{m}_{t_0-1}(i\!:\!i+1) \neq 11$, and it is easy to see by considering various possible values of $\mathbf{n}_{t_0-1}(i-1\!:\!i+1)$ that this cannot happen. ∎

In the remainder of Section 5 we study the ETIS states of the SSM considered as a process with state space $\widehat{X}^*$. Theorem 5.1 justifies this choice for states with density $\hat{\rho} > 1$. Moreover, the state $\hat{\mu}^{(2)}$ introduced in Section 1 was shown in Section 4 to be an ETIS state of density $\hat{\rho} = 1$ for the SSM; $\hat{\mu}^{(2)}$ is supported on $\widehat{X}^*$ (and is the only such state). It is convenient to include this state in our study, so that from now on we assume that $\hat{\rho} \geq 1$.

The dynamical rules of the SSM simplify on $\widehat{X}^*$ as follows: if $\mathbf{n}_t(i)$ is *0* or *1* then in the transition to $\mathbf{n}_{t+1}$ a particle moves from each of the sites $i+1$ and $i-1$ to i, while if $\mathbf{n}(j), \mathbf{n}(j+1) \geq 2$ a particle moves from j to $j+1$ or the reverse, each with probability $1/2$. Since at each time step a particle must move across each bond, the parity $(-1)^{\mathbf{n}(i)}$ of the stack height at each site i is conserved. Let $S := \{0,1\}^{\mathbb{Z}}$ be the space of *parity sequences*, define the *parity map* $\mathcal{P} : \widehat{X}^* \to S$ by $\mathcal{P}(\mathbf{n})(i) = (1-(-1)^{\mathbf{n}(i)})/2$, and for $\sigma \in S$ define the *parity sector* $\widehat{X}^*_\sigma \subset \widehat{X}^*$ by $\widehat{X}^*_\sigma := \mathcal{P}^{-1}(\{\sigma\})$, so that $\mathbf{n} \in \widehat{X}^*_\sigma$ iff, for all i, $\mathbf{n}(i)$ and $\sigma(i)$ have the same parity.

Since for each $\sigma \in S$ the parity sector $\widehat{X}^*_\sigma$ is invariant under the dynamics, we obtain by restriction a dynamical system on each $\widehat{X}^*_\sigma$. In Section 5.1 below

we discuss the stationary states in the *even sector* $\widehat{X}_e^*$, where e is the parity sequence satisfying $e(i) = 0$ for all i, and in Section 5.2 and Appendix B show how these give rise to all the ETIS states on $\widehat{X}^*$.

5.1 ETIS states in the even sector

Our next result describes a family of ETIS states for the SSM in the even sector; the proof will be given shortly.

Theorem 5.2 *For each $\hat{\rho}_e \geq 1$ there is an ETIS state $\mu_e^{(\hat{\rho}_e)}$ on $\widehat{X}_e^*$, of density $\hat{\rho}_e$; $\mu_e^{(\hat{\rho}_e)}$ is a grand-canonical Gibbs state for the statistical-mechanical system with state space $\widehat{X}^*$ and one-body potential*

$$V(n) := 2\ln 2\, \delta_{n0}. \tag{5.1}$$

For $\hat{\rho}_e = 1$, $\mu_e^{(\hat{\rho}_e)} = \hat{\mu}^{(2)}$ and hence is ergodic but not weakly mixing. For $\hat{\rho}_e > 1$, $\mu_e^{(\hat{\rho}_e)}$ is mixing.

As we will see in Section 5.2 and Appendix B, for a full discussion of the ETIS states of the SSM on $\widehat{X}^*$ we need to know all stationary states on $\widehat{X}_e^*$, both TI and non-TI (should any of the latter exist). We make the

Conjecture 5.3 *For each $\hat{\rho}_e \geq 1$, $\mu_e^{(\hat{\rho}_e)}$ is the unique stationary state of the SSM on $\widehat{X}_e^*$ with density $\hat{\rho}_e$.*

In Section 5.2 we will discuss the ETIS states on $\widehat{X}^*$ under the assumption that Conjecture 5.3 holds, and in Appendix B turn to the general case.

We begin our discussion of Theorem 5.2 with the consideration of the SSM on a ring of $2L + 1$ sites, indexed by $I_L := \{-L, \ldots, L\}$; the configuration space $\widehat{X}_e^{*(L)}$ is the set of elements $\mathbf{n} \in \mathbb{Z}_+^{I_L}$ for which $\mathbf{n}(i)$ is even for all i and for which no two adjacent sites both have height zero. For the moment we take a fixed number N of particles, with N even and $N \geq 2L + 2$, with corresponding configuration space $\widehat{X}_e^{*(N,L)} \subset \widehat{X}_e^{*(L)}$. For $\mathbf{n} \in \widehat{X}_e^{*(N,L)}$ let $z(\mathbf{n})$ be the number of sites i with $\mathbf{n}(i) = 0$. In a transition from $\mathbf{n}_t$ to $\mathbf{n}_{t+1}$ the direction of particle movement across $2z(\mathbf{n}_t)$ bonds is determined and across the remainder is chosen randomly; moreover, a given transition can occur via at most one set of these choices unless $\mathbf{n}_t = \mathbf{n}_{t+1}$, in which case there are two possibilities (this occurs iff $z(\mathbf{n}_t) = 0$). Thus the probability $P(\mathbf{n}_t, \mathbf{n}_{t+1})$ of such a transition, if nonzero, is $2^{-(2L+1-2z(\mathbf{n}_t))+\delta_{\mathbf{n}_t,\mathbf{n}_{t+1}}}$. Then a TIS state $\mu^{(N,L)}$

is given by $\mu^{(N,L)}(\mathbf{n}) = \mathscr{Z}_{N,L}^{-1} 2^{-2z(\mathbf{n})}$, with $\mathscr{Z}_{N,L}^{-1}$ a normalizing constant, since because $P(\mathbf{n}, \mathbf{n}') = 0$ iff $P(\mathbf{n}', \mathbf{n}) = 0$, $\mu^{(N,L)}$ satisfies the detailed balance condition $\mu^{(N,L)}(\mathbf{n})P(\mathbf{n}', \mathbf{n}) = \mu^{(N,L)}(\mathbf{n}')P(\mathbf{n}, \mathbf{n}')$. It is straightforward to check that the dynamics permits transition from any configuration in $\widehat{X}_e^{*(N,L)}$ to any other, so that $\mu^{(N,L)}$ is the unique TIS state.

The state $\mu^{(N,L)}$ is a Gibbs measure arising from the one-particle potential $V(n)$ of (5.1), that is, $\mu^{(N,L)}(\mathbf{n}) = \mathscr{Z}_{N,L}^{-1} \prod_{i=-L}^{L} e^{-V(\mathbf{n}(i))}$. (One may also view $\mu^{(N,L)}$ as a Gibbs measure on the space of all N-particle configurations, with one- and two-body hard core potentials, that is, formal potentials taking infinite values, that impose the restrictions of $\widehat{X}_e^{*(L)}$.) In order to pass to the $L \to \infty$ limit it is convenient to consider the grand canonical measure with fugacity $\zeta \geq 0$:

$$\mu^{(\zeta,L)} := \Xi_{L,\zeta}^{-1} \sum_{\mathbf{n} \in \widehat{X}_e^{*(L)}} \zeta^{\sum_i \mathbf{n}(i) - 2L - 2} e^{-\sum_i V(\mathbf{n}(i))}, \tag{5.2}$$

with $\Xi_{\zeta,L}^{-1}$ again a normalizing constant.

Lemma 5.4 *The limiting measure $\mu^{(\zeta,\infty)} = \lim_{L\to\infty} \mu^{(\zeta,L)}$ exists for $0 \leq \zeta < 1$ and is a TIS state of density $1/(1-\zeta)$ for the SSM on $\widehat{X}_e^*$. Moreover, $\mu^{(0,\infty)} = \mu^{(2)}$ and $\mu^{(\zeta,\infty)}$ is mixing if $\zeta > 1$.*

Proof: The case $\zeta = 0$ follows immediately from the fact, evident from (5.2), that $\mu^{(0,L)}$ gives equal weight to the $2L + 1$ configurations in $\widehat{X}_e^{*(N,L)}$ with $\sum_i \mathbf{n}(i) = 2L + 2$. From now on we assume that $0 < \zeta < 1$.

We can prove the existence of $\mu^{(\zeta,\infty)}$, and also calculate many of its properties, using the standard transfer matrix formalism; we give only a sketch. Let us think of $\ell^2 = \{(x_i)_{i=0}^{\infty} \mid \sum x_i^2 < \infty\}$ as a space of column vectors, with u^T denoting the transpose of the vector u, and define $u, v \in \ell^2$ by $u_i := \delta_{i0}$ and $v_0 := 0$, $v_i := \zeta^i$ if $i \geq 1$. Let $\mathcal{T} := (uv^T + vu^T)/2 + vv^T$; then for $\mathbf{n} = 2\mathbf{i} \in \widehat{X}_e^{*(L)}$,

$$\mu^{(\zeta,L)}(\mathbf{n}) = \Xi_{\zeta,L}^{-1} \mathcal{T}_{\mathbf{i}(-L)\mathbf{i}(-L+1)} \mathcal{T}_{\mathbf{i}(-L+1)\mathbf{i}(-L+2)} \cdots \mathcal{T}_{\mathbf{i}(L-1)\mathbf{i}(L)} \mathcal{T}_{\mathbf{i}(L)\mathbf{i}(-L)}, \tag{5.3}$$

with $\Xi_{\zeta,L}$ the trace of $\mathcal{T}^{2L+1}$. $\mathcal{T}$ is a rank 2 operator with nonzero eigenvalues $\lambda_1 = \zeta/(2(1-\zeta))$ and $\lambda_2 = -\zeta/(2(1+\zeta))$; the eigenvector associated with λ_1, the larger in magnitude, is $w = (\zeta/(1+\zeta))u + v$, so that

$$\lim_{n\to\infty} \lambda_1^{-n} \mathcal{T}^n = \|w\|^{-2} w^T w. \tag{5.4}$$

19

Thus for $\mathbf{m} = 2\mathbf{i} \in \mathbb{Z}_+^{\{-K,\cdots K\}}$ and E the event that $\mathbf{n}(-K\!:\!K) = \mathbf{m}(-K\!:\!K)$ we have

$$\mu^{(\zeta,\infty)}(E) := \lim_{L\to\infty} \mu^{(\zeta,L)}(E)$$
$$= \mathcal{Y}_{\zeta,K}^{-1} w_{\mathbf{i}(-K)} \mathcal{T}_{\mathbf{i}(-K)\mathbf{i}(-K+1)} \cdots \mathcal{T}_{\mathbf{i}(K-1)\mathbf{i}(K)} w_{\mathbf{i}(K)} \qquad (5.5)$$

with

$$\mathcal{Y}_{\zeta,K} = \lambda_1^{2K} \|w\|^2 = \frac{\zeta^{2K+2}}{(1+\zeta)^2 2^{2K-1}(1-\zeta)^{2K+1}}. \qquad (5.6)$$

Taking $K = 0$ in (5.5) we find that

$$\mu^{(\zeta,\infty)}(\mathbf{n}(0) = 2\mathbf{i}) = \mathcal{Y}_{\zeta,0}^{-1} w_{\mathbf{i}}^2 = \begin{cases} (1-\zeta)/2, & \text{if } \mathbf{i} = 0, \\ (1+\zeta)^2(1-\zeta)\zeta^{2\mathbf{i}-2}/2, & \text{otherwise,} \end{cases} \qquad (5.7)$$

from which we find the density $\hat{\rho}(\zeta) := \mu^{(\zeta,\infty)}(\mathbf{n}(0)) = 1/(1-\zeta)$. From (5.4) and (5.5) it follows that if $f, g : \widehat{X}^* \to \mathbb{R}$ each depend only on the values of the configuration at a finite number of sites then $\mu^{(\zeta,\infty)}(f\tau^n g) \sim Ce^{-(\lambda_1-|\lambda_2|)n}$ as $n \to \infty$, so that $\mu^{(\zeta,\infty)}$ is mixing.

The stationarity of $\mu^{(\zeta,\infty)}$ can be verified from the explicit formulas (5.5) and (5.6), but it is simpler to argue from the stationarity of $\mu^{(\zeta,L)}$. Take $L > K > 0$, suppose that $A^{(K)} \subset \widehat{X}_e^{*(K)}$, and let A, respectively $A^{(L)}$, be the set of $\mathbf{n} \in \widehat{X}_e^*$, respectively $\mathbf{n} \in \widehat{X}_e^{*(L)}$, such that $\mathbf{n}(-K-1\!:\!K+1) \in A^{(K)}$. If Q and $Q^{(L)}$ are the transition kernels for the SSM on $\widehat{X}^*$ and $\widehat{X}^{*(L)}$, respectively, then the stationarity of $\mu^{(\zeta,L)}$ implies that

$$\int_{\widehat{X}^{*(L)}} Q^{(L)}(\mathbf{n}, A^{(L)}) \mu^{(\zeta,L)}(d\mathbf{n}) = \mu^{(\zeta,L)}(A^{(L)}). \qquad (5.8)$$

Now $Q^{(L)}(\mathbf{n}, A^{(L)})$ depends only on $\mathbf{n}(-K-1\!:\!K+1)$ and $Q^{(L)}(\mathbf{n}, A^{(L)}) = Q(\mathbf{n}', A)$ if $\mathbf{n}(-K-1\!:\!K+1) = \mathbf{n}'(-K-1\!:\!K+1)$, so that taking the $L \to \infty$ limit in (5.8) yields $\int_{\widehat{X}^*} Q(\mathbf{n}, A) \mu^{(\zeta,\infty)}(d\mathbf{n}) = \mu^{(\zeta,\infty)}(A)$, the stationarity of $\mu^{(\zeta,\infty)}$. ∎

Proof of Theorem 5.2: The theorem follows immediately from Lemma 5.4 via $\nu_e^{(\hat{\rho}_e)} = \nu^{((\hat{\rho}_e-1)/\hat{\rho}_e),\infty)}$. ∎

Remark 5.5 There is an alternative way to describe the state $\mu^{(\zeta,\infty)}$ (and hence also $\mu_e^{\rho_e}$). Consider the image of $\mu^{(\zeta,\infty)}$ under the map $F : \widehat{X}^* \to \{0,1\}^{\mathbb{Z}}$

defined by $F(\mathbf{n})(j) = \min\{1, \mathbf{n}(j)\}$, which effectively classes sites simply as occupied or empty. The image measure $F_\mu^{(\zeta, \infty)}$ is again Gibbisan, with no interactions other than the exclusion of configurations containing adjacent holes, so that it is, after an interchange of the roles of particles and holes, the equilibrium state of the familiar *nearest-neighbor hard core* model. The holes in this system have effective fugacity $(1 - \zeta^2)/(2\zeta)^2$ (relative to a fugacity of 1 for the particles) and from (5.7) the density of holes is $(1 - \zeta)/2$. The full state $\mu^{(\zeta, \infty)}$ is then obtained by first conditioning $F(\mathbf{n}) = \eta$ for some $\eta \in \{0, 1\}^\infty$ with no adjacent holes, distributed according to $F_*\mu^{(\zeta, \infty)}$, and then distributing particles on each site j for which $\eta(j) = 1$ independently, with distribution $\mu\{\mathbf{n}(j) = 2i\} = \zeta^{2i-1}/(1 - \zeta^2)$, $i = 1, 2, \ldots$.

5.2 ETIS states for the SSM

We now discuss the passage from stationary states of the SSM on the even sector to general high-density ETIS states, specifically, to states with $\hat{\rho} > 1$ as well as the special state $\mu^{(2)}$ with $\hat{\rho} = 1$. By Theorem 5.1, these are precisely the ETIS states on $\widehat{X}^*$. Let us define $\gamma : \widehat{X}_e^* \times S \to \widehat{X}^*$ by $\gamma(\mathbf{m}, \sigma) = \mathbf{m} + \sigma$ (with component-wise addition: $(\mathbf{m} + \sigma)(i) := \mathbf{m}(i) + \sigma(i)$). Note that for fixed $\sigma \in S$, $\gamma(\cdot, \sigma)$ is a bijection of $\widehat{X}_e^*$ with $\widehat{X}_\sigma^*$. We define the dynamics in $\widehat{X}_e^* \times S$ to be constant on S.

Lemma 5.6 *(a) A measure μ on $\widehat{X}^*$ is an ETIS state for the SSM iff $\tilde{\mu} := \gamma_*^{-1}\mu$ is an ETIS state on $\widehat{X}_e^* \times S$.*
(b) Each ETIS state $\tilde{\mu}$ of density $\hat{\rho}$ on $\widehat{X}_e^* \times S$ has the form $\tilde{\mu}(d\mathbf{n}\, d\sigma) = \mu_\sigma(d\mathbf{n})\lambda(d\sigma)$, where λ is an ergodic TI probability measure on S and the μ_σ, $\sigma \in S$, are stationary probability measures on $\widehat{X}_e^*$ satisfying $\mu_{\tau\sigma} = \tau_*\mu_\sigma$. Moreover, if λ has density κ then λ-almost all μ_σ have density $\hat{\rho}_e = \hat{\rho} - \kappa$.

Proof: (a) The map γ clearly commutes with translations and with the dynamics. Thus γ_* and γ_*^{-1} carry TIS states to TIS states, and they clearly preserve extremality.
(b) The form $\tilde{\mu}(d\mathbf{n}\, d\sigma) = \mu_\sigma(d\mathbf{n})\lambda(d\sigma)$ is immediate, with λ the probability measure on S giving the distribution of σ and μ_σ the conditional probability measure on $\widehat{X}_e^*$ given σ. Stationarity of $\tilde{\mu}$ implies stationarity of each μ_σ, and translation invariance of $\tilde{\mu}$ yields translation invariance of λ and the relation $\mu_{\tau\sigma} = \tau_*\mu_\sigma$. λ must be ergodic, since a decomposition of λ as a convex combination of TI measures would yield immediately a decomposition of μ

21

in terms of TIS measures. Finally, since $\tilde{\mu}$-a.s. configurations have density $\hat{\rho}$, λ-a.s. each μ_σ must have density $\hat{\rho}_e = \hat{\rho} - \kappa$. ∎

Throughout the remainder of this section we assume that Conjecture 5.3 holds. See Appendix B for an analysis of the situation when this assumption is not valid.

Theorem 5.7 *Suppose that the $\mu_e^{(\hat{\rho}_e)}$ of Theorem 5.2 are the only stationary states of the SSM on $\widehat{X}_e^*$, i.e., that Conjecture 5.3 holds. Then the ETIS states with density $\hat{\rho} > 1$ are precisely the states $\mu^{(\hat{\rho},\lambda)} := \gamma_*(\mu_e^{(\hat{\rho}_e)} \times \lambda)$, with λ an ergodic TI probability measure on S of density $\kappa \leq \hat{\rho} - 1$ and $\hat{\rho}_e = \hat{\rho} - \kappa$.*

Before giving the proof we note an immediate consequence of this result and Theorem 2.1.

Corollary 5.8 *Under the hypotheses of Theorem 5.7 the ETIS states of the F-SSEP with density $\rho > 1/2$ are the states $\Phi_\phi(\mu^{(\hat{\rho},\lambda)})$; $\Phi_\phi(\mu^{(\hat{\rho},\lambda)})$ has density $\hat{\rho}/(1 + \hat{\rho})$.*

Proof of Theorem 5.7: By Lemma 5.6 it suffices to prove that the ETIS states on $\widehat{X}_e^* \times S$ are the states $\mu_e^{(\hat{\rho}_e)} \times \lambda$ described in the theorem. The latter are clearly TIS. If $\tilde{\mu}$ is an ETIS state on $\widehat{X}_e^* \times S$ of density $\hat{\rho}$ then Lemma 5.6 and Conjecture 5.3 imply that $\tilde{\mu} = \mu_e^{(\hat{\rho}-\kappa)} \times \lambda$. Conversely, a decomposition of a state $\mu_e^{(\hat{\rho}_e)} \times \lambda$ as described in the theorem into extremal components must, by this and the ergodicity of λ, be trivial. ∎

Remark 5.9 (a) The states $\mu^{(\hat{\rho},\lambda)}$ of the SSM with $\hat{\rho} > 1$ are defined in Theorem 5.7; it is natural then to define also $\mu^{(1,\delta_e)} := \gamma_*(\mu_e^{(\hat{\rho}_e)} \times \delta_e) = \mu^{(2)}$, with $\delta_e \in \overline{\mathcal{M}}(S)$ the point mass on the zero configuration e.
(b) It follows from the theorem that, whenever $\mu^{(\hat{\rho},\lambda)}$ is defined, the density κ of the measure λ satisfies $0 \leq \kappa \leq \max\{1 - \hat{\rho}, 1\}$. The case $\hat{\rho} = 1 + \kappa$, corresponding, in the notation of the theorem, to $\hat{\rho}_e = 1$, is of particular interest; see (c) and (d) below.
(c) $\mu^{(\hat{\rho},\lambda)}$ is extremal in the class of TIS states, but it may not be ergodic under translations. For example, if $\sigma \in S$ has even period p and density κ, and λ is the superposition of the point masses on σ and its translates, then if $\hat{\rho} = 1 + \kappa$ (i.e., if $\hat{\rho}_e = 1$), $\mu^{(\hat{\rho},\lambda)}$ is a superposition of two ergodic measures, these being the superpositions of the translates of $\mathbf{n}^* + \sigma$ and

22

$\tau \mathbf{n}^* + \sigma$, respectively. When in general can such non-ergodicity occur? First, if $\hat{\rho}_e > 1$ then $\mu_e^{(\hat{\rho}_e)} \times \lambda$, as the product of a mixing and an ergodic measure, is ergodic, and hence so is $\mu^{(\hat{\rho}_e + \kappa, \lambda)}$. On the other hand, if $\hat{\rho}_e = 1$ we use the fact that $\mu_e^{(\hat{\rho}_e)} \times \lambda$ will be ergodic iff the eigenvalue 1 of the translation operator in $L^2(\widehat{X}_e^* \times S, \mu_e^{(\hat{\rho}_e)} \times \lambda)$ is simple; since translation on $L^2(\widehat{X}_e^*, \mu_e^{(1)})$ has simple eigenvalues of ± 1 and no others, we conclude that $\mu^{(\hat{\rho}, \lambda)}$ is ergodic iff the translation operator on $L^2(S, \lambda)$ does not have the eigenvalue -1.

(d) If λ has density κ then, $\mu^{(1+\kappa, \lambda)}$-a.s., each configuration $\mathbf{n}$ has sites with 0 or 1 particle(s) alternating with sites with 2 or 3 particles, and the dynamics carries such a configuration to the one obtained from it by the substitutions $0 \to 2$, $1 \to 3$, $2 \to 0$, and $3 \to 1$. The evolution of these configurations is thus periodic, of period 2.

We finally observe that the states $\mu^{(\hat{\rho}, \lambda)}$ for fixed $\hat{\rho}_e = \hat{\rho} - \kappa$ (where as usual κ is the density of λ) form a λ-family (see Definition 2.3), which we denote $\widehat{\mathcal{F}}_{\hat{\rho}_e}$. The indexing map $\widehat{\Psi}_{\hat{\rho}_e} : \mathcal{L} \to \widehat{\mathcal{F}}_{\rho_e}$ is given by $\widehat{\Psi}_{\hat{\rho}_e}(\lambda) = \mu^{(\hat{\rho}_e + \kappa, \lambda)}$, where we have made the identification $\mathcal{L} = \overline{\mathcal{M}}(S)$. From Corollary 5.8, then, the ETIS states of the F-SSEP with density $\rho > 1/2$, together with the state $\mu^{(2)}$, form the λ-families $\mathcal{F}_{\hat{\rho}_e} = \Phi_*(\widehat{\mathcal{F}}_{\hat{\rho}_e})$.

6 Summary

The results of this paper are summarized, under the assumption that Conjecture 5.3 holds, by Figure 2, which gives a symbolic depiction of the relations among the λ-families of regular ETIS states of the SSM. The heavy black lines denote λ-families. $\widehat{\mathcal{F}}_{\text{left}}$ and $\widehat{\mathcal{F}}_{\text{right}}$ have been separated for visibility, but in fact both lie at $\hat{\rho} = 1$. The state $\hat{\mu}^{(1)}$ belongs to all three of the families $\widehat{\mathcal{F}}_{\text{low}}$, $\widehat{\mathcal{F}}_{\text{left}}$, and $\widehat{\mathcal{F}}_{\text{right}}$, and the state $\hat{\mu}^{(2)}$ to $\widehat{\mathcal{F}}_{\text{left}}$, $\widehat{\mathcal{F}}_{\text{right}}$, and $\widehat{\mathcal{F}}^{(1)}$ (see Remark 5.9(a)). κ denotes the density of a measure $\lambda \in \overline{\mathcal{M}}(S)$; this variable is relevant only for the state $\hat{\mu}^{(2)}$ (for which $\kappa = 0$) and for states with $\hat{\rho} > 1$. The shaded region is filled with the λ-families $\widehat{\mathcal{F}}_{\hat{\rho}_e}$; the family for $\hat{\rho}_e = 1$ and three other representative families ($\hat{\rho}_e = r_1, r_2, r_3$) are shown.
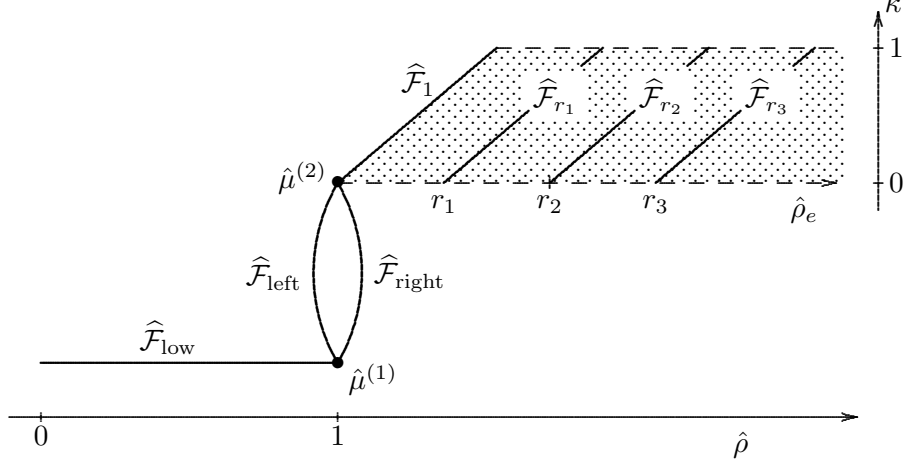
Figure 2: Symbolic picture of the set of ETIS states of the SSM.

A   Equivalence of TI measures under substitutions

In this appendix we give a construction which will be used at several points in the paper. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are countable alphabets, that $\widehat{Y} = \mathcal{S}^{\mathbb{Z}}$ and $Y = \mathcal{T}^{\mathbb{Z}}$ with typical elements $\zeta \in \widehat{Y}$ and $\eta \in Y$, and that for each $s \in \mathcal{S}$ we specify a finite sequence $\chi(s) = t_s(1) \ldots t_s(k(s))$ of elements of $\mathcal{T}$ (that is, a word of $k(s)$ letters in the alphabet $\mathcal{T}$). Then we define $\phi : \widehat{Y} \to Y$ to be the map which substitutes $\chi(s)$ for s, that is,

$$\phi(\ldots \zeta(-1)\zeta(0)\zeta(1)\zeta(2) \ldots) = \ldots \chi(\zeta(-1))\chi(\zeta(0))\chi(\zeta(1))\chi(\zeta(2)) \ldots,$$
(A.1)

with $\chi(\zeta(1))$ beginning at site 1, so that $\phi(\zeta)(1) = t_{\zeta(1)}(1)$. For $s \in \mathcal{S}$ we define $\widehat{Y}_s = \{\zeta \in \widehat{Y} \mid \zeta(1) = s\}$ and $Y_s = \phi(\widehat{Y}_s)$; for $0 \le j \le k(s) - 1$ we set $Y_{sj} = \tau^{-j} Y_s$ and define $\phi_{sj} : \widehat{Y}_s \to Y_{sj}$ by $\phi_{sj} = \tau^{-j} \phi|_{\widehat{Y}_s}$. From now on we assume that the $\chi(s)$ are such that the sets Y_{sj} are pairwise disjoint.

We call a TI measure $\hat{\nu}$ on $\widehat{Y}$ *regular* if $\hat{\nu}(k(\zeta(1)))$ is finite, i.e., if $Z_{\hat{\nu}} := \sum_{s \in \mathcal{S}} k(s)\hat{\nu}(\widehat{Y}_s) < \infty$, and a TI measure ν on Y *regular* if it is supported on $\bigcup_{s \in \mathcal{S}} \bigcup_{j=0}^{k(s)-1} Y_{sj}$, the smallest translation invariant subset of $\mathcal{T}^{\mathbb{Z}}$ containing $\phi(\widehat{Y})$. Let $\mathcal{M}(\widehat{Y})$ denote the space of regular TI states on $\widehat{Y}$ and $\mathcal{M}(Y)$ the space of regular TI states on Y. If $\hat{\nu} \in \mathcal{M}(\widehat{Y})$ and $\nu \in \mathcal{M}(Y)$ define $\hat{\nu}_s = \hat{\nu}|_{\widehat{Y}_s}$

and $\nu_s = \nu\big|_{Y_s}$.

Theorem A.1 *For $\hat{\nu} \in \mathcal{M}(\widehat{Y})$ define the measure $\Phi(\hat{\nu})$ on Y by*

$$\Phi(\hat{\nu}) := Z_{\hat{\nu}}^{-1} \sum_{s \in \mathcal{S}} \sum_{j=0}^{k(s)-1} \phi_{sj*} \hat{\nu}_s. \tag{A.2}$$

Then Φ is a bijection of $\mathcal{M}(\widehat{Y})$ with $\mathcal{M}(Y)$, and for $\nu \in \mathcal{M}(Y)$, $\Phi^{-1}(\nu) = \widetilde{Z}_\nu^{-1} \sum_{s \in \mathcal{S}} \phi_{s0*}^{-1} \nu_s$, where $\widetilde{Z}_\nu = \sum_{s \in \mathcal{S}} \nu(Y_s)$. $\Phi(\hat{\nu})$ is ergodic iff $\hat{\nu}$ is.

Proof: This is straightforward to check. The final statement follows from the fact that $\hat{\nu} = \sum c_\alpha \hat{\nu}_\alpha$ if and only if $\Phi(\hat{\nu}) = \sum c_\alpha' \Phi(\hat{\nu}_\alpha)$, where $\hat{\nu}_\alpha \in \mathcal{M}(\widehat{Y})$ and $c_\alpha' = Z_{\hat{\nu}_\alpha} c_\alpha / Z_{\hat{\nu}}$. ∎

We now suppose that we are given dynamical rules in the spaces $\widehat{Y}$ and Y, that is, translation invariant Markov processes with state spaces $\widehat{Y}$ and Y, specified by respective TI transition kernels $\widehat{Q}(\zeta, A)$ and $Q(\eta, B)$. We will say that $\widehat{Q}$ or Q *preserves ergodicity* if $\hat{\nu}\widehat{Q}$ (respectively νQ) is ergodic whenever $\hat{\nu}$ (respectively ν) is; *preserving regularity* is defined similarly.

We next want to give a condition which will imply that the mapping Φ preserves these dynamics, i.e., that $\Phi(\hat{\nu}\widehat{Q}) = \Phi(\hat{\nu})Q$; we will need some further notation. Let $\overline{Y} \subset \widehat{Y} \times Y$ be the set of pairs (ζ, η) such that η is a (possibly trivial) translate of $\phi(\zeta)$, and let $\hat{\pi}$ and π be the projections of $\overline{Y}$ onto the first and second components, respectively, of $\widehat{Y} \times Y$.

Definition A.2 A τ-*coupling* of $\widehat{Q}$ and Q is a Markov transition kernel $\overline{Q}$ with state space $\overline{Y}$ such that for $(\zeta, \eta) \in \overline{Y}$, $\widehat{A} \subset \widehat{Y}$, and $A \subset Y$,

$$\overline{Q}\big((\zeta, \eta), \hat{\pi}^{-1}(\widehat{A})\big) = \widehat{Q}(\zeta, \widehat{A}), \tag{A.3}$$

$$\overline{Q}\big((\zeta, \eta), \pi^{-1}(A)\big) = Q(\eta, A). \tag{A.4}$$

Equivalently, a Markov transition kernel $\overline{Q}$ with state space $\overline{Y}$ is a τ-coupling of $\widehat{Q}$ and Q provided that for any Markov process (ζ_t, η_t) with transition kernel $\overline{Q}$, ζ_t and η_t are Markov processes with transition kernels $\widehat{Q}$ and Q, respectively.

Remark A.3 To show that there is a τ-coupling of Q and $\widehat{Q}$ it suffices to find a restricted transition probability $\overline{Q}\big((\zeta, \phi(\zeta)), \cdot\big)$ which satisfies (A.3)

and (A.4) when $\eta = \phi(\zeta)$. For $\overline{Q}$ can then be extended to the rest of $\overline{Y}$ by setting $\overline{Q}((\zeta,\eta),\cdot) := \overline{\tau}_*^q \overline{Q}((\zeta,\phi(\zeta)),\cdot)$ when $\eta = \tau^q\phi(\zeta)$, choosing q, when periodicity of $\phi(\zeta)$ necessitates a choice, to be the minimal nonnegative q with $\eta = \tau^q\phi(\zeta)$. Here $\overline{\tau}$ acts on $\overline{Y}$ via $\overline{\tau}(\zeta,\eta) := (\zeta,\tau\eta)$.

Theorem A.4 *Suppose that Q and $\widehat{Q}$ are TI Markov transition kernels on $\widehat{Y}$ and Y, respectively, which preserve ergodicity and regularity and for which there exists a τ-coupling $\overline{Q}$. Then for any TI state $\hat{\nu} \in \mathcal{M}(\widehat{Y})$ and any $n \geq 1$, $\Phi(\hat{\nu}\widehat{Q}^n) = \Phi(\hat{\nu})Q^n$.*

Corollary A.5 *If Q and $\widehat{Q}$ are as in Theorem A.4 then Φ is a bijection of $\mathcal{M}_s(\widehat{Y})$ with $\mathcal{M}_s(Y)$, i.e., $\hat{\nu} \in \mathcal{M}(\widehat{Y})$ is stationary for $\widehat{Q}$ if and only if $\Phi(\hat{\nu})$ is stationary for Q.*

The corollary is of course immediate. The idea of the proof of Theorem A.4 is taken from [1] and [12]:

Proof of Theorem A.4: It suffices to verify the result for $\hat{\nu}$ ergodic. Then since $\widehat{Q}$ and Q preserve ergodicity, as does Φ, $\Phi(\hat{\nu}\widehat{Q}^n)$ and $\Phi(\hat{\nu})Q^n$ are ergodic, so that these two measures are either equal or mutually singular. Hence to prove their equality it suffices to find a nonzero measure λ with $\lambda \leq \Phi(\hat{\nu}\widehat{Q}^n)$ and $\lambda \leq \Phi(\hat{\nu})Q^n$.

Let $\overline{Q}$ be a τ-coupling of $\widehat{Q}$ and Q with state space $\overline{Y}$, as in Definition A.2, define $\psi : \widehat{Y} \to \overline{Y}$ by $\psi(\zeta) := (\zeta,\phi(\zeta))$, and for $\hat{\nu} \in \mathcal{M}(\widehat{Y})$ let $\overline{\nu} := \psi_*\hat{\nu}$, so that $\hat{\pi}_*\overline{\nu} = \hat{\nu}$ and $\pi\overline{\nu} = \phi_*\hat{\nu}$. Fix $n \geq 1$ and let $q \in \mathbb{Z}$ be such that $(\overline{\nu}\overline{Q}^n)(C_q) > 0$, where $C_q := \{(\zeta,\eta) \in \overline{Y} \mid \eta = \tau^q\phi(\zeta)\}$, and define $\overline{\nu}^{(n,q)} := \mathbf{1}_{C_q}\overline{\nu}\overline{Q}^n$, $\nu' := \phi_*\hat{\pi}_*\overline{\nu}^{(n,q)}$, and $\nu'' := \pi_*\overline{\nu}^{(n,q)}$. We claim that (i) $\nu'' = \tau_*^q\nu'$ and that, for an appropriate constant $c > 0$, (ii) $c\nu' \leq \Phi(\hat{\nu}\widehat{Q}^n)$ and (iii) $c\nu'' \leq \Phi(\hat{\nu})Q^n$. Since $\Phi(\hat{\nu})Q^n$ is TI, the proof is completed by taking $\lambda = c\nu'$.

It remains to prove the claim. (i) follows from the definition of C_q. From Definition A.2, $\hat{\pi}_*(\overline{\nu}\overline{Q}^n) = \hat{\nu}\widehat{Q}^n$ and $\pi_*(\overline{\nu}\overline{Q}^n) = (\phi_*\hat{\nu})Q^n$, and with this and (A.2) we have

$$\nu' = \phi_*\hat{\pi}_*\overline{\nu}^{(n,q)} \leq \phi_*\hat{\pi}_*(\overline{\nu}\overline{Q}^n) = \phi_*(\hat{\nu}\widehat{Q}^n) \leq Z_{\hat{\nu}\widehat{Q}^n}\Phi(\hat{\nu}\widehat{Q}^n) \qquad (A.5)$$

and

$$\nu'' = \pi_*\overline{\nu}^{(n,q)} \leq \pi_*(\overline{\nu}\overline{Q}^n) = (\phi_*\hat{\nu})Q^n \leq Z_{\hat{\nu}}\Phi(\hat{\nu})Q^n. \qquad (A.6)$$

This verifies parts (ii) and (iii) of the claim, with $c := \min\{Z_{\hat{\nu}\widehat{Q}^n}^{-1}, Z_{\hat{\nu}}^{-1}\}$. $\blacksquare$

26

In the remainder of this appendix we discuss the applications we make of these results. Theorem A.1 is used in Section 4; the substitution maps there are denoted ϕ_{left} and ϕ_{right}. Theorems A.1 and A.4 are used to obtain a correspondence between the stationary states of the Symmetric Stack Model (SSM) and of the Facilitated Simple Symmetric Exclusion Process (F-SSEP), a correspondence first introduced in Section 1 (see that section and Section 2 for the definition of the transition kernels $\widehat{Q}$ and Q for these models) and used throughout the paper. In this application $\mathcal{S} = \mathbb{Z}_+$, $\mathcal{T} = \{0,1\}$, and for $n \in \mathbb{Z}_+$, $\chi(n) := 0\,1^n$; we write $\widehat{X} := \mathbb{Z}_+^{\mathbb{Z}}$ and $X = \{0,1\}^{\mathbb{Z}}$ but keep the notation $\phi : \widehat{X} \to X$ for the substitution map obtained from χ. The general definition of regularity of measures given above corresponds in this case to the definition given in Section 2, and it is clear that $\widehat{Q}$ and Q preserve regularity.

Lemma A.6 $\widehat{Q}$ *and* Q *preserve ergodicity.*

Proof: We consider $\widehat{Q}$ (the proof for Q is similar), and so must show that if $\hat{\nu} \in \overline{\mathcal{M}}(\widehat{X})$ then $\hat{\nu}\widehat{Q}$ is ergodic. We define the probability space (Ω, P) by $\Omega := \widehat{X} \times \{0,1\}^{\mathbb{Z}}$, $P := \hat{\nu} \times \kappa$, where κ is the Bernoulli measure with parameter $1/2$, and write a typical element of Ω as $(\mathbf{n}, \alpha)$. Then we can introduce a concrete realization on Ω of one step of the $\widehat{Q}$ process, from $\mathbf{n}_0$ distributed as $\hat{\nu}$ to $\mathbf{n}_1$ distributed as $\hat{\nu}\widehat{Q}$, as follows. Recall that if $\mathbf{n}_0$ has a short stack on either i or $i+1$ then the movement, or non-movement. of a particle across the bond $\langle i, i+1 \rangle$, in passing from $\mathbf{n}_0$ to $\mathbf{n}_1$, is determined by the rule given in Section 1; we supplement this rule by requiring that if $\mathbf{n}_0$ has tall stacks at both i and $i+1$ then a particle moves from site i to site $i+1$ if $\alpha(i) = 1$ and from $i+1$ to i if $\alpha(i) = 0$.

As the product of measures which are respectively ergodic and mixing under translations, P is ergodic under translations. But then $\hat{\nu}\widehat{Q} = \mathbf{n}_{1*}P$ is the covariant image of an ergodic measure and hence is ergodic. ∎

Theorem A.7 *There exists a τ-coupling $\overline{Q}$ of the Markov transitions kernels $\widehat{Q}$ and Q for the SSM and F-SSEP.*

Proof: As in the definition of $\widehat{Q}$ and Q in Section 2 we give the transition rules for a Markov process $(\mathbf{n}_t, \eta_t)$ on $\overline{X} \subset \widehat{X} \times X$ (see Definition A.7), leaving the specification of $\overline{Q}$ as an easy exercise. By Remark A.3 it suffices to consider only the transition from $(\mathbf{n}_0, \eta_0)$ to $(\mathbf{n}_1, \eta_1)$ for $\eta_0 = \phi(\mathbf{n}_0)$. As a

preliminary, for $\mathbf{n} \in \widehat{X}$ we define the map $K = K_{\mathbf{n}}$, with $K : \mathbb{Z} \to \mathbb{Z}$, so that $K(i)$ is the starting point of the word $\chi(\zeta(i))$ in the substitution (A.1):

$$K(i) = \begin{cases} 1 + \sum_{j=1}^{i-1}(\mathbf{n}(j) + 1), & \text{if } i \geq 1, \\ 1 - \sum_{j=i}^{0}(\mathbf{n}(j) + 1), & \text{if } i \leq 0. \end{cases} \tag{A.7}$$

For the dynamics under $\overline{Q}$ we allow $\mathbf{n}_0$ to evolve to $\mathbf{n}_1$ according the the $\widehat{Q}$ dynamics, and let $\mathbf{n}_1$ and η_0 determine η_1 as follows. Write $K = K_{\mathbf{n}_1}$. Then, if in passing from $\mathbf{n}_0$ to $\mathbf{n}_1$ a particle jumps from site $i-1$ to site i, then in passing from η_0 to η_1 a particle jumps from site $K(i) - 1$ to site $K(i)$, while if a particle jumps from site i to site $i-1$ in passing from $\mathbf{n}_0$ to $\mathbf{n}_1$ then one jumps from site $K(i) + 1$ to site $K(i)$ in passing from η_0 to η_1. It is straightforward to verify that then $(\mathbf{n}_1, \eta_1) \in \overline{X}$ and with η_1 distributed according to $Q(\eta_0, \cdot)$. ∎

With Lemma A.6 and Theorem A.7 we can apply Corollary A.5 to obtain the correspondence of the F-SSEP and SSM stationary states. The result is summarized in Theorem 2.1.

B Possible ETIS states of the SSM for $\hat{\rho} > 1$.

In this appendix we discuss, in the case where Conjecture 5.3 is not satisfied, the passage from stationary states of the SSM on the even sector $\widehat{X}_e^*$ to general ETIS states of the SSM with density $\hat{\rho} > 1$. By Theorem 5.1, the latter all have support on $\widehat{X}^*$; moreover, as observed in Section 5, the stationary states on $\widehat{X}^*$ include in addition only $\hat{\mu}^{(2)}$. Let $\mathcal{N}_{\hat{\rho}_e}$ denote the family of stationary states on $\widehat{X}_e^*$ with density $\hat{\rho}_e$, that is, states ν for which ν-a.e. $\mathbf{n} \in \widehat{X}_e^*$ satisfies

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathbf{n}_i = \lim_{N \to \infty} \frac{1}{N} \sum_{i=-N}^{-1} \mathbf{n}_i = \hat{\rho}_e. \tag{B.1}$$

Let $\overline{\mathcal{N}}_{\hat{\rho}_e}$ be the extremal elements of $\mathcal{N}_{\hat{\rho}_e}$. We analyze completely the structure of ETIS states on $\widehat{X}^*$, in terms of the states of $\overline{\mathcal{N}}_{\hat{\rho}_e}$, in two cases: when for each $\hat{\rho}_e$, $\overline{\mathcal{N}}_{\hat{\rho}_e}$ contains only TI states (Theorem B.1), and when for each $\hat{\rho}_e$, $\overline{\mathcal{N}}_{\hat{\rho}_e}$ is countable (Theorem B.5). The first case is quite simple:

Theorem B.1 *If all stationary states on $\widehat{X}_e^*$ are translation invariant then every ETIS state μ on $\widehat{X}^*$ of density $\hat{\rho}$ is of the form $\gamma_*(\mu_* \times \lambda)$, with λ an ergodic TI measure on S of density κ and $\mu_* \in \overline{\mathcal{N}}_{\hat{\rho}_e}$, where $\hat{\rho}_e = \hat{\rho} - \kappa$. Conversely, every state of this form is ETIS.*

Proof: If μ is an ETIS state on $\widehat{X}^*$ of density $\hat{\rho}$ then Lemma 5.6 implies that $\mu = \gamma_* \tilde{\mu}$ with $\tilde{\mu}(d\mathbf{n}\, d\sigma) = \mu_\sigma(d\mathbf{n})\lambda(d\sigma)$ and $\mu_{\tau\sigma} = \tau_* \mu_\sigma$. But if all stationary states on $\widehat{X}_e^*$ are TI then $\mu_\sigma = \mu_{\tau\sigma}$ for all σ, so that the ergodicity of λ implies that μ_σ is independent of σ, λ-a.s., and extremality of μ implies that this state, μ_*, must be ETIS on $\widehat{X}_e^*$. The converse is clear. ∎

We now consider the possibility of non-TI stationary states on $\widehat{X}_e^*$. In the next definition we introduce a class of stationary states on $\widehat{X}_e^* \times S$ which we call *basic* states, and a further restriction of this class to *irreducible* basic states.

Definition B.2 Let λ be an ergodic TI measure on S, let $\nu \in \overline{\mathcal{N}}_{\hat{\rho}_e}$ have period $n = n(\nu)$ under translation, and let m and q be positive integers such that $n = qm$.
(a) A (λ, m)-*partition* of S is an ordered family $A = \left(A_i\right)_{i=0}^{m-1}$ of subsets of S such that $\bigcup_i A_i = S$ and $A_i \cap A_j = \emptyset$ for $0 \le i < j \le m-1$, both up to sets of λ-measure zero, and such that the family is cyclically permuted by translation: $\tau(A_k) = A_{(k+1) \bmod m}$. Translations act on such partitions via $(\tau A)_k = A_{(k+1) \bmod m}$.
(b) Let A be a (λ, m)-partition of S, and let $\nu^{(q)} := q^{-1}\sum_{i=0}^{q-1} \tau_*^{im}\nu$. Then $\tilde{\mu}^{(\lambda,\nu,A)}$ is the state on $\widehat{X}_e^* \times S$ with $\tilde{\mu}^{(\lambda,\nu,A)}(d\mathbf{n}\, d\sigma) = \mu_\sigma^{(\nu,A)}(d\mathbf{n})\lambda(d\sigma)$, where $\mu_\sigma^{(\nu,A)} = \tau_*^k \nu^{(q)}$ iff $\sigma \in A_k$, i.e.,

$$\mu_\sigma^{(\nu,A)} = \sum_{k=0}^{m-1} \mathbf{1}_{A_k}(\sigma)\tau_*^k \nu^{(q)}. \tag{B.2}$$

$\tilde{\mu}^{(\lambda,\nu,A)}$ and $\gamma_* \tilde{\mu}^{(\lambda,\nu,A)}$ will be called *basic states*.
(c) The basic state $\tilde{\mu}^{(\lambda,\nu,A)}$ is *reduced* by the basic state $\tilde{\mu}^{(\lambda,\nu,A')}$ if A' is a proper refinement of A; $\tilde{\mu}^{(\lambda,\nu,A)}$ is then called *reducible*, and we say also that $\gamma_* \tilde{\mu}^{(\lambda,\nu,A)}$ is reducible. If $\tilde{\mu}^{(\lambda,\nu,A)}$ and $\gamma_* \tilde{\mu}^{(\lambda,\nu,A)}$ are not reducible then they are *irreducible*.

Observe that whether or not $\tilde{\mu}^{(\lambda,\nu,A)}$ is reducible depends only on λ, A, and n, the period of the orbit of ν under the action of τ_*: $\tilde{\mu}^{(\lambda,\nu,A)}$ is reducible

29

precisely when there is a (λ, m')-partition A' of S such that m' divides n and A' is a proper refinement of A. Equivalently, $\tilde{\mu}^{(\lambda,\nu,A)}$ is irreducible when the action of τ^n on A_0, equipped with the invariant measure $\lambda\big|_{A_0}$, is ergodic.

We next give some simple consequences of Definition B.2.

Lemma B.3 *Let $\tilde{\mu}^{(\lambda,\nu,A)}$ be a basic state. Then:*
(a) $\tilde{\mu}^{(\lambda,\nu,A)}$ is stationary.
(b) $\tilde{\mu}^{(\lambda,\tau_*\nu,\tau A)}(d\mathbf{n}\, d\sigma) = \tilde{\mu}^{(\lambda,\nu,A)}(d\mathbf{n}\, d\sigma)$.
(c) For any $\sigma \in S$, $\tau_*\mu_\sigma^{(\nu,A)} = \mu_{\tau\sigma}^{(\nu,A)} = \mu_\sigma^{(\nu,\tau^{-1}A)}$. In particular, $\tilde{\mu}^{(\lambda,\nu,A)}$ is TI.
Moreover, if λ has density κ then $\tilde{\mu}^{(\lambda,\nu,A)}$ has density $\hat{\rho}_e + \kappa$.
(d) If the basic state $\tilde{\mu}^{(\lambda,\nu,A')}$ reduces $\tilde{\mu}^{(\lambda,\nu,A)}$, and $A_0' \subset A_j$, then $A_i = \bigcup_{k=0}^{p-1} A'_{(i-j+mk) \bmod m}$ for $i = 0, \ldots, m-1$, where $m = |A|$ and $p = |A'|/|A|$.
Moreover, $\tilde{\mu}^{(\lambda,\nu,A)} = p^{-1} \sum_{k=0}^{p-1} \tilde{\mu}^{(\lambda,\nu,\tau^{-j-mk}A')}$.
(e) If $\tilde{\mu}^{(\lambda,\nu,A)}$ is irreducible then it cannot be written as a convex combination of other $\tilde{\mu}^{(\lambda',\nu',A')}$; more generally, we cannot have

$$\tilde{\mu}^{(\lambda,\nu,A)} = \int \tilde{\mu}^\beta \, \alpha(d\beta), \tag{B.3}$$

with α a probability measure on triples $\beta = (\lambda', \nu', A')$ which assigns zero probability to (λ, ν, A).

Proof: (a), (b), and (c) are immediate.
(d) The first statement follows from $A'_k = \tau^k A_0' \subset \tau^k A_j = A_{(k+j) \bmod m}$. For the second it suffices to prove that $\mu_\sigma^{(\nu,A)} = p^{-1} \sum_{k=0}^{p-1} \mu_\sigma^{(\nu,\tau^{-j-mk}A')}$ for all σ, and this follows from the first and (B.2), by straightforward rearrangement of sums.
(e) If (B.3) holds then ergodicity of λ implies that $\lambda' = \lambda$ α-a.s., and thus, λ-a.s.,

$$\mu_\sigma^{(\nu,A)} = \int \mu_\sigma^\beta \, \alpha(d\beta), \tag{B.4}$$

where $\mu_\sigma^\beta := \mu_\sigma^{(\nu',A')}$. Now we can use $\lim_{L\to\infty} (2L+1)^{-1} \sum_{i=-L}^{L} \tau_*^i \mu_\sigma^{(\nu,A)} = \nu^{(n)}$, the corresponding formula for $\mu_\sigma^{(\nu'A')}$, and the bounded convergence theorem, to conclude from (B.4) that $\nu^{(n)} = \int \nu'^{(n')} \alpha(d\beta)$ (with n' the period of ν'). Moreover, since ν is extremal, $\nu^{(n)}$ is ETIS. But then, since $\nu'^{(n')}$ is TIS, ν' must be a translate of ν α-a.s.

Without loss of generality, then, we may assume that $\nu' = \nu$, and with that it suffices to show that $A_0' \subset A_0$ (because τ acts cyclically on A and all

30

A'). Let G_i, $i = 0, \ldots, n-1$, support $\tau_*^i \nu$, with G_i and G_j disjoint for $i \neq j$. We have from (B.3) that

$$0 = \tilde{\mu}^{(\lambda, \nu, A)}(G_0 \times A_0^c) = \int \tilde{\mu}^{\beta}(G_0 \times A_0^c)\, \alpha(d\beta), \qquad \text{(B.5)}$$

so that for α-a.e. β, $\tilde{\mu}^{\beta}(G_0 \times A_0^c) = 0$ and so also $\tilde{\mu}^{\beta}(G_0 \times (A_0^c \cap A_0')) = 0$. But for $\sigma \in A_0'$, $\mu_\sigma^{(\nu, A')} = \nu^{(q')}$ (with $q' = n'/|A'|$), and so

$$0 = \tilde{\mu}^{\beta}(G_0 \times (A_0^c \cap A_0')) = \nu^{(q')}(G_0)\lambda(A_0^c \cap A_0') = \frac{1}{q'}\lambda(A_0^c \cap A_0'). \qquad \text{(B.6)}$$

Thus $A_0' \subset A_0$. ∎

Lemma B.4 *If $\overline{\mathcal{N}}_{\rho_e}$ is countable for each $\hat{\rho}_e \geq 1$ then each ETIS state on $\widehat{X}^*$ is an irreducible basic state.*

Proof: We write $\overline{\mathcal{N}}_{\hat{\rho}_e} = \{\nu_j\}_{j \in \mathbb{J}}$, with $\mathbb{J}$ either $\mathbb{Z}$ or a finite set $\{1, 2, \ldots, J\}$, and allow the translation operator τ to act on $\mathbb{J}$ via $\nu_{\tau j} = \tau_* \nu_j$. Let $\bar{\mathbb{J}}$ denote the set of orbits in $\mathbb{J}$ under translation.

Now let μ be an ETIS state on $\widehat{X}$ of density $\hat{\rho}$. Then from Lemma 5.6 we know that $\mu = \gamma_* \tilde{\mu}$ with $\tilde{\mu}(d\mathbf{n}\, d\sigma) = \mu_\sigma(d\mathbf{n})\lambda(d\sigma)$, λ ergodic, μ_σ stationary, and $\mu_{\tau\sigma} = \tau_* \mu_\sigma$. For each $\sigma \in S$ we have $\mu_\sigma = \sum_{j \in \mathbb{J}} a_{\sigma, j} \nu_j$, with coefficients $a_{\sigma, j}$ which, from translation invariance, satisfy $a_{\tau\sigma, \tau j} = a_{\sigma, j}$. Now note that if $c \in \bar{\mathbb{J}}$ then $Z_\sigma^{(c)} := \sum_{j \in c} a_{\sigma, j}$ is TI and hence constant λ-a.s.; from now on we write this as $Z^{(c)}$. Define $\mu_\sigma^{(c)}$ by $\mu_\sigma^{(c)} := (Z^{(c)})^{-1} \sum_{j \in c} a_{\sigma, j} \nu_j$ and $\tilde{\mu}^{(c)}$ by $\tilde{\mu}^{(c)}(d\mathbf{n}\, d\sigma) = \mu_\sigma^{(c)}(d\mathbf{n})\lambda(d\sigma)$, so that $\tilde{\mu} = \sum_{c \in \bar{\mathbb{J}}} Z^{(c)} \tilde{\mu}^{(c)}$; since the $\tilde{\mu}^{(c)}$ are TIS, the extremality of $\tilde{\mu}$ implies that $\tilde{\mu} = \tilde{\mu}^{(c_0)}$ for some $c_0 \in \bar{\mathbb{J}}$.

We claim that c_0 must be a finite set. For otherwise we may fix $j_0 \in c_0$ and define $f : S \to \mathbb{Z}$ by

$$f(\sigma) := \min\{i \in \mathbb{Z} \mid a_{\sigma, \tau^i j_0} = \max_{i' \in \mathbb{Z}} a_{\sigma, \tau^{i'} j_0}\}; \qquad \text{(B.7)}$$

f satisfies $f(\tau\sigma) = f(\sigma) + 1$, so that by the translation invariance of λ the sets $f^{-1}(\{k\})$, $k \in \mathbb{Z}$, have equal λ-measure, a contradiction.

Thus we are reduced to the case where c_0 contains n elements, that is, if $j_0 \in c$ then n is the minimal period for ν_{j_0}, and

$$\mu_\sigma = \sum_{i=0}^{n-1} a_{\sigma, \tau^i j_0} \nu_{\tau^i j_0}. \qquad \text{(B.8)}$$

31

We regard $a_\sigma := (a_{\sigma,\tau^i j_0})_{i=0}^{n-1}$ as an element of $\mathbb{R}^n$ and let translations act on $\mathbb{R}^n$ via $\tau(b_0, \ldots, b_{n-1}) = (b_{n-1}, b_0, \ldots, b_{n-2})$, so that $a_{\tau\sigma} = \tau a_\sigma$. The orbit of a_σ under this action is independent of translations in σ and hence constant, λ-a.s.; let θ denote this orbit.

$|\theta|$ is a positive integer m satisfying $n = qm$ for some integer q. Let $\theta = \{v_0, \ldots, v_{m-1}\}$, with $\tau v_k = v_{(k+1) \bmod m}$, let $A_k := \{\sigma \mid a_\sigma = v_k\}$, and note that if $w \in \mathbb{R}^m$ is defined by $w_i = qv_{0,i}$ then w is independent of σ and $\sum_{i=0}^{m-1} w_i = 1$. Then from (B.8) we find, using (B.2), that $\mu_\sigma = \sum_{r=0}^{m-1} w_i \mu_\sigma^{(\nu_{j_0}, \tau^{-i}A)}$. Thus

$$\tilde{\mu} = \sum_{r=0}^{m-1} w_i \tilde{\mu}^{(\lambda, \nu_{j_0}, \tau^{-i}A)}. \tag{B.9}$$

But since $\tilde{\mu}$ is extremal, precisely one of the w_i can be nonzero, so that $\tilde{\mu}$ is basic. It then follows from Lemma B.3 (d) that $\tilde{\mu}$, and hence μ, is irreducible basic. ∎

Now we can give the second main result of this appendix.

Theorem B.5 *If $\overline{\mathcal{N}}_{\rho_e}$ is countable for each $\hat{\rho}_e \geq 1$ then the ETIS states on $\widehat{X}^*$ are precisely the basic irreducible states.*

Proof: Lemma B.4 tells us that every ETIS state is an irreducible basic state. It follows from this that the decomposition of any irreducible basic state into ETIS components must be of the form (B.3) and hence, by Lemma B.3 (e), that such a state must be ETIS. ∎

References

[1] A. Ayyer, S. Goldstein, J. L. Lebowitz, and E. R. Speer, Stationary States of the One-Dimensional Facilitated Asymmetric Exclusion Process.

[2] J. Baik, G. Barraquand, I. Corwin, and T. Suidan. Facilitated Exclusion Process. *Computation and Combinatorics in Dynamics, Stochastics and Control*, 1–35, Abel Symp. **13**, Springer, Cham, 2018.

[3] Urna Basu and P. K. Mohanty, Active-Absorbing-State Phase Transition Beyond Directed Percolation: A Class of Exactly Solvable Models. *Phys. Rev. E* **79**, 041143 (2009).

[4] Oriane Blondel, Clément Erignoux, Makiko Sasada, and Marielle Simon, Hydrodynamic limit for a Facilitated Exclusion Process. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* **56**, 667714 (2020).

[5] Oriane Blondel, Clément Erignoux, and Marielle Simon, Stefan Problem for a Non-Ergodic Facilitated Exclusion Process. *Probability and Mathematical Physics* **2**, 127–178 (2021).

[6] Dayne Chen and Linjie Zhao, The Limiting Behavior of the FTASEP with Product Bernoulli Initial Distribution. arXiv:1801.10612v1 [math PR].

[7] M. R. Evans and T. Hannye, Nonequilibrium Statistical Mechanics of the Zero-Range Process and Related Models. *J. Phys. A: Math. Gen.* bf 38 R195 (2005).

[8] P. A. Ferrari, E. Presutti, and M. E. Vares, Local Equilibrium for a One-Dimensional Zero Range Process. *Stoch. Proc. Appl* **26**, 31-45 (1987).

[9] Alan Gabel, P. L. Krapivsky, and S. Redner, Facilitated Asymmetric Exclusion. *Phys. Rev. Lett.* **105**, 210603 (2010).

[10] A. Gabel and S. Redner, Cooperativity-Driven Singularities in Asymmetric Exclusion, *J. Stat. Mech.* **2011**, P06008 (2011).

[11] S. Goldstein, J. L. Lebowitz and E. R. Speer, Exact Solution of the F-TASEP . J. Stat. Mech. 123202 (2019).

[12] S. Goldstein, J. L. Lebowitz and E. R. Speer, The Discrete-Time Facilitated Totally Asymmetric Simple Exclusion Process. arXiv:2003.04995 [math-ph]. To appear in *Pure and Applied Functional Analysis*.

[13] Charles M. Grinstead and J. Laurie Snell, *Introduction to Probability, Revised Edition.* American Mathematical Society, Providence, 2012. Available at `https://chance.dartmouth.edu/teaching_aids/books_articles/probability_book/book.html`.

[14] Daniel Hexner and Dov Levine, Hyperuniformity of Critical Absorbing States. *Phys. Rev. Lett.* **114**, 110602 (2015).

[15] Olav Kallenberg, *Foundations of Modern Probability, Second Edition.* Springer, New York, 2002.

[16] Stefano Martiniani, Paul M. Chaikin, and Dov Levine, Quantifying Hidden Order out of Equilibrium. *Phys. Rev. X* **9**, 011031 (2019).

[17] Mário J. de Oliveira, Conserved Lattice Gas Model with Infinitely Many Absorbing States in One Dimension. *Phys. Rev. E* **71**, 016112 (2005).

[18] Michela Rossi, Romualdo Pastor-Satorras, and Alessandro Vespignani, Universality Class of Absorbing Phase Transitions with a Conserved Field. *Phys. Rev. Lett.* **85**, 1803 (2000).

[19] Vladas Sidoravicius and Augusto Teixeira, Absorbing-State Transition for Stochastic Sandpiles and Activated Random Walks. *Electron. J. Probab.* **22**, no. 33, 135 (2017).