

A Microscopic Derivation of the Quantum Mechanical Formal Scattering Cross Section

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Abstract: We prove that the empirical distribution of crossings of a “detector” surface by scattered particles converges in appropriate limits to the scattering cross section computed by stationary scattering theory. Our result, which is based on Bohmian mechanics and the flux-across-surfaces theorem, is the first derivation of the cross section starting from first microscopic principles.

1. Introduction

The central quantity in a scattering experiment is the empirical cross section, which reflects the number of particles that are scattered in a given solid angle per unit time. In this paper we shall derive the theoretical prediction for the cross section starting from a microscopic model describing a realistic scattering situation. We confine ourselves to the case of potential scattering of a nonrelativistic, (spinless) quantum particle and leave the many-particle case for future research. This paper is in fact a technical elaboration and continuation of our article “Scattering theory from microscopic first principles” [9].

The common approaches to the foundations of scattering theory take for granted that “an experimentalist generally prepares a state ... at $t \rightarrow -\infty$, and then measures what this state looks like at $t \rightarrow +\infty$ ” (cf. [25], p. 113), meaning that the asymptotic expressions are “all there is,” as if they are not the asymptotic expressions of some other formula, however complicated, describing the scattering situation as it really is, namely happening at finite distances and at finite times. Thus a truly microscopic derivation starting from first principles must provide firstly a formula for the empirical cross section, which by the law of large numbers approximates its expectation value, and which is computed from the underlying theory. Secondly, that formula should apply to the realistic finite-times and finite-distances situation, from which eventually the usual Born formula should emerge by taking appropriate limits.¹

¹ For a detailed discussion of the scattering regime see [8].

We shall present a Bohmian analysis of the scattering cross section. With a particle trajectory we can ask for example whether or not that trajectory eventually crosses a distant spherical surface and if it does when and where it first crosses that surface. Similarly, for a beam of particles we can ask for the number of particles in the beam that first crosses the surface in a given solid angle Σ . From a Bohmian perspective it appears reasonable to identify this number with detection events in a scattering experiment. We thus model in this paper the measured cross section using the number $N^*(\Sigma)$ of first crossings of Σ . This will of course depend on many parameters encoding the experimental setup, e.g. the distances R and L of the detector and the particle source from the scattering center, the details of the beam including its profile A and the wave functions of the particles in the beam, as well as on the length of the time interval τ during which the particles are emitted. We shall show in this paper that when these parameters are suitably scaled, $\frac{N^*(\Sigma)}{\tau}$ is well approximated by the usual Born formula for the scattering cross section in terms of the T -matrix, i.e.,

$$\lim \frac{N^*(\Sigma)}{\tau} = 16\pi^4 \int_{\Sigma} |T(k_0\omega, \mathbf{k}_0)|^2 d\Omega, \quad (1)$$

where $\hbar\mathbf{k}_0$ is the initial momentum of the particles.

The paper is organized as follows: We collect first some mathematical notions and facts as well as recent results of scattering theory. In Sect. 3 we define the relevant random variables associated with the surface-crossings of a single particle and relate their distribution to the quantum probability current density. In Sect. 4 we model the beam by a suitable point process and in Sect. 5 we define $N^*(\Sigma)$ in terms of this point process. A precise description of the limit procedure will be presented in Sect. 6. Our main results, Theorem 1 and 2, are stated in Sect. 7 and are proven in Sect. 8.

2. The Mathematical Framework of Potential Scattering

We list those results of scattering theory (e.g. [2, 7, 11, 14, 16, 18–20, 22]) which are essential for the proof of Theorem 1 and Theorem 2 in Sect. 8.

We use the usual description of a nonrelativistic spinless one-particle system by the Hamiltonian H (we use natural units $\hbar = m = 1$),

$$H := -\frac{1}{2}\Delta + V(\mathbf{x}) =: H_0 + V(\mathbf{x}),$$

with the real-valued potential $V \in (V)_n$, defined as follows:

Definition 1. V is in $(V)_n$, $n=2,3,4,\dots$, if

- (i) $V \in L^2(\mathbb{R}^3)$,
- (ii) V is locally Hölder continuous except, perhaps, at a finite number of singularities,
- (iii) there exist positive numbers δ , C , R_0 such that

$$|V(\mathbf{x})| \leq C\langle x \rangle^{-n-\delta} \text{ for } x \geq R_0,$$

where $\langle \cdot \rangle := (1 + (\cdot)^2)^{\frac{1}{2}}$.

Under these conditions (see e.g. [16]) H is self-adjoint on the domain $D(H) = D(H_0) = \{f \in L^2(\mathbb{R}^3) : \int |k^2 \hat{f}(\mathbf{k})|^2 d^3k < \infty\}$ ($k = |\mathbf{k}|$), where $\hat{f} := \mathcal{F}f$ is the Fourier transform

$$\hat{f}(\mathbf{k}) := (2\pi)^{-\frac{3}{2}} \int e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) d^3x. \quad (2)$$

Let $U(t) = e^{-iHt}$. Since H is self-adjoint on the domain $D(H)$, $U(t)$ is a strongly continuous one-parameter unitary group on $L^2(\mathbb{R}^3)$. Let $\phi \in D(H)$. Then $\phi_t \equiv U(t)\phi \in D(H)$ and satisfies the Schrödinger equation

$$i \frac{\partial}{\partial t} \phi_t(\mathbf{x}) = H\phi_t.$$

In a typical scattering experiment the scattered particles move almost freely far away from the scattering center. ‘‘Far away’’ in position space can also be phrased as ‘‘long before’’ and ‘‘long after’’ the scattering event takes place. So for the ‘‘scattering states’’ ψ there are asymptotes $\psi_{\text{in}}, \psi_{\text{out}}$ defined by

$$\begin{aligned} \lim_{t \rightarrow -\infty} \|e^{-iH_0t} \psi_{\text{in}}(\mathbf{x}) - e^{-iHt} \psi(\mathbf{x})\| &= 0, \\ \lim_{t \rightarrow \infty} \|e^{-iH_0t} \psi_{\text{out}}(\mathbf{x}) - e^{-iHt} \psi(\mathbf{x})\| &= 0. \end{aligned} \quad (3)$$

From this it is natural to define the wave operators $\Omega_{\pm} : L^2(\mathbb{R}^3) \rightarrow \text{Ran}(\Omega_{\pm})$ by the strong limits

$$\Omega_{\pm} := \text{s-lim}_{t \rightarrow \pm\infty} e^{iHt} e^{-iH_0t}. \quad (4)$$

These wave operators map the incoming and outgoing asymptotes to their corresponding scattering states. Ikebe [14] proved that for a potential $V \in (V)_n$ the wave operators exist and have the range

$$\text{Ran}(\Omega_{\pm}) = \mathcal{H}_{\text{cont}}(H) = \mathcal{H}_{\text{a.c.}}(H).$$

(This property is called asymptotic completeness.) Hence, the scattering states consist of states with absolutely continuous spectrum and the singular continuous spectrum of H is empty. In addition Ikebe [14] showed that the Hamiltonian has no positive eigenvalues. Then we have for every $\psi \in \mathcal{H}_{\text{a.c.}}(H)$ asymptotes $\psi_{\text{in}}, \psi_{\text{out}} \in L^2(\mathbb{R}^3)$ with

$$\Omega_- \psi_{\text{in}} = \psi = \Omega_+ \psi_{\text{out}}. \quad (5)$$

On $D(H_0)$ the wave operators satisfy the so-called intertwining property

$$H\Omega_{\pm} = \Omega_{\pm}H_0,$$

while on $\mathcal{H}_{\text{a.c.}}(H) \cap D(H)$ we have that

$$H_0\Omega_{\pm}^{-1} = \Omega_{\pm}^{-1}H. \quad (6)$$

The scattering operator $S : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ is given by

$$S := \Omega_+^{-1}\Omega_-,$$

while using the identity I , the T -operator is given by

$$T := S - I. \quad (7)$$

If the system is asymptotically complete, the ranges of the wave operators are equal and thus S is unitary. Since the wave operator maps a scattering state onto its asymptotic state, the scattering operator maps the incoming asymptote ψ_{in} onto the corresponding out state ψ_{out} . The formula for the T -matrix, which holds in the L^2 -sense, is given by (see e.g., Theorem XI.42 in [19])

$$\widehat{T}g(\mathbf{k}) = -2\pi i \int_{\mathbf{k}'=k} T(\mathbf{k}, \mathbf{k}') \widehat{g}(\mathbf{k}') k' d\Omega', \quad (8)$$

for $g \in \mathcal{S}(\mathbb{R}^3)$ (Schwartz space) such that \widehat{g} has support in a spherical shell.² $T(\mathbf{k}, \mathbf{k}')$ is given by (see e.g., [19]):

$$T(\mathbf{k}, \mathbf{k}') = (2\pi)^{-3} \int e^{-i\mathbf{k}\cdot\mathbf{x}} V(\mathbf{x}) \varphi_-(\mathbf{x}, \mathbf{k}') d^3x, \quad (9)$$

where φ_- (as well as φ_+) are eigenfunctions of H defined by Lemma 1 below. Since the eigenfunctions φ_{\pm} are bounded and continuous (cf. Lemma 2), we can conclude that $T(\mathbf{k}, \mathbf{k}')$ is bounded and continuous on $\mathbb{R}^3 \times \mathbb{R}^3$, if the potential is in $(V)_3$. Then the formula (8) can be proved for $g \in \mathcal{S}(\mathbb{R}^3)$ without any restriction on the momentum support by the same method as in [19].

We will need the time evolution of a state $\psi \in \mathcal{H}_{\text{a.c.}}(H)$ with the Hamiltonian H . Its diagonalization on $\mathcal{H}_{\text{a.c.}}(H)$ is given by the eigenfunctions φ_{\pm} :

$$\left(-\frac{1}{2}\Delta + V(\mathbf{x})\right)\varphi_{\pm}(\mathbf{x}, \mathbf{k}) = \frac{k^2}{2}\varphi_{\pm}(\mathbf{x}, \mathbf{k}). \quad (10)$$

Inverting $(-\frac{1}{2}\Delta - \frac{k^2}{2})$ one obtains the Lippmann-Schwinger equation. We recall the main parts of a result on this due to Ikebe in [14] which is collected in the present form in [22].

Proposition 1. *Let $V \in (V)_2$. Then for any $\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}$ there are unique solutions $\varphi_{\pm}(\cdot, \mathbf{k}) : \mathbb{R}^3 \rightarrow \mathbb{C}$ of the Lippmann-Schwinger equations*

$$\varphi_{\pm}(\mathbf{x}, \mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{1}{2\pi} \int \frac{e^{\mp ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} V(\mathbf{x}') \varphi_{\pm}(\mathbf{x}', \mathbf{k}) d^3x', \quad (11)$$

which satisfy the boundary conditions $\lim_{|\mathbf{x}|\rightarrow\infty} (\varphi_{\pm}(\mathbf{x}, \mathbf{k}) - e^{i\mathbf{k}\cdot\mathbf{x}}) = 0$, which are also classical solutions of the stationary Schrödinger equation (10), and are such that:

(i) For any $f \in L^2(\mathbb{R}^3)$ the generalized Fourier transforms³

$$(\mathcal{F}_{\pm}f)(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \text{l. i. m.} \int \varphi_{\pm}^*(\mathbf{x}, \mathbf{k}) f(\mathbf{x}) d^3x$$

exist in $L^2(\mathbb{R}^3)$.

² In [19] Equation (8) was proven outside an “exceptional set”. For our class of potentials the “exceptional set” is empty. The additional factor $\frac{1}{2}$ in [19] comes from the different definition of H_0 .

³ l. i. m. \int is a shorthand notation for $\lim_{R\rightarrow\infty} \int_{B_R}$, where s-lim denotes the limit in the L^2 -norm and B_R a ball with radius R around the origin.

(ii) $\text{Ran}(\mathcal{F}_\pm) = L^2(\mathbb{R}^3)$. Moreover $\mathcal{F}_\pm : \mathcal{H}_{a.c.}(H) \rightarrow L^2(\mathbb{R}^3)$ are unitary and the inverses of these unitaries are given by

$$(\mathcal{F}_\pm^{-1} f)(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \text{l. i. m.} \int \varphi_\pm(\mathbf{x}, \mathbf{k}) f(\mathbf{k}) d^3 k.$$

(iii) For any $f \in L^2(\mathbb{R}^3)$ the relations $\Omega_\pm f = \mathcal{F}_\pm^{-1} \mathcal{F} f$ hold, where \mathcal{F} is the ordinary Fourier transform given by (2).

(iv) For any $f \in \text{D}(H) \cap \mathcal{H}_{a.c.}(H)$ we have:

$$Hf(\mathbf{x}) = \left(\mathcal{F}_\pm^{-1} \frac{k^2}{2} \mathcal{F}_\pm f \right) (\mathbf{x}),$$

and therefore for any $f \in \mathcal{H}_{a.c.}(H)$,

$$e^{-iHt} f(\mathbf{x}) = \left(\mathcal{F}_\pm^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}_\pm f \right) (\mathbf{x}).$$

In order to apply stationary phase methods we will need estimates on the derivatives of the generalized eigenfunctions:

Proposition 2. Let $V \in (V)_n$ for some $n \geq 3$. Then:

(i) $\varphi_\pm(\mathbf{x}, \cdot) \in C^{n-2}(\mathbb{R}^3 \setminus \{0\})$ for all $\mathbf{x} \in \mathbb{R}^3$ and the partial derivatives⁴
 $\partial_{\mathbf{k}}^\alpha \varphi_\pm(\mathbf{x}, \mathbf{k})$, $|\alpha| \leq n - 2$, are continuous with respect to \mathbf{x} and \mathbf{k} .

If, in addition, zero is neither an eigenvalue nor a resonance of H , then

(ii) $\sup_{\mathbf{x} \in \mathbb{R}^3, \mathbf{k} \in \mathbb{R}^3} |\varphi_\pm(\mathbf{x}, \mathbf{k})| < \infty$,

for any α with $|\alpha| \leq n - 2$ there is a $c_\alpha < \infty$ such that

(iii) $\sup_{\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}} |\kappa^{|\alpha|-1} \partial_{\mathbf{k}}^\alpha \varphi_\pm(\mathbf{x}, \mathbf{k})| < c_\alpha \langle x \rangle^{|\alpha|}$, with $\kappa := \frac{k}{\langle k \rangle}$,

and for any $l \in \{1, \dots, n - 2\}$ there is a $c_l < \infty$ such that

(iv) $\sup_{\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}} \left| \frac{\partial^l}{\partial k^l} \varphi_\pm(\mathbf{x}, \mathbf{k}) \right| < c_l \langle x \rangle^l$, where $\frac{\partial}{\partial k}$ is the radial partial derivative in \mathbf{k} -space.

Remark 1. This proposition, except the assertion (iii), was proved in [22], Theorem 3.1. Assertion (iii) repairs a false statement in Theorem 3.1 which did not include the necessary $\kappa^{|\alpha|-1}$ factor, which we have in (iii). For $|\alpha| = 1$, which was the important case in that paper, there is however no difference. We have commented on the proof of this corrected version in [11].

Remark 2. Zero is a resonance of H if there exists a solution f of $Hf = 0$ such that $\langle x \rangle^{-\gamma} f \in L^2(\mathbb{R}^3)$ for any $\gamma > \frac{1}{2}$ but not for $\gamma = 0$.⁵ The appearance of a zero eigenvalue or resonance can be regarded as an exceptional event: For a Hamiltonian $H = H_0 + cV$, $c \in \mathbb{R}$, this can only happen for c in a discrete subset of \mathbb{R} , see [1], p. 20 and [15], p. 589.

As a simple consequence of Proposition 2 we obtain

⁴ We use the usual multi-index notation: $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \in \mathbb{N}_0$, $\partial_{\mathbf{k}}^\alpha f(\mathbf{k}) : \partial_{k_1}^{\alpha_1} \partial_{k_2}^{\alpha_2} \partial_{k_3}^{\alpha_3} f(\mathbf{k})$ and $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$.

⁵ There are various definitions, see e.g. [26], p. 552, [1], p.20 and [15], p. 584.

Corollary 1. *Let $V \in (V)_3$ and let zero be neither an eigenvalue nor a resonance of H . Then the T -matrix defined by (9) is a bounded and continuous function on $\mathbb{R}^3 \times \mathbb{R}^3$. Moreover, if $V \in (V)_n$, for some $n \geq 3$ we have for all multi-indices α with $|\alpha| \leq n - 3$ a constant $c_\alpha > 0$ such that*

$$\sup_{\mathbf{k}' \in \mathbb{R}^3, \mathbf{k} \in \mathbb{R}^3 \setminus \{0\}} \kappa^{|\alpha|-1} |\partial_{\mathbf{k}}^\alpha T(\mathbf{k}', \mathbf{k})| \leq c_\alpha. \quad (12)$$

With the regularity of the generalized eigenfunctions one can prove the flux-across-surfaces theorem. The quantum probability current density (=quantum flux density) is given by

$$\mathbf{j}^{\psi_t}(\mathbf{x}) := -\frac{i}{2}(\psi_t^*(\mathbf{x})\nabla\psi_t(\mathbf{x}) - \psi_t(\mathbf{x})\nabla\psi_t^*(\mathbf{x})). \quad (13)$$

For $\psi_t(\mathbf{x})$ a solution of the Schrödinger equation we have the identity

$$\frac{\partial |\psi_t(\mathbf{x})|^2}{\partial t} + \operatorname{div} \mathbf{j}^{\psi_t}(\mathbf{x}) = 0,$$

which has the form of a continuity equation. The flux-across-surfaces theorem can be naturally proven for the following class of wave functions (in the following definition we have the Fourier transform of ψ_{out} , $\widehat{\psi}_{\text{out}}(\mathbf{k}) = \int \varphi_+(\mathbf{x}, \mathbf{k})\psi(\mathbf{x})d^3x$ (cf. Proposition 1), in mind):

Definition 2. *A function $f : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{C}$ is in \mathcal{G}^+ if there is a constant $C \in \mathbb{R}_+$ with:*

$$\begin{aligned} |f(\mathbf{k})| &\leq C\langle k \rangle^{-15}, \\ |\partial_{\mathbf{k}}^\alpha f(\mathbf{k})| &\leq C\langle k \rangle^{-6}, \quad |\alpha| = 1, \\ |\kappa \partial_{\mathbf{k}}^\alpha f(\mathbf{k})| &\leq C\langle k \rangle^{-5}, \quad |\alpha| = 2, \quad \kappa = \frac{k}{\langle k \rangle}, \\ \left| \frac{\partial^2}{\partial k^2} f(\mathbf{k}) \right| &\leq C\langle k \rangle^{-3}. \end{aligned}$$

With this definition we have

Proposition 3. *(Flux-across-surfaces theorem (FAST)). Suppose $V \in (V)_4$ and that zero is neither a resonance nor an eigenvalue of H . Suppose $\widehat{\psi}_{\text{out}}(\mathbf{k}) \in \mathcal{G}^+$ and let $\psi = \Omega_+\psi_{\text{out}}$. Then $\psi_t(\mathbf{x}) = e^{-iHt}\psi(\mathbf{x})$ is continuously differentiable except at the singularities of V , for any measurable set $\Sigma \subseteq S^2$ and any $T \in \mathbb{R}$ $\mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\sigma dt$ is absolutely integrable on $R\Sigma \times [T, \infty)$ for R sufficiently large and*

$$\lim_{R \rightarrow \infty} \int_T^\infty \int_{R\Sigma} \mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\sigma dt = \lim_{R \rightarrow \infty} \int_T^\infty \int_{R\Sigma} |\mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\sigma| dt = \int_{C_\Sigma} |\widehat{\psi}_{\text{out}}(\mathbf{k})|^2 d^3k, \quad (14)$$

where $R\Sigma := \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = R\boldsymbol{\omega}, \boldsymbol{\omega} \in \Sigma\}$, $C_\Sigma := \{\mathbf{k} \in \mathbb{R}^3 : \frac{\mathbf{k}}{k} \in \Sigma\}$ is the cone given by Σ and $d\sigma$ is the outward-directed surface element on RS^2 .

The proof can be found in [11].

The FAST plays a crucial role in the proof of our main results, Theorem 1 and Theorem 2. Its importance for scattering theory was first pointed out in [6].

3. The Quantum Flux, Crossing Statistics and Bohmian Mechanics

In Bohmian mechanics, see [5], the particle has a position \mathbf{Q}_t that evolves via the equations

$$\begin{aligned} \frac{d}{dt} \mathbf{Q}_t &= \mathbf{v}^{\psi_t}(\mathbf{Q}_t) = \text{Im} \frac{\nabla \psi_t}{\psi_t}(\mathbf{Q}_t), \\ i \frac{\partial}{\partial t} \psi_t(\mathbf{x}) &= H \psi_t(\mathbf{x}). \end{aligned} \quad (15)$$

According to the quantum equilibrium hypothesis ([10], Born's law), the positions of particles in an ensemble of particles each having wave function ψ are always $|\psi|^2$ -distributed. Note that if \mathbf{Q}_0 is $|\psi_0|^2$ -distributed then \mathbf{Q}_t is $|\psi_t|^2$ -distributed.

Under two assumptions we have the $|\psi_0|^2$ almost-sure existence and uniqueness of the Bohmian dynamics:

A 1. The initial wave function ψ_0 is normalized, $\|\psi_0\| = 1$, and $\psi_0 \in C^\infty(H) = \bigcap_{n=1}^{\infty} D(H^n)$.

A 2. The potential V is in V_2 and C^∞ except, perhaps, at a finite number of singularities.

(See Berndl et al. [4], Theorem 3.1 and Corollary 3.2 for the proof, as well as Theorem 3 and Corollary 4 in [23]. The conditions in [4, 23] are much more general. In our context, however, we have to restrict to the case where $V \in (V)_2$.) Hence, depending on the initial position $\mathbf{q}_0 \in \Omega_0$, where Ω_0 is the set of ‘‘good’’ points, the particle has the trajectory $\mathbf{Q}_t^\psi(\mathbf{q}_0)$. On the set of ‘‘good’’ points, $\psi_0(\mathbf{x})$ is different from zero and is differentiable. The complement $\mathbb{R}^3 \setminus \Omega_0$ of Ω_0 has measure 0 (with respect to $|\psi_0|^2$).

Given a trajectory $\mathbf{Q}_t^\psi(\mathbf{q}_0)$, $\mathbf{q}_0 \in \Omega_0$, we can define the number of crossings in a natural way. For the surface $R\Sigma \subset R\mathbb{S}^2$ with unit and normal vector $\mathbf{n}(\mathbf{x}) = \frac{\mathbf{x}}{x}$, $\mathbf{x} \in R\Sigma$ we define $N_+^\psi(R\Sigma)$ on Ω_0 by:

$$N_+^\psi(R\Sigma)(\mathbf{q}_0) := \left| \left\{ t \geq 0 \mid \mathbf{Q}_t^\psi(\mathbf{q}_0) \in R\Sigma \text{ and } \dot{\mathbf{Q}}_t^\psi(\mathbf{q}_0) \cdot \mathbf{n}(\mathbf{Q}_t^\psi(\mathbf{q}_0)) > 0 \right\} \right|, \quad (16)$$

the number of crossings of the trajectory $\mathbf{Q}_t^\psi(\mathbf{q}_0)$ through $R\Sigma$ in the direction of the orientation in the time interval $[0, \infty)$ (‘‘problematical crossings’’ where the velocity is ‘‘orthogonal’’ to the orientation of $R\Sigma$ have measure zero and need not concern us, see [3], p. 28-34). If $N_+^\psi(R\Sigma)(\mathbf{q}_0) \geq 1$, we can define $t_{\text{exit}}^{R\Sigma}$ as the time when the particle crosses the surface $R\Sigma$ in the positive direction for the first time:

$$t_{\text{exit}}^{R\Sigma}(\mathbf{q}_0) := \min \left\{ t \geq 0 \mid \mathbf{Q}_t^\psi(\mathbf{q}_0) \in R\Sigma \text{ and } \dot{\mathbf{Q}}_t^\psi(\mathbf{q}_0) \cdot \mathbf{n}(\mathbf{Q}_t^\psi(\mathbf{q}_0)) > 0 \right\}. \quad (17)$$

In the case that the particle does not cross the surface in the positive direction, we set

$$t_{\text{exit}}^{R\Sigma}(\mathbf{q}_0) := \infty, \text{ if } N_+^\psi(R\Sigma)(\mathbf{q}_0) = 0. \quad (18)$$

Analogously to (16) we have $N_-^\psi(R\Sigma)$, the number of crossings in the opposite direction. For convenience we define $N_+^\psi(R\Sigma)$ and $N_-^\psi(R\Sigma)$ on the whole of \mathbb{R}^3 by setting

$N_+^\psi(R\Sigma) = N_-^\psi(R\Sigma) = 0$ for all $\mathbf{q}_0 \in \mathbb{R}^3 \setminus \Omega_0$. Then we can define the number of signed crossings on \mathbb{R}^3 by

$$N_{\text{sig}}^\psi(R\Sigma) := N_+^\psi(R\Sigma) - N_-^\psi(R\Sigma). \quad (19)$$

The total number of crossings defined on \mathbb{R}^3 is then

$$N_{\text{tot}}^\psi(R\Sigma) := N_+^\psi(R\Sigma) + N_-^\psi(R\Sigma). \quad (20)$$

These quantities are random variables on the space \mathbb{R}^3 of initial conditions, see [3], Lemma 4.2. The expectation values of $N_{\text{sig}}^\psi(R\Sigma)$ and $N_{\text{tot}}^\psi(R\Sigma)$ are given by flux integrals and are finite, see Proposition 4 below. This means that $N_{\text{sig}}^\psi(R\Sigma)$ and $N_{\text{tot}}^\psi(R\Sigma)$ are almost surely finite. Before we give a precise statement we argue heuristically for the connection between the quantum flux and the expectation values. For a particle to cross an infinitesimal surface $d\boldsymbol{\sigma} := \mathbf{n}d\sigma$ in a time interval $[t, t + dt)$, it must be at time t in the appropriate cylinder of size $|\mathbf{v}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma} dt|$. The probability is therefore

$$|\psi_t(\mathbf{x})|^2 |\mathbf{v}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma} dt| = |\mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma} dt|.$$

Because the intervals are infinitesimal, we have for $N_{\text{sig}}^\psi(dt, d\boldsymbol{\sigma}) \in \{-1, 0, 1\}$,⁶ where the sign will be the same as that of $\mathbf{j} \cdot d\boldsymbol{\sigma}$. Therefore $\mathbb{E}(N_{\text{sig}}^\psi(dt, d\boldsymbol{\sigma})) = \mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma} dt$ and integration over $R\Sigma$ and $[0, \infty)$ yields (21). The precise statement is:

Proposition 4. *Let A1 and A2 be satisfied. In addition suppose that the conditions of Proposition 3 are satisfied. Then for sufficiently large R the expectation values of $N_{\text{sig}}^\psi(R\Sigma)$ and $N_{\text{tot}}^\psi(R\Sigma)$ are finite and*

$$\mathbb{E}(N_{\text{sig}}^\psi(R\Sigma)) = \int_0^\infty \int_{R\Sigma} \mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma} dt, \quad (21)$$

$$\mathbb{E}(N_{\text{tot}}^\psi(R\Sigma)) = \int_0^\infty \int_{R\Sigma} |\mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma}| dt. \quad (22)$$

The proof of Proposition 4 can be found in [3], pp. 34–37, and under slightly different conditions in [24]. The results in the references hold under more general conditions on the surfaces.

Consider now a scattering situation where we want to calculate the number of first crossings. The detector corresponds to the surface $R\Sigma := \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = R\boldsymbol{\omega}, \boldsymbol{\omega} \in \Sigma \subset S^2\} \subset RS^2$. Then we define $N_{\text{det}}^\psi([0, \infty), R, \Sigma)$ to be equal to one if the particle with the wave function $\psi_0 = \psi$ is “detected” in $[0, \infty)$ and zero otherwise. More precisely,

$$N_{\text{det}}^\psi(R, \Sigma) : \mathbb{R}^3 \rightarrow \{0, 1\},$$

⁶ $N_{\text{sig}}^\psi(dt, d\boldsymbol{\sigma})$ is the number of signed crossings in the time interval $[t, t + dt)$ through the surface $d\boldsymbol{\sigma}$.

$$N_{\det}^{\psi}(R, \Sigma)(\mathbf{q}_0) := \begin{cases} 1, & \text{if } q_0 \leq R, t_{\text{exit}}^{RS^2} < \infty \text{ and } \mathbf{Q}_{t_{\text{exit}}^{RS^2}}^{\psi}(\mathbf{q}_0) \in R\Sigma, \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

The definition is motivated by the idea that particles are detected when they cross the boundary RS^2 for the first time. Using the fact that RS^2 is closed we can estimate

$$\left| N_{\det}^{\psi}(R, \Sigma) - N_{\text{sig}}^{\psi}(R\Sigma) \right| \leq N_{-}^{\psi}(RS^2)$$

so that by the triangle inequality

$$\left| \mathbb{E}(N_{\det}^{\psi}(R, \Sigma)) - \mathbb{E}(N_{\text{sig}}^{\psi}(R\Sigma)) \right| \leq \mathbb{E}(N_{-}^{\psi}(RS^2)). \quad (24)$$

With (19), (20) and Proposition 4 we obtain for the right-hand side of (24),

$$\begin{aligned} \mathbb{E}(N_{-}^{\psi}(RS^2)) &= \frac{1}{2} \mathbb{E} \left(N_{\text{tot}}^{\psi}(RS^2) - N_{\text{sig}}^{\psi}(RS^2) \right) \\ &= \frac{1}{2} \int_0^{\infty} \int_{RS^2} (|\mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma}| - \mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma}) dt. \end{aligned} \quad (25)$$

If $\mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma} \geq 0$ for all $d\boldsymbol{\sigma} \in RS^2$ and $t > 0$ then we have by (24) and (25) that $\mathbb{E}(N_{\text{sig}}^{\psi}(R\Sigma)) = \mathbb{E}(N_{\det}^{\psi}(R\Sigma))$. In general $\mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma}$ does not have to be positive, but the flux-across-surfaces theorem (Proposition 3) ensures that the flux is asymptotically outwards. Thus we can estimate the difference between $\mathbb{E}(N_{\text{sig}}^{\psi}(R\Sigma))$ and $\mathbb{E}(N_{\det}^{\psi}(R\Sigma))$ for all ψ which satisfy the flux-across-surfaces theorem using (24) and (25),

$$\left| \mathbb{E}(N_{\text{sig}}^{\psi}(R\Sigma)) - \mathbb{E}(N_{\det}^{\psi}(R, \Sigma)) \right| \leq \frac{1}{2} \int_0^{\infty} \int_{RS^2} (|\mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma}| - \mathbf{j}^{\psi_t}(\mathbf{x}) \cdot d\boldsymbol{\sigma}) dt \xrightarrow{R \rightarrow \infty} 0. \quad (26)$$

In particular under the hypotheses of Proposition 3 and the general assumptions A1 and A2 we obtain asymptotic equality between the expectation values $\mathbb{E}(N_{\det}^{\psi}(R, \Sigma))$ and $\mathbb{E}(N_{\text{sig}}^{\psi}(R\Sigma))$.

4. A Model for the Beam

In a scattering situation a beam of particles is scattered off a target. We now wish to focus on the beam. We take the beam to be produced by a particle source located in the plane Y_L perpendicular to the x_3 -axis:

$$Y_L := \{-L\mathbf{e}_3 + \mathbf{a} \mid \mathbf{a} \perp \mathbf{e}_3\}, \quad L > 0.$$

The particles are created with wave functions $\psi \in \mathcal{H}_{\text{a.c.}}$ translated to the plane Y_L . Calling $\psi_{\mathbf{y}}$ the translation of ψ by \mathbf{y} , the ‘‘centers’’ of the translated wave functions, with which we are concerned, are located at

$$\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 - L\mathbf{e}_3 \in Y_L$$

and are uniformly distributed in a bounded region $A \subset Y_L$ with area $|A|$. We call A the beam profile. The momentum distribution of the wave function is concentrated around the momentum $\mathbf{k}_0 \parallel \mathbf{e}_3$.

Remark 3. This model of a beam, in which the particles have random impact parameters and are scattered off a single target “particle,” is equivalent to the more realistic description of the scattering situation, in which all the target particles are randomly distributed (e.g., in a foil) and the incoming particles have the very same impact parameter, provided coherent and multiple-scattering effects are neglected (see e.g. [17], p. 214).

The translated wave function $\psi_{\mathbf{y}}$ of a wave function $\psi \in \mathcal{H}_{\text{a.c.}}$ will not in general be in $\mathcal{H}_{\text{a.c.}}$, but can have a part in $\mathcal{H}_{\text{p.p.}}$. This is problematical for the application of our general results (see Sect. 9). To avoid this difficulty, we assume:

A 3. The Hamiltonian $H = -\frac{1}{2}\Delta + V$ has no bound states, i.e. $\mathcal{H}_{\text{p.p.}} = \{0\}$.

Then $\psi_{\mathbf{y}} \in \mathcal{H}_{\text{a.c.}}, \forall \mathbf{y} \in \mathbb{R}^3$.

We specify now more precisely the model for the beam, which has been already mentioned in [9]. The particles are created with wave functions ψ at random times $t \in \mathbb{R}_+$ and where the wave function of a particle is shifted randomly by the uniformly distributed “impact parameter” $\mathbf{y} \in A$, the “center” of the wave function at the moment of emission. In Bohmian mechanics the initial position $\mathbf{q} \in \mathbb{R}^3$ of the particle determines its trajectory. The initial position is $|\psi_{\mathbf{y}}|^2$ -distributed. We shall not need many stochastic details about the beam. The reader may think of a Poisson point process with points in

$$\Lambda = \mathbb{R}^+ \times A \times \mathbb{R}^3,$$

with a point $\lambda = (t, \mathbf{y}, \mathbf{q}) \in \Lambda$ representing a particle with wave function

$$\psi_{\mathbf{y}}(\mathbf{x}) \equiv \psi(\mathbf{x} - \mathbf{y}), \quad \mathbf{y} \in A \quad (27)$$

emitted at the time $t \in \mathbb{R}^+$ and with initial position $\mathbf{q} \in \mathbb{R}^3$. We shall consider a general point process $(\Lambda^*, \mathfrak{F}, \mathbb{P})$ built on $(\Lambda, \mathfrak{B}(\Lambda), \mu)$, where $\lambda^* \in \Lambda^*$ represents a configuration of countably many points in Λ , i.e.

$$\lambda^* = \{\lambda\}, \quad \lambda \in \Lambda, \quad \lambda^* \text{ countable.}$$

For the number of points

$$\chi_B^*(\lambda^*) \equiv \sum_{\lambda \in \lambda^*} \chi_B(\lambda)$$

in a set $B \in \mathfrak{B}(\Lambda)$, where χ_B is the indicator function of the set B , we have that

$$\mathbb{E}(\chi_B^*) = \mu(B), \quad (28)$$

where the intensity measure μ on $\mathfrak{B}(\Lambda)$ is given by

$$d\mu = |\psi(\mathbf{x} - \mathbf{y})|^2 \chi_A(\mathbf{y}) dt d^2 y d^3 x. \quad (29)$$

Remark 4. For a Poisson process we would have, in addition to (28), that

$$\mathbb{P}(\chi_B^* = k) = \exp(-\mu(B)) \frac{\mu(B)^k}{k!} \quad (30)$$

as well as the independence of χ_A^* and χ_B^* , for $A \cap B = \emptyset$, $A, B \in \mathfrak{B}(\Lambda)$.

We shall assume that the point process is ergodic in the following sense: For any $B \in \mathfrak{B}(\Lambda)$ let

$$B(\tau) := \{(t, \mathbf{y}, \mathbf{q}) \in B \mid t \in [0, \tau)\}. \quad (31)$$

Then for any $\epsilon > 0$,

$$\lim_{\tau \rightarrow \infty} \mathbb{P} \left(\left| \frac{\chi_{B(\tau)}^*}{\tau} - \mathbb{E} \left(\frac{\chi_{B(\tau)}^*}{\tau} \right) \right| \geq \epsilon \right) = 0, \quad (32)$$

with $\mathbb{E} \left(\chi_{B(\tau)}^* \right)$ given by (28).

Remark 5. Because of the independence property (cf. Remark 4), (32) holds for the case of a Poisson process.

Remark 6. The point process has unit density in the following sense: Let $C \subset A$, $\tau > 0$ and $B := [0, \tau) \times C \times \mathbb{R}^3$ be given. Then with (32) for any $\epsilon > 0$,

$$\lim_{\tau \rightarrow \infty} \mathbb{P} \left(\left| \frac{\chi_B^*}{|C|\tau} - \mathbb{E} \left(\frac{\chi_B^*}{|C|\tau} \right) \right| \geq \epsilon \right) = 0, \quad (33)$$

and

$$\mathbb{E} \left(\frac{\chi_B^*}{|C|\tau} \right) = \frac{1}{|C|\tau} \mu(B) = 1. \quad (34)$$

5. The Definition of the Scattering Cross Section

We shall now start to define $N^*(\tau, R, A, L, \psi, \Sigma)$, the number of detected particles. To simplify the notation we do not always indicate the dependence of N^* on A, L and ψ . Sometimes we will also suppress the dependence on R and Σ . We define first $N_{\text{det}}(\tau, R, \Sigma)$ for a single particle corresponding to $\lambda = (t, \mathbf{y}, \mathbf{q})$ by

$$N_{\text{det}}(\tau, R, \psi, \Sigma) : \Lambda \rightarrow \{0, 1\},$$

$$N_{\text{det}}(\tau, R, \psi, \Sigma)(\lambda) := \chi_{[0, \tau)}(t) N_{\text{det}}^{\psi_y}(R, \Sigma)(\mathbf{q}), \quad (35)$$

where $N_{\text{det}}^{\psi_y}(R, \Sigma)(\mathbf{q})$ is defined by (23). The characteristic function ensures that no particle is counted which is emitted after the time τ . Note that ψ_y must satisfy condition A1 (Sect. 3) to ensure that $N_{\text{det}}^{\psi_y}(R, \Sigma)(\mathbf{q})$ is well defined. Then

$$N^*(\tau, R, A, L, \psi, \Sigma) : \Lambda^* \rightarrow \mathbb{N}_0,$$

$$N^*(\tau, R, A, L, \psi, \Sigma)(\lambda^*) = \sum_{\lambda \in \lambda^*} N_{\text{det}}(\tau, R, \psi, \Sigma)(\lambda). \quad (36)$$

The empirical scattering cross section $\sigma_{\text{emp}}(\Sigma)$ for the solid angle Σ is the random variable⁷

$$\sigma_{\text{emp}}(\Sigma) := \frac{N^*(\tau, R, A, L, \psi, \Sigma)}{\tau}, \quad (37)$$

which by the law of large numbers (for the Poisson case and by the ergodicity assumption (32) for the general case) should approximate for large τ in \mathbb{P} -probability its corresponding \mathbb{P} -expectation value. The expected value of (37) is then the theoretically predicted cross section. This theoretically predicted cross section involves a very complicated formula which is not very explicit, cf. (47) and Remark 7. It depends of course on the detection directions Σ , the potential V and the approximate momentum \mathbf{k}_0 of the particles in the beam, but depends also on the other details of the experimental setup such as R , A , L and the detailed specification of ψ . By taking the scaling limit described in the next section, we shall arrive at (1), which does not depend on these additional details.

6. The Scaling of the Parameters

According to the usual asymptotic picture of scattering theory where the particles are prepared long before and are detected long after the scattering event has occurred, the preparation and detection should be far away from the scattering center. That means the limits $R \rightarrow \infty$ and $L \rightarrow \infty$ have to be taken. However, increasing L has the (undesirable) effect of an increased spreading of the beam, which reduces the beam intensity in the scattering region. To maintain the beam intensity in the scattering region we must widen the beam profile A as $L \rightarrow \infty$. The idealization of an incoming plane wave corresponds to particles with a narrow distribution in momentum space, i.e., to a limit in which the Fourier transform of the initial wave function becomes more and more concentrated around a fixed initial wave vector \mathbf{k}_0 . For a detailed discussion of the scattering regime see [8].

The limits for the parameters L , A , and ψ will be combined by simultaneously scaling them using a small parameter ϵ : We introduce L^ϵ , A^ϵ and ψ^ϵ , whose precise dependence on ϵ will be given below, and consider the cross section corresponding to (37), depending on ϵ , R , τ ,

$$\sigma_{\text{emp}}^\epsilon(\Sigma) = \frac{N^*(\tau, R, A^\epsilon, L^\epsilon, \psi^\epsilon, \Sigma)}{\tau}, \quad (38)$$

to which the limit $\epsilon \rightarrow 0$ is to be applied.

However, the limit $R \rightarrow \infty$ is taken before we take $\epsilon \rightarrow 0$; this is because we must have that the diameter of the beam profile A is much smaller than R , since otherwise unscattered particles will often contribute to what should be the cross section for scattered particles. For convenience, we first take the limit $\tau \rightarrow \infty$, required for the stabilization of the empirical cross section produced by the law of large numbers. We are thus led to consider a limit for the cross section of the form

$$\sigma(\Sigma) = \lim_{\epsilon \rightarrow 0} \lim_{R \rightarrow \infty} \lim_{\tau \rightarrow \infty} \sigma_{\text{emp}}^\epsilon(\Sigma). \quad (39)$$

⁷ We shall ignore the dimension factor [unit area · unit time] which comes from the normalization of (37) by the unit density $\frac{1}{[\text{unit area} \cdot \text{unit time}]}$ of the underlying point process, cf. Remark 6. One can also normalize by the beam density, i.e. with the number of detected particles (by a detector in the beam with a surface perpendicular to the beam axis) per unit time and unit area, in front of the target. In the scattering regime, i.e. if the parameters are suitably scaled (cf. Section 6), the beam will have unit density in front of the target. We shall not elaborate on this further in this paper, see however [8].

The precise definition of L^ϵ , A^ϵ and ψ^ϵ , used in our main results, is the following:

$$\psi^\epsilon(\mathbf{x}) = \epsilon^{\frac{3}{2}} e^{i\mathbf{k}_0 \cdot \mathbf{x}} \psi(\epsilon \mathbf{x}), \quad (40)$$

with the Fourier transform

$$\widehat{\psi}^\epsilon(\mathbf{k}) = \epsilon^{-\frac{3}{2}} \widehat{\psi}\left(\frac{\mathbf{k} - \mathbf{k}_0}{\epsilon}\right). \quad (41)$$

The particle source is located on Y_{L^ϵ} , with

$$L^\epsilon = \frac{L}{\epsilon^l}, \quad l > 2. \quad (42)$$

For the beam profile $A^\epsilon \subset Y_{L^\epsilon}$ we take the circular region

$$A^\epsilon = \{\mathbf{x} \in \mathbb{R}^3 \mid \sqrt{x_1^2 + x_2^2} < \frac{D^\epsilon}{2} \text{ and } x_3 = L^\epsilon\} \quad (43)$$

with the beam diameter D^ϵ given by

$$D^\epsilon = \frac{D}{\epsilon^d}, \quad d > 2l - 3. \quad (44)$$

(One might be inclined to consider a scattering experiment in which the diameter of the beam is much smaller than the distance of the particle source from the scattering center. Indeed, if $2 < l < 3$, $d < l$ is consistent with (44). Hence, such a scenario is covered by our results.)

7. The Scattering Cross Section Theorem

We can now formulate our main results. Our basic assumptions are that $V \in (V)_5$ (Definition 1), A2 (Sect. 3), A3 (no bound states, Sect. 4) and hat for all ϵ small enough ψ_y^ϵ is “good” for all $\mathbf{y} \in A^\epsilon$ in the sense that it satisfies A1 (Sect. 3) as well as the condition for the FAST (Prop. 3). Moreover, we need to assume that the potential has no zero energy resonances. However, instead of invoking the implicit condition on ψ that the ψ_y^ϵ are “good,” we impose stronger but more explicit conditions on ψ , namely that $\psi \in C_0^\infty(\mathbb{R}^3)$ (Theorem 2) or $\psi \in \mathcal{S}$ (Theorem 1), with corresponding additional conditions on the potential (Definitions 4 and 3, respectively).

Definition 3. V is in \mathcal{V} if

- (i) the Hamiltonian $H = -\frac{1}{2}\Delta + V$ has no bound states, i.e. $\mathcal{H}_{p.p.} = \{0\}$,
- (ii) the Hamiltonian $H = -\frac{1}{2}\Delta + V$ has no zero energy resonances,
- (iii) V is a C^∞ -function on \mathbb{R}^3 ,
- (iv) V and its derivatives of all orders are uniformly bounded in \mathbf{x} : For all multi-indices α there exist an $M_\alpha < \infty$ such that $|\partial_{\mathbf{x}}^\alpha V(\mathbf{x})| < M_\alpha$ for all $\mathbf{x} \in \mathbb{R}^3$,
- (v) there exist positive numbers δ and C such that

$$|V(\mathbf{x})| \leq C \langle x \rangle^{-5-\delta} \text{ for all } \mathbf{x} \in \mathbb{R}^3.$$

Theorem 1. *Let ψ be a normalized vector in $\mathcal{S}(\mathbb{R}^3)$ and suppose that V is in \mathcal{V} . Furthermore, suppose that the point process $(\Lambda^*, \mathfrak{F}, \mathbb{P})$ satisfies (28), (29) and the ergodic assumption (32). Let $\mathbf{k}_0 \parallel \mathbf{e}_3$ with $k_0 > 0$ and suppose that $\mathbf{k}_0 \notin C_\Sigma$. Then σ_{emp}^ϵ is well defined and (recalling (1))*

$$\sigma_{emp}^\epsilon(\Sigma) = \frac{N^*(\tau, R, A^\epsilon, L^\epsilon, \psi^\epsilon, \Sigma)}{\tau} \xrightarrow[\epsilon \rightarrow 0, R \rightarrow \infty, \tau \rightarrow \infty]{\mathbb{P}} \sigma(\Sigma) = \int_{\Sigma} \sigma^{diff}(\boldsymbol{\omega}) d\Omega, \quad (45)$$

where $\sigma^{diff}(\boldsymbol{\omega}) = 16\pi^4 |T(k_0 \boldsymbol{\omega}, \mathbf{k}_0)|^2$ and $\xrightarrow{\mathbb{P}}$ denotes convergence in probability.

Definition 4. V is in \mathcal{V}' if

- (i) the Hamiltonian $H = -\frac{1}{2}\Delta + V$ has no bound states, i.e. $\mathcal{H}_{p.p.} = \{0\}$,
- (ii) the Hamiltonian $H = -\frac{1}{2}\Delta + V$ has no zero energy resonances,
- (iii) V is in $(V)_5$,
- (iv) V is C^∞ except, perhaps, at a finite number of singularities.

Under these conditions we obtain

Theorem 2. *Let ψ be a normalized vector in $C_0^\infty(\mathbb{R}^3)$ and let V be in \mathcal{V}' . Furthermore, suppose that the point process $(\Lambda^*, \mathfrak{F}, \mathbb{P})$ satisfies (28), (29) and the ergodic assumption (32). Let $\mathbf{k}_0 \parallel \mathbf{e}_3$ with $k_0 > 0$ and suppose that $\mathbf{k}_0 \notin C_\Sigma$. Then σ_{emp}^ϵ is well defined and (45) of Theorem 1 holds.*

8. Proof of Theorem 1 and Theorem 2

During the proof in this section and in the appendix $0 < c < \infty$ will denote a constant whose value can change during a calculation—even within the same equation or inequality.

If either $V \in \mathcal{V}$ and $\psi \in \mathcal{S}(\mathbb{R}^3)$ or $V \in \mathcal{V}'$ and $\psi \in C_0^\infty$, then (if ψ is normalized) the ψ_y^ϵ are “good” for all $\mathbf{y} \in A^\epsilon$ for all ϵ small enough. That the ψ_y^ϵ satisfy the conditions for the FAST follows from Lemma 1 below, and that they satisfy A1 is easily seen: For the case $V \in \mathcal{V}$ and $\psi \in \mathcal{S}(\mathbb{R}^3)$ the conclusion follows from a simple computation, and if $V \in \mathcal{V}'$ and $\psi \in C_0^\infty$ it suffices to observe that by choosing ϵ small enough the wave function ψ_y^ϵ has, for all $\mathbf{y} \in A^\epsilon$, no overlap with the singularities of the potential.

N^* is thus well defined by (36), and we can take the first limit in (45) using the following

Proposition 5. *Suppose that ψ_y^ϵ satisfies A1 for all $\mathbf{y} \in A^\epsilon$ and that the potential satisfies A2. Furthermore, suppose that the point process $(\Lambda^*, \mathfrak{F}, \mathbb{P})$ satisfies (28), (29) and the ergodic assumption (32). Then the number of detected particles $N^*(\tau)$ obeys the law of large numbers, i.e. for all $\delta > 0$,*

$$\lim_{\tau \rightarrow \infty} \mathbb{P} \left(\left| \frac{N^*(\tau, \Sigma)}{\tau} - \gamma \right| \geq \delta \right) = 0, \quad (46)$$

where

$$\gamma = \int_{A^\epsilon} \mathbb{E} \left(N_{det}^{\psi_y^\epsilon}(\Sigma) \right) d^2 y. \quad (47)$$

Remark 7. $\gamma = \gamma(\Sigma)$ is in fact the cross section which would be measured in an experiment. The remaining limits in (45) applied to γ yield the cross section $\sigma(\Sigma)$. If the basic point process is a Poisson process with $[0, \tau) = \mathbb{R}^+$ the times of detection in Σ form a Poisson process with intensity γ . Moreover, in the scattering regime, the detailed detection events, involving times and directions, form a Poisson process on $\mathbb{R}^+ \times S^2$ with intensity $\sigma^{\text{diff}}(\omega)$.

Proof. By the definition (36) of N^* we have that

$$N^*(\tau)(\lambda^*) = \chi_{B(\tau)}^*(\lambda^*) = \sum_{\lambda \in \lambda^*} \chi_{B(\tau)}(\lambda), \quad (48)$$

with $B(\tau)$ given by

$$B(\tau) = \{(t, \mathbf{y}, \mathbf{q}) \in \Lambda \mid N_{\text{det}}(\tau, \Sigma)(t, \mathbf{y}, \mathbf{q}) = 1\}. \quad (49)$$

It thus follows from (28) and (29) that

$$\begin{aligned} \mathbb{E}(N^*(\tau)) &= \mu(B(\tau)) = \int \chi_{[0, \tau)}(t) N_{\text{det}}^{\psi_y^\epsilon}(\Sigma)(\mathbf{q}) d\mu \\ &= \tau \int_{A^\epsilon} \mathbb{E}\left(N_{\text{det}}^{\psi_y^\epsilon}(\Sigma)\right) d^2y = \tau \gamma. \end{aligned} \quad (50)$$

The proposition follows from the ergodicity assumption (32). \square

It is not easy to calculate the expectation value γ (cf. (47)) directly. However, as we shall show below, using the FAST we can approximate (47) by

$$\int_{A^\epsilon} \mathbb{E}\left(N_{\text{sig}}^{\psi_y^\epsilon}(R\Sigma)\right) d^2y, \quad (51)$$

where the integrand of (51) is given by an integral over the flux (cf. (21)), an expression that we can more easily handle. We will show in Lemma 2 below that $\mathbb{E}\left(N_{\text{sig}}^{\psi_y^\epsilon}(R\Sigma)\right)$ is absolutely integrable over A^ϵ .

We introduce now a class of scattering states \mathcal{G} for which we can show that the corresponding asymptotes are in the set \mathcal{G}^+ , i.e. that they satisfy the FAST.

Definition 5. A function $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ is in \mathcal{G}^0 if⁸

$$\begin{aligned} f &\in \mathcal{H}_{a.c.}(H) \cap C^8(H), \\ \langle x \rangle^2 H^n f(\mathbf{x}) &\in L^2(\mathbb{R}^3), \quad n \in \{0, 1, 2, \dots, 8\}, \\ \langle x \rangle^4 H^n f(\mathbf{x}) &\in L^2(\mathbb{R}^3), \quad n \in \{0, 1, 2, 3\}. \end{aligned}$$

Then $\mathcal{G} := \bigcup_{t \in \mathbb{R}} e^{-iHt} \mathcal{G}^0$.

We state now the important lemma that ensures that the ψ_y^ϵ satisfy the FAST.

⁸ $C^8(H) := \bigcap_{n=1}^8 D(H^n)$

Lemma 1. *Suppose $V \in (V)_4$ and that zero is neither a resonance nor an eigenvalue of H . Then*

$$\psi(\mathbf{x}) \in \mathcal{G} \Rightarrow \widehat{\psi}_{out}(\mathbf{k}) = \mathcal{F}\left(\Omega_+^{-1}\psi\right)(\mathbf{k}) \in \mathcal{G}^+.$$

The proof is adapted from [12] and can be found in the appendix.

Remark 8. For other mapping properties between ψ and ψ_{out} , which are not applicable in our case, see [26].

For $\psi \in \mathcal{S}$ and $V \in \mathcal{V}$ or $\psi \in C_0^\infty(\mathbb{R}^3)$ and $V \in \mathcal{V}'$ we have that $\psi_y^\epsilon \in C^\infty(H)$ for all $\mathbf{y} \in A^\epsilon$ and ϵ small enough. By (i) in the definition of \mathcal{V} or \mathcal{V}' (Definition 3 or 4) there are no bound states. Hence $\psi_y^\epsilon \in \mathcal{H}_{a.c.}(H) \cap C^8(H)$, and one easily sees that $\psi_y^\epsilon \in \mathcal{G}$. Thus by Lemma 1 and Proposition 3 the ψ_y^ϵ satisfy the FAST for all $\mathbf{y} \in A^\epsilon$ and ϵ small enough.

We now show that $\mathbb{E}\left(N_{sig}^{\psi_y^\epsilon}(R\Sigma)\right)$ is absolutely integrable over A^ϵ .

Lemma 2. *Suppose that $\psi \in \mathcal{S}$ and $V \in \mathcal{V}$ or that $\psi \in C_0^\infty(\mathbb{R}^3)$ and $V \in \mathcal{V}'$. Then there exist M and $R_0 > 0$ such that for ϵ small enough*

$$\int_0^\infty \int_{RS^2} |\mathbf{j}^{\psi_{y,t}^\epsilon}(\mathbf{x}) \cdot d\sigma| dt < M, \quad \forall \mathbf{y} \in A^\epsilon, \forall R > R_0. \quad (52)$$

For the proof see the Appendix. From now on we assume that $R > R_0$.

By Lemma 1, Proposition 3, Proposition 4 and Lemma 2 we see that (51) is a well defined expression. Moreover, by (26) the difference between $\mathbb{E}\left(N_{det}^{\psi_y^\epsilon}(R, \Sigma)\right)$ and $\mathbb{E}\left(N_{sig}^{\psi_y^\epsilon}(R\Sigma)\right)$ vanishes in the limit $R \rightarrow \infty$, and using Lemma 2 we easily see by the dominated convergence theorem that the same conclusion holds for the integrals themselves. Thus, by Proposition 5, the limit $\sigma(\Sigma)$ in Theorem 1 is given by

$$\begin{aligned} \sigma(\Sigma) &= \lim_{\epsilon \rightarrow 0} \lim_{R \rightarrow \infty} \gamma = \lim_{\epsilon \rightarrow 0} \lim_{R \rightarrow \infty} \int_{A^\epsilon} \mathbb{E}\left(N_{det}^{\psi_y^\epsilon}(R, \Sigma)\right) d^2y \\ &= \lim_{\epsilon \rightarrow 0} \int_{A^\epsilon} \lim_{R \rightarrow \infty} \mathbb{E}\left(N_{sig}^{\psi_y^\epsilon}(R\Sigma)\right) d^2y \\ &= \lim_{\epsilon \rightarrow 0} \int_{A^\epsilon} \lim_{R \rightarrow \infty} \int_{R\Sigma} \mathbf{j}^{\psi_{y,t}^\epsilon}(\mathbf{x}) \cdot d\sigma dt d^2y. \end{aligned} \quad (53)$$

Using Lemma 1 and Proposition 3 we get instead of (53),

$$\sigma(\Sigma) = \lim_{\epsilon \rightarrow 0} \int_{C_\Sigma} \int_{A^\epsilon} |\widehat{|\Omega_+^{-1}\psi_y^\epsilon(\mathbf{k})|^2} d^2y d^3k = \lim_{\epsilon \rightarrow 0} \int_{C_\Sigma} \int_{A^\epsilon} |\widehat{|\Omega_-^{-1}\psi_y^\epsilon(\mathbf{k})|^2} d^2y d^3k. \quad (54)$$

The formula for $S = T + I$ is given by (8) and (9). To exploit this formula we write instead of (54):

$$\sigma(\Sigma) = \lim_{\epsilon \rightarrow 0} \int_{C_\Sigma} \int_{A^\epsilon} |\mathcal{F}\left(S(\Omega_-^{-1}\psi_y^\epsilon - \psi_y^\epsilon) + T\psi_y^\epsilon + \psi_y^\epsilon\right)(\mathbf{k})|^2 d^2y d^3k. \quad (55)$$

By the triangle equality we see that (55) yields

$$\sigma(\Sigma) = \lim_{\epsilon \rightarrow 0} \int_{C_\Sigma} \int_{A^\epsilon} |\mathcal{F}(T\psi_y^\epsilon)(\mathbf{k})|^2 d^2 y d^3 k, \quad (56)$$

provided

$$\lim_{\epsilon \rightarrow 0} \int_{A^\epsilon} \|\Omega_-^{-1} \psi_y^\epsilon - \psi_y^\epsilon\|^2 d^2 y = 0 \quad (57)$$

and

$$\lim_{\epsilon \rightarrow 0} \int_{C_\Sigma} \int_{A^\epsilon} |\widehat{\psi}_y^\epsilon(\mathbf{k})|^2 d^2 y d^3 k = 0. \quad (58)$$

Remark 9. In [9] the “sufficient condition” for proceeding from (54) to (56) was insufficient.

We will establish now (57) and (58). We start with (58). Suppose that Σ is such that $\mathbf{k}_0 \notin C_\Sigma$. With (41) we have then that

$$\begin{aligned} \int_{C_\Sigma} \int_{A^\epsilon} |\widehat{\psi}_y^\epsilon(\mathbf{k})|^2 d^2 y d^3 k &= \epsilon^{-3} \int_{C_\Sigma} \int_{A^\epsilon} \left| \widehat{\psi} \left(\frac{\mathbf{k} - \mathbf{k}_0}{\epsilon} \right) \right|^2 d^2 y d^3 k \\ &= \int_{\frac{1}{\epsilon}(C_\Sigma - \mathbf{k}_0)} \int_{A^\epsilon} |\widehat{\psi}(\mathbf{k})|^2 d^2 y d^3 k. \end{aligned} \quad (59)$$

Since $\mathbf{k}_0 \notin C_\Sigma$ there exists a $\delta > 0$ such that

$$|\mathbf{k} - \mathbf{k}_0| > \delta \quad \text{for all } \mathbf{k} \in C_\Sigma. \quad (60)$$

Using that $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^3)$ (we will use that $|\widehat{\psi}| \leq ck^{-(d+2)}$), the last integral in (59) can be estimated by

$$\int_{\frac{1}{\epsilon}(C_\Sigma - \mathbf{k}_0)} \int_{A^\epsilon} |\widehat{\psi}(\mathbf{k})|^2 d^2 y d^3 k \leq \int_{k > \frac{\delta}{\epsilon}} \int_{A^\epsilon} |\widehat{\psi}(\mathbf{k})|^2 d^2 y d^3 k \leq \frac{c}{\epsilon^{2d}} \int_{k > \frac{\delta}{\epsilon}} \frac{1}{k^{2d+4}} d^3 k \leq c\epsilon, \quad (61)$$

from which (58) follows.

Since Ω_- is a partial isometry, (57) is equivalent to

$$\lim_{\epsilon \rightarrow 0} \int_{A^\epsilon} \|\Omega_- \psi_y^\epsilon - \psi_y^\epsilon\|^2 d^2 y = 0, \quad (62)$$

which is the content of the following

Lemma 3. *Let zero be neither an eigenvalue nor a resonance of H and suppose that $V \in (V)_5$. Let $\psi \in \mathcal{S}(\mathbb{R}^3)$ and let $k_0 > 0$. Then*

$$\lim_{\epsilon \rightarrow 0} \int_{A^\epsilon} \|\Omega_- \psi_y^\epsilon - \psi_y^\epsilon\|^2 d^2 y = 0. \quad (63)$$

Remark 10. Under the additional condition that $\text{supp } \widehat{\psi} \subset P_{\mathbf{e}_3}^\alpha$ for some $\alpha \in (0, \frac{\pi}{2})$, where $P_{\mathbf{e}_3}^\alpha := \{\mathbf{k} \in \mathbb{R}^3 : \mathbf{k} \cdot \mathbf{e}_3 > k \cos \alpha\}$, $0 < \alpha < \frac{\pi}{2}$ (this is a convenient condition, see e.g. [2], Lemma 7.17), one can prove in a manner similar to the way we prove Lemma 3 that the following holds:

$$\lim_{L \rightarrow \infty} \int_{Y_L} \|\Omega_- \psi_y - \psi_y\|^2 d^2 y = 0. \quad (64)$$

It is well known that the integrand in (64) tends to zero for large y (see e.g. [2], Corollary 8.17, [19], Theorem XI.33, and [21], Theorem 2.20).

Proof of Lemma 3. We have that

$$\|\Omega_- \psi_y^\epsilon - \psi_y^\epsilon\|^2 = 1 - (\psi_y^\epsilon, \Omega_- \psi_y^\epsilon) + c.c. \quad (65)$$

Since $\Omega_- \psi = \mathcal{F}_-^{-1} \widehat{\psi}(\mathbf{k})$ for any $\psi \in L^2(\mathbb{R}^3)$ (Proposition 1, (iii)) we obtain for the r.h.s. of (65):

$$1 - \int (\psi_y^\epsilon)^*(\mathbf{x}) (2\pi)^{-3/2} \int \widehat{\psi}_y^\epsilon(\mathbf{k}) \varphi_-(\mathbf{x}, \mathbf{k}) d^3 k d^3 x + c.c. \quad (66)$$

Writing

$$\varphi_-(\mathbf{x}, \mathbf{k}) = e^{i\mathbf{k} \cdot \mathbf{x}} - \eta_-(\mathbf{x}, \mathbf{k}), \quad (67)$$

and since $\|\psi_y^\epsilon\|^2 = 1$, we then find that

$$\|\Omega_- \psi_y^\epsilon - \psi_y^\epsilon\|^2 = \int (\psi_y^\epsilon)^*(\mathbf{x}) (2\pi)^{-3/2} \int \widehat{\psi}_y^\epsilon(\mathbf{k}) \eta_-(\mathbf{x}, \mathbf{k}) d^3 k d^3 x + c.c. \quad (68)$$

We shall divide the \mathbf{k} -integration into two parts with the help of smooth (C^∞) mollifiers $0 \leq f_1(\mathbf{k}) \leq 1$ and $0 \leq f_2(\mathbf{k}) \leq 1$ satisfying

$$f_1(\mathbf{k}) = \begin{cases} 1, & \text{for } |\mathbf{k} - \mathbf{k}_0| < \frac{k_0}{3}, \\ 0, & \text{for } |\mathbf{k} - \mathbf{k}_0| \geq \frac{k_0}{2}, \end{cases} \quad (69)$$

$$f_2(\mathbf{k}) := 1 - f_1(\mathbf{k}).$$

Using (69) we obtain for (68)

$$\begin{aligned} \|\Omega_- \psi_y^\epsilon - \psi_y^\epsilon\|^2 &= \int (\psi_y^\epsilon)^*(\mathbf{x}) (2\pi)^{-3/2} \int \widehat{\psi}_y^\epsilon(\mathbf{k}) (f_1 + f_2)(\mathbf{k}) \eta_-(\mathbf{x}, \mathbf{k}) d^3 k d^3 x + c.c. \\ &= \int (\psi_y^\epsilon)^*(\mathbf{x}) (2\pi)^{-3/2} \int \widehat{\psi}_y^\epsilon(\mathbf{k}) f_1(\mathbf{k}) \eta_-(\mathbf{x}, \mathbf{k}) d^3 k d^3 x \\ &\quad + \int (\psi_y^\epsilon)^*(\mathbf{x}) (2\pi)^{-3/2} \int \widehat{\psi}_y^\epsilon(\mathbf{k}) f_2(\mathbf{k}) \eta_-(\mathbf{x}, \mathbf{k}) d^3 k d^3 x \\ &\quad + c.c. =: I_1 + I_2 + c.c. \\ &\leq 2|I_1| + 2|I_2|. \end{aligned} \quad (70)$$

Observing that $\psi \in \mathcal{S}(\mathbb{R}^3)$ we estimate $|I_2|$ by using that for any $n > 0$ $|\widehat{\psi}(\mathbf{k})| \leq \frac{c}{k^n}$ and that $|\eta_-(\mathbf{x}, \mathbf{k})| \leq 1 + |\varphi_-(\mathbf{x}, \mathbf{k})| \leq c$ (Proposition 2 (ii)) as well as (40), (41) and (69):

$$\begin{aligned} |I_2| &\leq \frac{c}{\epsilon^3} \int |\psi(\mathbf{x} - \mathbf{y})| (2\pi)^{-3/2} \int_{|\mathbf{k} - \mathbf{k}_0| \geq \frac{k_0}{3}} \left| \widehat{\psi} \left(\frac{\mathbf{k} - \mathbf{k}_0}{\epsilon} \right) \right| d^3 k d^3 x \\ &\leq \frac{c}{\epsilon^3} \int_{|\mathbf{k}| \geq \frac{k_0}{3}} \left| \widehat{\psi} \left(\frac{\mathbf{k}}{\epsilon} \right) \right| d^3 k \leq c \epsilon^{n-3} \int_{|\mathbf{k}| \geq \frac{k_0}{3}} \frac{1}{k^n} d^3 k = c \epsilon^{n-3}, \end{aligned} \quad (71)$$

if $n \geq 4$.

Lemma 3 concerns the integration of I_1 and I_2 over A^ϵ . With (71) we obtain that

$$\int_{A^\epsilon} |I_2| d^2 y \leq c \epsilon^{n-3-2d}, \quad (72)$$

which tends to zero if we choose n large enough. We are left with showing that

$$\lim_{\epsilon \rightarrow 0} \int_{A^\epsilon} |I_1| d^2 y = 0, \quad (73)$$

and for this it suffices to prove that

$$\lim_{\epsilon \rightarrow 0} \int_{Y_{L^\epsilon}} |I_1| d^2 y = 0. \quad (74)$$

Recalling the Lippmann-Schwinger equation (11), i.e. that

$$\eta_-(\mathbf{x}, \mathbf{k}) = \frac{1}{2\pi} \int \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} V(\mathbf{x}') \varphi_-(\mathbf{x}', \mathbf{k}),$$

we find that

$$I_1 = \frac{1}{(2\pi)^{\frac{5}{2}}} \int (\psi_y^\epsilon)^*(\mathbf{x}) \int \widehat{\psi}_y^\epsilon(\mathbf{k}) f_1(\mathbf{k}) \int \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} V(\mathbf{x}') \varphi_-(\mathbf{x}', \mathbf{k}) d^3 x' d^3 k d^3 x. \quad (75)$$

Since the integrand in (75) is absolutely integrable over $\mathbf{x}, \mathbf{x}', \mathbf{k}$ (because $\psi \in \mathcal{S}(\mathbb{R}^3)$, $V \in (V)_5$; cf. Lemma 2, (ii)) we are free to interchange these integrations and more generally change integration variables as convenient. Using $(\psi_y^\epsilon)^*(\mathbf{x}) = (\psi^\epsilon)^*(\mathbf{x} - \mathbf{y})$, $\widehat{\psi}_y^\epsilon(\mathbf{k}) = \widehat{\psi}^\epsilon(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{y}}$ we obtain that

$$\begin{aligned} I_1 &= \frac{1}{(2\pi)^{\frac{5}{2}}} \int_{\mathbb{R}^3} (\psi^\epsilon)^*(\mathbf{x} - \mathbf{y}) \int_{\mathbb{R}^3} \widehat{\psi}^\epsilon(\mathbf{k}) f_1(\mathbf{k}) \\ &\quad \times \int_{\mathbb{R}^3} \frac{e^{ik|\mathbf{x} - \mathbf{x}'| - i\mathbf{k} \cdot \mathbf{y}}}{|\mathbf{x} - \mathbf{x}'|} V(\mathbf{x}') \varphi_-(\mathbf{x}', \mathbf{k}) d^3 x' d^3 k d^3 x. \end{aligned} \quad (76)$$

Making the change of variables $\mathbf{x} \rightarrow \mathbf{x} - \mathbf{y}$ and using $\mathbf{y} = (y_1, y_2, -L^\epsilon)$ we obtain

$$I_1 = \frac{1}{(2\pi)^{\frac{5}{2}}} \int_{\mathbb{R}^3} (\psi^\epsilon)^*(\mathbf{x}) \int_{\mathbb{R}^3} \widehat{\psi}^\epsilon(\mathbf{k}) f_1(\mathbf{k}) \int_{\mathbb{R}^3} \frac{e^{ik|\mathbf{y}+\mathbf{x}-\mathbf{x}'|-ik_1y_1-ik_2y_2+ik_3L^\epsilon}}{|\mathbf{y}+\mathbf{x}-\mathbf{x}'|} V(\mathbf{x}') \times \varphi_-(\mathbf{x}', \mathbf{k}) d^3x' d^3k d^3x. \quad (77)$$

Introducing as shorthand notation (no change of variables) $\tilde{\mathbf{y}} = \mathbf{y} + \mathbf{x} - \mathbf{x}'$, $\mathbf{a} := \mathbf{x} - \mathbf{x}'$, $b_3 := -L^\epsilon + a_3$ and letting (r, ϑ) be the polar coordinates for $(\tilde{y}_1, \tilde{y}_2)$, with \mathbf{e}_r the corresponding radial unit vector ($\perp \mathbf{e}_3$), this becomes

$$\begin{aligned} I_1 &= \frac{1}{(2\pi)^{\frac{5}{2}}} \int_{\mathbb{R}^3} (\psi^\epsilon)^*(\mathbf{x}) \int_{\mathbb{R}^3} \widehat{\psi}^\epsilon(\mathbf{k}) f_1(\mathbf{k}) \\ &\quad \times \int_{\mathbb{R}^3} \frac{e^{ik\sqrt{\tilde{y}_1^2+\tilde{y}_2^2+(-L^\epsilon+a_3)^2}-ik_1\tilde{y}_1-ik_2\tilde{y}_2+ik_3L^\epsilon}}{|\tilde{\mathbf{y}}|} e^{ik_1a_1+ik_2a_2} \cdot V(\mathbf{x}') \varphi_-(\mathbf{x}', \mathbf{k}) d^3x' d^3k d^3x \\ &= \frac{1}{(2\pi)^{\frac{5}{2}}} \int_{\mathbb{R}^3} (\psi^\epsilon)^*(\mathbf{x}) \int_{\mathbb{R}^3} \widehat{\psi}^\epsilon(\mathbf{k}) f_1(\mathbf{k}) \\ &\quad \times \int_{\mathbb{R}^3} \frac{e^{ik\sqrt{r^2+b_3^2}-ik\sin\vartheta r\cos\beta+ik\cos\vartheta L^\epsilon}}{\sqrt{r^2+b_3^2}} e^{ik_1a_1+ik_2a_2} \cdot V(\mathbf{x}') \varphi_-(\mathbf{x}', \mathbf{k}) d^3x' d^3k d^3x, \end{aligned} \quad (78)$$

with $k \sin \vartheta = |\mathbf{k}_p| = \sqrt{k_1^2 + k_2^2}$, $k_3 = k \cos \vartheta$, where ϑ ($0 \leq \vartheta \leq \pi$) is the angle between \mathbf{k} and \mathbf{e}_3 and β is the angle between $\mathbf{k}_p = (k_1, k_2, 0)$ and \mathbf{e}_r . Moreover, there is an angle $0 < \alpha < \frac{\pi}{2}$ such that

$$\vartheta \leq \alpha, \text{ i.e. } \cos \alpha \leq \cos \vartheta \leq 1, \quad 0 \leq \sin \vartheta \leq \sin \alpha, \quad 0 < \alpha < \frac{\pi}{2} \quad (79)$$

for all \mathbf{k} 's in the support of f_1 (cf. (69)).

We introduce now spherical coordinates $(k, \boldsymbol{\omega})$ for \mathbf{k} as integration variables and do first the integration over k (note that β is not k -dependent). Since $\widehat{\psi}^\epsilon \in \mathcal{S}(\mathbb{R}^3)$, f_1 is smooth and $\frac{\partial}{\partial k} \varphi_-(\mathbf{x}', \mathbf{k})$ is uniformly bounded in k (Proposition 2 (iv)), we can do two integration by parts with respect to k and obtain that

$$\begin{aligned} I_1 &= -\frac{1}{(2\pi)^{\frac{5}{2}}} \int_{\mathbb{R}^3} (\psi^\epsilon)^*(\mathbf{x}) \int_{\mathbb{R}^3} V(\mathbf{x}') \\ &\quad \times \int_0^\infty \int_{S^2} \frac{\partial^2}{\partial k^2} \left(\widehat{\psi}^\epsilon(\mathbf{k}) f_1(\mathbf{k}) \varphi_-(\mathbf{x}', \mathbf{k}) e^{ik_1a_1+ik_2a_2} k^2 \right) \cdot \frac{e^{ik\lambda}}{\sqrt{r^2+b_3^2}\lambda^2} dk d\Omega d^3x' d^3x, \end{aligned} \quad (80)$$

where

$$\lambda := r \left(\sqrt{1 + \frac{b_3^2}{r^2}} - \sin \vartheta \cos \beta \right) + \cos \vartheta L^\epsilon. \quad (81)$$

To estimate the derivatives of the functions $f_1(\mathbf{k})\varphi_-(\mathbf{x}', \mathbf{k})$ we use Proposition 2, (iv) and the smoothness of $f_1(\mathbf{k})$. We introduce a multi-index notation

$$i := (i_1, i_2, i_3, i_4), \quad i_m \in \mathbb{N}_0, \quad |i| := i_1 + i_2 + i_3 + i_4, \quad j := (j_1, j_2, j_3) \text{ analogously.}$$

With $k_l = \kappa_l k$, $\kappa_l \in [-1, 1]$, $l = 1, 2$ we obtain that

$$\begin{aligned} & \left| \frac{\partial^2}{\partial k^2} (f_1(\mathbf{k})\varphi_-(\mathbf{x}', \mathbf{k})\widehat{\psi}^\epsilon(\mathbf{k})k^2 e^{ik_1 a_1 + ik_2 a_2}) \right| \\ & \leq 2 \sum_{|i|=2} \left| \frac{\partial^{i_1}}{\partial k^{i_1}} (f_1(\mathbf{k})\varphi_-(\mathbf{x}', \mathbf{k})) \right| \left| \frac{\partial^{i_2}}{\partial k^{i_2}} (\widehat{\psi}^\epsilon(\mathbf{k})k^2) \right| \left| \frac{\partial^{i_3}}{\partial k^{i_3}} (e^{i\kappa_1 k a_1}) \right| \left| \frac{\partial^{i_4}}{\partial k^{i_4}} (e^{i\kappa_2 k a_2}) \right| \\ & \leq c \sum_{|i|=2} (1+x')^{i_1} \left| \frac{\partial^{i_2}}{\partial k^{i_2}} (\widehat{\psi}^\epsilon(\mathbf{k})k^2) \right| |\kappa_1 a_1|^{i_3} |\kappa_2 a_2|^{i_4} \\ & \leq c \sum_{|i|=2} (1+x')^{i_1} \left| \frac{\partial^{i_2}}{\partial k^{i_2}} (\widehat{\psi}^\epsilon(\mathbf{k})k^2) \right| a^{i_3} a^{i_4} \\ & \leq c \sum_{|i|=2} (1+x')^{i_1} \left| \frac{\partial^{i_2}}{\partial k^{i_2}} (\widehat{\psi}^\epsilon(\mathbf{k})k^2) \right| |\mathbf{x} - \mathbf{x}'|^{i_3+i_4} \\ & \leq c \sum_{|j|=2} (1+x')^{j_1} \left| \frac{\partial^{j_2}}{\partial k^{j_2}} (\widehat{\psi}^\epsilon(\mathbf{k})k^2) \right| |\mathbf{x} - \mathbf{x}'|^{j_3}. \end{aligned} \quad (82)$$

With (79) we may assume that λ in (81) is bounded below,

$$\lambda \geq r(1 - \sin \alpha) + L^\epsilon \cos \alpha \geq \lambda_{\min} := \eta(r + L^\epsilon), \quad (83)$$

with $\eta := \min((1 - \sin \alpha), \cos \alpha) > 0$. Using (83) and (82) in (80) we obtain that

$$\begin{aligned} M & := \int_{Y_{L^\epsilon}} |I_1| d^2 y \leq c \sum_{|j|=2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} |\psi^\epsilon(\mathbf{x})| \\ & \quad \times \int_{\mathbb{R}^3} |V(\mathbf{x}')| \int_{S^2} \int_0^\infty \frac{1}{\sqrt{r^2 + b_3^2 \lambda_{\min}^2}} \left| \frac{\partial^{j_2}}{\partial k^{j_2}} (\widehat{\psi}^\epsilon(\mathbf{k})k^2) \right| |\mathbf{x} - \mathbf{x}'|^{j_3} (1+x')^{j_1} \\ & \quad \times dk d\Omega d^3 x' d^3 x d^2 y. \end{aligned} \quad (84)$$

Since the integrand of the right-hand side of (84) is positive, we may perform the change of integration variables $(y_1, y_2) \rightarrow (\tilde{y}_1, \tilde{y}_2) \rightarrow (r, \theta)$, as well as freely interchange the

order of integrations. With (83) we then obtain that

$$\begin{aligned}
 M &\leq c \sum_{|j|=2} \int_{\mathbb{R}^3} |\psi^\epsilon(\mathbf{x})| \int_{\mathbb{R}^3} |V(\mathbf{x}')| \\
 &\quad \times \int_{S^2} \int_0^\infty \int_0^\infty \int_0^{2\pi} \frac{1}{\sqrt{r^2 + b_3^2 \lambda_{\min}^2}} \left| \partial_k^{j_2} \left(\widehat{\psi}^\epsilon(\mathbf{k}) k^2 \right) \right| |\mathbf{x} - \mathbf{x}'|^{j_3} (1+x')^{j_1} r d\theta dr dk d\Omega d^3 x' d^3 x \\
 &\leq c \sum_{|j|=2} \int_{\mathbb{R}^3} |\psi^\epsilon(\mathbf{x})| \int_{\mathbb{R}^3} |V(\mathbf{x}')| \\
 &\quad \times \int_{S^2} \int_0^\infty \int_0^\infty \frac{1}{\eta^2(r+L^\epsilon)^2} \left| \partial_k^{j_2} \left(\widehat{\psi}^\epsilon(\mathbf{k}) k^2 \right) \right| |\mathbf{x} - \mathbf{x}'|^{j_3} (1+x')^{j_1} r dr dk d\Omega d^3 x' d^3 x \\
 &= \frac{c}{\eta^2 L^\epsilon} \sum_{|j|=2} \int_{\mathbb{R}^3} |\psi^\epsilon(\mathbf{x})| \int_{\mathbb{R}^3} |V(\mathbf{x}')| \\
 &\quad \times \int_{S^2} \int_0^\infty \left| \partial_k^{j_2} \left(\widehat{\psi}^\epsilon(\mathbf{k}) k^2 \right) \right| |\mathbf{x} - \mathbf{x}'|^{j_3} (1+x')^{j_1} dk d\Omega d^3 x' d^3 x. \tag{85}
 \end{aligned}$$

Using that $|\mathbf{x} - \mathbf{x}'|^{j_3} \leq 2(x^{j_3} + x'^{j_3})$ for $j_3 = 1, 2$ we obtain that

$$\begin{aligned}
 M &\leq \frac{c}{L^\epsilon} \sum_{|j|=2} \int_{\mathbb{R}^3} |\psi^\epsilon(\mathbf{x})| (1+x)^{j_3} \int_{\mathbb{R}^3} \left| \partial_k^{j_2} \left(\widehat{\psi}^\epsilon(\mathbf{k}) k^2 \right) \right| \\
 &\quad \times \int_{\mathbb{R}^3} |V(\mathbf{x}')| (1+x')^{j_1+j_3} d^3 x' dk d\Omega d^3 x. \tag{86}
 \end{aligned}$$

Since $V \in (V)_5$ (so that $V \in L^2(\mathbb{R}^3)$ and $|V(\mathbf{x})| \leq Cx^{-5-\delta}$, $\delta > 0$, for $x > R_0$) and $j_1 + j_3 \leq 2$ the \mathbf{x}' integration is finite and we obtain (by dividing the integration region for \mathbf{x}' into two parts, $x' > R_0$ and $x' \leq R_0$)

$$M \leq \frac{c}{L^\epsilon} \sum_{j_2+j_3 \leq 2} \int_{\mathbb{R}^3} |\psi^\epsilon(\mathbf{x})| (1+x)^{j_3} \int_{\mathbb{R}^3} \left| \partial_k^{j_2} \left(\widehat{\psi}^\epsilon(\mathbf{k}) k^2 \right) \right| dk d\Omega d^3 x. \tag{87}$$

Using (40), (41) and that $\psi \in \mathcal{S}(\mathbb{R}^3)$ one finds by simple calculation that

$$\int_{\mathbb{R}^3} |\psi^\epsilon(\mathbf{x})| x^{j_3} d^3 x \leq \frac{c}{\epsilon^{\frac{3}{2}}} \frac{1}{\epsilon^{j_3}} \tag{88}$$

and

$$\int_{\mathbb{R}^3} \left| \partial_k^{j_2} \left(\widehat{\psi}^\epsilon(\mathbf{k}) k^2 \right) \right| dk d\Omega \leq c \epsilon^{\frac{3}{2}} \frac{1}{\epsilon^{j_2}}. \tag{89}$$

Since $j_2 + j_3 \leq 2$ we see with (88), (89) and (42) that for M in (87) we have for small ϵ the bound

$$M \leq \frac{c}{L^\epsilon \epsilon^2} = c\epsilon^{l-2}. \quad (90)$$

Since $l > 2$, this completes the proof of (63). \square

We can now proceed with the evaluation of (56). With (8) we obtain for (56)

$$\begin{aligned} \sigma(\Sigma) &= \lim_{\epsilon \rightarrow 0} \int_{C_{\Sigma} A^\epsilon} \int |\widehat{T\psi}_y^\epsilon(\mathbf{k})|^2 d^2 y d^3 k \\ &= \lim_{\epsilon \rightarrow 0} 4\pi^2 \int_{C_{\Sigma} A^\epsilon} \int \left| \int_{k'=k} e^{-i\mathbf{k}' \cdot \mathbf{y}} T(\mathbf{k}, \mathbf{k}') \widehat{\psi}^\epsilon(\mathbf{k}') k' d\Omega(\mathbf{k}') \right|^2 d^2 y d^3 k \\ &= \lim_{\epsilon \rightarrow 0} 4\pi^2 \int_{C_{\Sigma} y_p < \frac{D^\epsilon}{2}} \int \left| \int_{k'=k} e^{-i(k'_1 y_1 + k'_2 y_2 - k'_3 L^\epsilon)} T(\mathbf{k}, \mathbf{k}') \widehat{\psi}^\epsilon(\mathbf{k}') k' d\Omega(\mathbf{k}') \right|^2 dy_1 dy_2 d^3 k, \end{aligned} \quad (91)$$

where $\mathbf{y}_p := (y_1, y_2)$. We insert again the identity $f_1 + f_2 \equiv 1$ and obtain for $\sigma(\Sigma)$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} 4\pi^2 \int_{C_{\Sigma} y_p < \frac{D^\epsilon}{2}} \int \left| \int_{k'=k} e^{-i(k'_1 y_1 + k'_2 y_2 - k'_3 L^\epsilon)} T(\mathbf{k}, \mathbf{k}') \widehat{\psi}^\epsilon(\mathbf{k}') (f_1(\mathbf{k}') + f_2(\mathbf{k}')) k' d\Omega(\mathbf{k}') \right|^2 \\ \times dy_1 dy_2 d^3 k. \end{aligned} \quad (92)$$

Multiplying out we get four terms. The main term is

$$\lim_{\epsilon \rightarrow 0} 4\pi^2 \int_{C_{\Sigma} y_p < \frac{D^\epsilon}{2}} \int \left| \int_{k'=k} e^{-i(k'_1 y_1 + k'_2 y_2 - k'_3 L^\epsilon)} T(\mathbf{k}, \mathbf{k}') \widehat{\psi}^\epsilon(\mathbf{k}') f_1(\mathbf{k}') k' d\Omega(\mathbf{k}') \right|^2 dy_1 dy_2 d^3 k. \quad (93)$$

Before we evaluate (93) we show that the three other terms are zero. Noting that $T(\mathbf{k}, \mathbf{k}')$ is bounded (Corollary 1) and that $\psi \in \mathcal{S}(\mathbb{R}^3)$ we obtain that

$$\begin{aligned} \left| \int_{k'=k} e^{-i(k'_1 y_1 + k'_2 y_2 - k'_3 L^\epsilon)} T(\mathbf{k}, \mathbf{k}') \widehat{\psi}^\epsilon(\mathbf{k}') f_i(\mathbf{k}') k' d\Omega(\mathbf{k}') \right| &\leq \frac{c}{\epsilon^{\frac{3}{2}}} k, \quad i = 1, 2, \\ \left| \int_{k'=k} e^{-i(k'_1 y_1 + k'_2 y_2 - k'_3 L^\epsilon)} T(\mathbf{k}, \mathbf{k}') \widehat{\psi}^\epsilon(\mathbf{k}') f_2(\mathbf{k}') k' d\Omega(\mathbf{k}') \right| &\leq \frac{c}{\epsilon^{\frac{3}{2}}} k \int_{k'=k} \left| \widehat{\psi} \left(\frac{\mathbf{k}' - \mathbf{k}_0}{\epsilon} \right) \right| \\ &\times f_2(\mathbf{k}') d\Omega(\mathbf{k}'). \end{aligned} \quad (94)$$

Using (94), the difference between (93) and (92) is no greater than

$$\begin{aligned}
 & \frac{c}{\epsilon^3} \int_{C_\Sigma} \int_{y_p < \frac{D\epsilon}{2}} \int_{k'=k} \left| \widehat{\psi} \left(\frac{\mathbf{k}' - \mathbf{k}_0}{\epsilon} \right) \right| f_2(\mathbf{k}') k'^2 d\Omega(\mathbf{k}') d^2 y d^3 k \\
 & \leq \frac{c}{\epsilon^{3+2d}} \int_{\mathbb{R}^3} \left| \widehat{\psi} \left(\frac{\mathbf{k}' - \mathbf{k}_0}{\epsilon} \right) \right| f_2(\mathbf{k}') k'^2 d^3 k' \\
 & \leq \frac{c}{\epsilon^{3+2d}} \int_{|\mathbf{k}' - \mathbf{k}_0| \geq \frac{k_0}{3}} \left| \widehat{\psi} \left(\frac{\mathbf{k}' - \mathbf{k}_0}{\epsilon} \right) \right| k'^2 d^3 k'. \tag{95}
 \end{aligned}$$

Using that $|\widehat{\psi}(\mathbf{k})| \leq \frac{c}{k^n}$ for any $6 \leq n \in \mathbb{N}$, we see that the right-hand side in (95) is bounded by $c\epsilon^{n-3-2d}$, which tends to zero for sufficiently large n . Thus the three other terms are zero.

Since, as we shall show,

$$\begin{aligned}
 & \lim_{\epsilon \rightarrow 0} 4\pi^2 \int_{C_\Sigma} \int_{y_p \geq \frac{D\epsilon}{2}} \left| \int_{k'=k} e^{-i(k'_1 y_1 + k'_2 y_2 - k'_3 L\epsilon)} T(\mathbf{k}, \mathbf{k}') \widehat{\psi}^\epsilon(\mathbf{k}') f_1(\mathbf{k}') k' d\Omega(\mathbf{k}') \right|^2 \\
 & \quad \times dy_1 dy_2 d^3 k = 0, \tag{96}
 \end{aligned}$$

we may extend the \mathbf{y} -integration in (93) to all of \mathbb{R}^2 , so that

$$\begin{aligned}
 \sigma(\Sigma) &= \lim_{\epsilon \rightarrow 0} 4\pi^2 \int_{C_\Sigma} \int_{\mathbb{R}^2} \left| \int_{k'=k} e^{-i(k'_1 y_1 + k'_2 y_2 - k'_3 L\epsilon)} T(\mathbf{k}, \mathbf{k}') \widehat{\psi}^\epsilon(\mathbf{k}') f_1(\mathbf{k}') k' d\Omega(\mathbf{k}') \right|^2 \\
 & \quad \times dy_1 dy_2 d^3 k. \tag{97}
 \end{aligned}$$

Before establishing (96) we compute (97) with the help of the following

Lemma 4. *Let $0 < \alpha < \frac{\pi}{2}$ and $\delta > 0$ be given. Suppose that $\phi : \mathbb{R}^3 \rightarrow \mathbb{C}$ is a function with support in the sector $P_{\mathbf{e}_3}^\alpha := \{\mathbf{k} \in \mathbb{R}^3 : \mathbf{k} \cdot \mathbf{e}_3 > k \cos \alpha\}$ such that $\int_{k=\delta} |\phi(\mathbf{k})|^2 d\Omega(\mathbf{k}) < \infty$. Then*

$$\int_{\mathbb{R}^2} \left| \frac{1}{2\pi} \int_{k=\delta} e^{-i\mathbf{k} \cdot \mathbf{y}} \phi(\mathbf{k}) d\Omega(\mathbf{k}) \right|^2 d^2 y = \int_{k=\delta} |\phi(\mathbf{k})|^2 \frac{1}{kk_3} d\Omega(\mathbf{k}). \tag{98}$$

Remark 11. This lemma is proved in [2], Lemma 7.17. The integration over the impact parameter is crucial for the derivation and is a standard ingredient in the derivation of the scattering cross section.

Because of Corollary 1, $T(\mathbf{k}, \mathbf{k}')$ is bounded on $\mathbb{R}^3 \times \mathbb{R}^3$ and continuous on $\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\}$. Moreover, $\widehat{\psi}^\epsilon(\mathbf{k}) \in \mathcal{S}(\mathbb{R}^3)$ and $\widehat{\psi}^\epsilon(\mathbf{k}) f_1(\mathbf{k})$ has support in $P_{\mathbf{e}_3}^{\vartheta_2}$ with $0 < \vartheta_2 < \frac{\pi}{2}$. Hence, by Lemma 4, (97) becomes

$$\begin{aligned} \sigma(\Sigma) &= \lim_{\epsilon \rightarrow 0} 16\pi^4 \int_{C_\Sigma} \int_{k'=k} |T(\mathbf{k}, \mathbf{k}')|^2 |\widehat{\psi}^\epsilon(\mathbf{k}')|^2 |f_1(\mathbf{k}')|^2 \frac{1}{\cos \vartheta'} d\Omega(\mathbf{k}') d^3k \\ &= \lim_{\epsilon \rightarrow 0} 16\pi^4 \int_{\Sigma} \int_{\mathbb{R}^3} |T(k'\boldsymbol{\omega}, \mathbf{k}')|^2 |\widehat{\psi}^\epsilon(\mathbf{k}')|^2 |f_1(\mathbf{k}')|^2 \frac{1}{\cos \vartheta'} d^3k' d\Omega, \end{aligned} \quad (99)$$

where $k'_3 = k \cos \vartheta'$. Because $\text{supp } f_1(\mathbf{k}) \subset P_{\mathbf{e}_3}^{\vartheta_2}$ with $0 < \vartheta_2 < \frac{\pi}{2}$, there exists a $\delta > 0$ such that $\delta < \cos \vartheta'$. Hence the integral in (99) is finite (it is $\leq c \|\psi\|^2$). Thus, since clearly $|\widehat{\psi}^\epsilon(\mathbf{k})|^2 \rightarrow \delta(\mathbf{k} - \mathbf{k}_0)$ (in the sense that $\lim_{\epsilon \rightarrow 0} \int |\widehat{\psi}^\epsilon(\mathbf{k})|^2 g(\mathbf{k}) d^3k = g(\mathbf{k}_0)$ for any bounded continuous function g), and since $T(k'\boldsymbol{\omega}, \mathbf{k}')$, $f_1(\mathbf{k}')$ and $\frac{1}{\cos \vartheta'}$ are bounded and continuous as functions of \mathbf{k}' , we may conclude that

$$\sigma(\Sigma) = 16\pi^4 \int_{\Sigma} |T(k_0\boldsymbol{\omega}, \mathbf{k}_0)|^2 d\Omega. \quad (100)$$

The proof of Theorem 1 and Theorem 2 will thus be complete once we establish (96). Changing variables, (96) follows from

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} \int_{y_p \geq \frac{D}{2}} \frac{1}{\epsilon^{2d}} \left| \int_{k'=k} e^{-i(k'_1 \frac{y_1}{\epsilon^d} + k'_2 \frac{y_2}{\epsilon^d} - k'_3 L^\epsilon)} T(\mathbf{k}, \mathbf{k}') \widehat{\psi}^\epsilon(\mathbf{k}') f_1(\mathbf{k}') k' d\Omega(\mathbf{k}') \right|^2 dy_1 dy_2 d^3k = 0. \quad (101)$$

Equation (101) is the content of

Lemma 5. *Let $V \in (V)_5$, $\psi \in \mathcal{S}(\mathbb{R}^3)$ and suppose that $k_0 > 0$. Let $l > 2$, $d > 2l - 3$ and let M be given by (to simplify the notation we interchange \mathbf{k} and \mathbf{k}')*

$$M = M(y_1, y_2, \mathbf{k}', \epsilon) := \int_{k=k'} e^{-i(k_1 \frac{y_1}{\epsilon^d} + k_2 \frac{y_2}{\epsilon^d} - k_3 L^\epsilon)} T(\mathbf{k}', \mathbf{k}) \widehat{\psi}^\epsilon(\mathbf{k}) f_1(\mathbf{k}) k d\Omega(\mathbf{k}). \quad (102)$$

Then for any $D > 0$,

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} \int_{y_p \geq D} \frac{1}{\epsilon^{2d}} |M|^2 dy_1 dy_2 d^3k' = 0. \quad (103)$$

Proof. We will establish the following inequality (104) giving a bound on M : There exists a $c < \infty$ such that

$$|M|^2 \leq c \chi_{\left(\frac{k_0}{2}, \frac{3}{2}k_0\right)}(k') \frac{\epsilon^{4d+5-4l}}{y_p^4} \frac{1}{\left(1 + \frac{|k'-k_0|}{\epsilon}\right)^2}. \quad (104)$$

Assuming (104) we show now that (103) follows. Using (104), the integral in (103) is dominated by

$$\begin{aligned}
 \int_{\frac{k_0}{2} < k' < \frac{3}{2}k_0} \int_{y_p \geq D} c \frac{\epsilon^{2d+5-4l}}{y_p^4} \frac{1}{\left(1 + \frac{|k'-k_0|}{\epsilon}\right)^2} d^2 y d^3 k' &\leq c \epsilon^{2d+5-4l} \int_{-\infty}^{\infty} \frac{dk'}{\left(1 + \frac{|k'-k_0|}{\epsilon}\right)^2} \\
 &= c \epsilon^{2d+6-4l} \int_{-\infty}^{\infty} \frac{dk'}{(1 + |k'|)^2} \\
 &= c \epsilon^{2d+6-4l}. \tag{105}
 \end{aligned}$$

Since $d > 2l - 3$ there is a $\delta > 0$ such that $d = 2l - 3 + \delta$. Then (105) is of order $\epsilon^{2\delta}$ and (103) follows.

It thus remains to establish (104). Changing variables in (102) from ω to k_1, k_2 we obtain, with the Jacobian determinant $k'k_3$ with $k_3 = k_3(k_1, k_2) = \sqrt{k'^2 - k_1^2 - k_2^2}$ and $\mathbf{k}_+ = (k_1, k_2, k_3(k_1, k_2))$,

$$\begin{aligned}
 M &= \int \int_{k_1^2 + k_2^2 \leq k'^2} e^{-i(k_1 \frac{y_1}{\epsilon^d} + k_2 \frac{y_2}{\epsilon^d} - k_3 L^\epsilon)} T(\mathbf{k}', \mathbf{k}_+) \widehat{\psi}^\epsilon(\mathbf{k}_+) f_1(\mathbf{k}_+) k' \frac{1}{k'k_3} dk_1 dk_2 \\
 &= \frac{1}{\epsilon^{\frac{3}{2}}} \int \int_{k_1^2 + k_2^2 \leq k'^2} e^{-i(k_1 \frac{y_1}{\epsilon^d} + k_2 \frac{y_2}{\epsilon^d})} \left(T(\mathbf{k}', \mathbf{k}_+) \widehat{\psi} \left(\frac{\mathbf{k}_+ - \mathbf{k}_0}{\epsilon} \right) e^{ik_3 L^\epsilon} \frac{f_1(\mathbf{k}_+)}{k_3} \right) dk_1 dk_2 \\
 &=: \frac{1}{\epsilon^{\frac{3}{2}}} \int \int_{k_1^2 + k_2^2 \leq k'^2} e^{-i(k_1 \frac{y_1}{\epsilon^d} + k_2 \frac{y_2}{\epsilon^d})} g(k_1, k_2, \mathbf{k}', \epsilon) dk_1 dk_2. \tag{106}
 \end{aligned}$$

Performing two integration by parts with respect to $\mathbf{k}_p := (k_1, k_2)$, we obtain (using the fact that $f_1(\mathbf{k}_+)$ and its derivatives vanish on the boundary of the region of integration) that

$$\begin{aligned}
 |M| &= \frac{1}{\epsilon^{\frac{3}{2}}} \epsilon^d \left| \int \int_{k_p \leq k'} \left(\nabla_{\mathbf{k}_p} e^{-i(k_1 \frac{y_1}{\epsilon^d} + k_2 \frac{y_2}{\epsilon^d})} \right) \cdot \frac{\mathbf{y}_p}{y_p^2} f_1(\mathbf{k}_+) g(k_1, k_2, \mathbf{k}', \epsilon) dk_1 dk_2 \right| \\
 &= \frac{1}{\epsilon^{\frac{3}{2}}} \epsilon^d \left| \int \int_{k_p \leq k'} e^{-i(k_1 \frac{y_1}{\epsilon^d} + k_2 \frac{y_2}{\epsilon^d})} \frac{\mathbf{y}_p}{y_p^2} \cdot \nabla_{\mathbf{k}_p} g(k_1, k_2, \mathbf{k}', \epsilon) dk_1 dk_2 \right| \\
 &= \frac{1}{\epsilon^{\frac{3}{2}}} \epsilon^{2d} \left| \int \int_{k_p \leq k'} \left(\nabla_{\mathbf{k}_p} e^{-i(k_1 \frac{y_1}{\epsilon^d} + k_2 \frac{y_2}{\epsilon^d})} \right) \cdot \frac{\mathbf{y}_p}{y_p^2} \frac{\mathbf{y}_p}{y_p^2} \cdot \nabla_{\mathbf{k}_p} g(k_1, k_2, \mathbf{k}', \epsilon) dk_1 dk_2 \right|
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\epsilon^{\frac{3}{2}}} \epsilon^{2d} \left| \int \int_{k_p \leq k'} e^{-i(k_1 \frac{y_1}{\epsilon^d} + k_2 \frac{y_2}{\epsilon^d})} \frac{\mathbf{y}_p}{y_p^2} \cdot \nabla_{\mathbf{k}_p} \frac{\mathbf{y}_p}{y_p^2} \cdot \nabla_{\mathbf{k}_p} g(k_1, k_2, \mathbf{k}', \epsilon) dk_1 dk_2 \right| \\
&\leq \frac{1}{\epsilon^{\frac{3}{2}}} \frac{\epsilon^{2d}}{y_p^2} \int \int_{k_p \leq k'} \sum_{i,j=1}^2 |\partial_{k_i} \partial_{k_j} g(k_1, k_2, \mathbf{k}', \epsilon)| dk_1 dk_2. \tag{107}
\end{aligned}$$

We estimate now the derivatives of g on the support of f_1 . Note first that on $\text{supp } f_1$ $k_3 > k_0/2$. Using Corollary 1 we have for $i, j = 1, 2$ that

$$\begin{aligned}
\sup_{\mathbf{k}' \in \mathbb{R}^3, \mathbf{k}_+ \in \text{supp } f_1} |T(\mathbf{k}', \mathbf{k}_+)| &\leq c, & \sup_{\mathbf{k}' \in \mathbb{R}^3, \mathbf{k}_+ \in \text{supp } f_1} |\partial_{k_i} T(\mathbf{k}', \mathbf{k}_+)| &\leq c, \tag{108} \\
\sup_{\mathbf{k}' \in \mathbb{R}^3, \mathbf{k}_+ \in \text{supp } f_1} |\partial_{k_i} \partial_{k_j} T(\mathbf{k}', \mathbf{k}_+)| &\leq c.
\end{aligned}$$

To estimate the wave function $\widehat{\psi}\left(\frac{\mathbf{k}_+ - \mathbf{k}_0}{\epsilon}\right)$ and its derivatives we introduce the following notation:

$$P_{\mathbf{k}} := \frac{1}{1 + \frac{|\mathbf{k} - \mathbf{k}_0|}{\epsilon}}, \quad P_{\mathbf{k}'} := \frac{1}{1 + \frac{|\mathbf{k}' - \mathbf{k}_0|}{\epsilon}}. \tag{109}$$

Clearly

$$P_{\mathbf{k}} \leq P_{\mathbf{k}'}. \tag{110}$$

Since $\psi \in \mathcal{S}(\mathbb{R}^3)$, $\widehat{\psi}$ and its derivatives decay faster than the reciprocal of any polynomial, we can find for $\mathbf{k}_+ \in \text{supp } f_1$ and for $n \in \mathbb{N}$ suitable constants such that

$$\left| \widehat{\psi}\left(\frac{\mathbf{k}_+ - \mathbf{k}_0}{\epsilon}\right) \right| \leq c P_{\mathbf{k}_+}^n, \quad \left| \partial_{k_i} \widehat{\psi}\left(\frac{\mathbf{k}_+ - \mathbf{k}_0}{\epsilon}\right) \right| \leq \frac{c}{\epsilon} P_{\mathbf{k}_+}^n, \quad \left| \partial_{k_i} \partial_{k_j} \widehat{\psi}\left(\frac{\mathbf{k}_+ - \mathbf{k}_0}{\epsilon}\right) \right| \leq \frac{c}{\epsilon^2} P_{\mathbf{k}_+}^n. \tag{111}$$

The derivatives of the third factor $e^{-ik_3 L^\epsilon}$ of g can be estimated on $\text{supp } f_1$ as follows:

$$\left| e^{-ik_3 L^\epsilon} \right| \leq 1, \quad \left| \partial_{k_i} e^{-ik_3 L^\epsilon} \right| \leq L^\epsilon \frac{|k_i|}{|k_3|} \leq L^\epsilon |k_i|. \tag{112}$$

Since $|k_i| P_{\mathbf{k}_+} \leq \epsilon$, we obtain using (111) with $n = j + 1$ and (42) that

$$\left| \left(\partial_{k_i} e^{-ik_3 L^\epsilon} \right) \widehat{\psi}\left(\frac{\mathbf{k}_+ - \mathbf{k}_0}{\epsilon}\right) \right| \leq c L^\epsilon |k_i| P_{\mathbf{k}_+} P_{\mathbf{k}_+}^j \leq c L^\epsilon \epsilon P_{\mathbf{k}_+}^j = \frac{c}{\epsilon^{l-1}} P_{\mathbf{k}_+}^j, \quad j \text{ arbitrary}. \tag{113}$$

With a similar calculation we find that

$$\left| \left(\partial_{k_i} \partial_{k_j} e^{-ik_3 L^\epsilon} \right) \widehat{\psi}\left(\frac{\mathbf{k}_+ - \mathbf{k}_0}{\epsilon}\right) \right| \leq \frac{c}{\epsilon^{2l-2}} P_{\mathbf{k}_+}^j, \quad j \text{ arbitrary}, \tag{114}$$

and analogous estimates for terms which contains derivatives of $\widehat{\psi}\left(\frac{\mathbf{k}_+-\mathbf{k}_0}{\epsilon}\right)$. Clearly we have that

$$\sup_{\mathbf{k}_+ \in \text{supp } f_1} \left| \frac{f_1(\mathbf{k}_+)}{k_3} \right| \leq c, \quad \sup_{\mathbf{k}_+ \in \text{supp } f_1} \left| \partial_{k_i} \frac{f_1(\mathbf{k}_+)}{k_3} \right| \leq c, \quad \sup_{\mathbf{k}_+ \in \text{supp } f_1} \left| \partial_{k_i} \partial_{k_j} \frac{f_1(\mathbf{k}_+)}{k_3} \right| \leq c, \quad i, j = 1, 2. \quad (115)$$

Combining (108), (111)–(115) and using that $2l - 2 > 2$ since $l > 2$ we obtain for all $\mathbf{k}' \in \mathbb{R}^3$ and any $n \in \mathbb{N}$ that

$$\left| \partial_{k_i} \partial_{k_j} g(k_1, k_2, \mathbf{k}', \epsilon) \right| \leq \frac{c}{\epsilon^{2l-2}} P_{\mathbf{k}_+}^n, \quad (116)$$

for all (k_1, k_2) such that $\mathbf{k}_+ \in \text{supp } f_1$.

Reintroducing the original integration variable $\boldsymbol{\omega}$ we then have that

$$\begin{aligned} |M| &\leq \frac{c}{y_p^2} \epsilon^{2d-2l+\frac{1}{2}} \int_{k=k'} \chi_{\{f_1>0\}} P_{\mathbf{k}}^n k' k_3 d\Omega(\mathbf{k}) \\ &\leq \frac{c}{y_p^2} \epsilon^{2d-2l+\frac{1}{2}} \chi_{\left(\frac{k_0}{2}, \frac{3}{2}k_0\right)}(k') \int_{k=k', |\mathbf{k}-\mathbf{k}_0| < \frac{k_0}{2}} P_{\mathbf{k}}^n d\Omega(\mathbf{k}). \end{aligned} \quad (117)$$

Choosing $n = 4$ in (117) and splitting $P_{\mathbf{k}}^4$ into

$$P_{\mathbf{k}}^4 = P_{\mathbf{k}}^1 P_{\mathbf{k}}^3 \leq P_{\mathbf{k}}^1 P_{\mathbf{k}}^3, \quad (118)$$

we obtain that

$$|M| \leq \frac{c}{y_p^2} \epsilon^{2d+\frac{1}{2}-2l} \chi_{\left(\frac{k_0}{2}, \frac{3}{2}k_0\right)}(k') P_{k'}^1 \int_{k=k', |\mathbf{k}-\mathbf{k}_0| < \frac{k_0}{2}} P_{\mathbf{k}}^3 d\Omega(\mathbf{k}). \quad (119)$$

Moreover, it is easy to see that

$$\int_{k=k', |\mathbf{k}-\mathbf{k}_0| < \frac{k_0}{2}} P_{\mathbf{k}}^3 d\Omega(\mathbf{k}) \leq c \int_{\mathbb{R}^2} \frac{1}{\left(1 + \frac{k_p}{\epsilon}\right)^3} dk_1 dk_2 \leq c\epsilon^2. \quad (120)$$

Thus

$$|M| \leq \frac{c}{y_p^2} \epsilon^{2d+\frac{5}{2}-2l} \chi_{\left(\frac{k_0}{2}, \frac{3}{2}k_0\right)}(k') P_{k'}^1 \quad (121)$$

and (104) follows. This completes the proof of Lemma 5. \square

9. Summary and Outlook

The purpose of this paper has been to rigorously derive the standard formula for the scattering cross section starting from a microscopic model of a scattering experiment. While the use of Bohmian mechanics is crucial for our result, we would like to stress that major parts of our proof are vital even from an orthodox point of view. These parts concern in particular the replacement of the incoming asymptote by its scattering state (cf. Lemma 3 and Remark 10) and the flux-across-surfaces theorem in a formulation which depends only on the smoothness of the scattering state (cf. Proposition 3, Lemma 1 and [11]).

Several problems have been left for future work, which we shall mention here.

- **Bound states:** Our assumption A3 arises from the problem that in general the translation of the initial wave function by the impact parameter \mathbf{y} —which is needed for the averaging over the beam profile—will produce wave functions which have a component in the bound states. One would then have to show that asymptotically the crossing statistics are induced by the “relevant part” ψ' of the wave function, namely

$$\psi' := P\psi,$$

where P is the projection onto the absolutely continuous subspace $\mathcal{H}_{\text{a.c.}}(H)$ and is given by

$$P := \Omega_- \Omega_-^*.$$

Note that by using Lemma 3 one can also show that

$$\lim_{L \rightarrow \infty} \int_{Y_L} \|P\psi_{\mathbf{y}} - \psi_{\mathbf{y}}\|^2 d^2y = 0, \quad (122)$$

i.e., that the bound state component is small in an L^2 -sense. This is however not directly applicable.

- It would of course be desirable to derive the crossing statistics for many particles guided in general by an entangled wave function both for the noninteracting case and eventually even for interacting particles [13].
- We are currently working [8] on a detailed formulation of the conditions characterizing the scattering regime, which turns out to be surprisingly intricate. What we have shown here is that the simplest limiting procedure that brings the experimental arrangement into the scattering regime yields the standard formula of formal scattering theory. This formula should of course hold much more generally—more or less for all limits corresponding to the scattering regime—but establishing that this is so remains a formidable challenge.

10. Appendix

Proof of Lemma 1. Let $\psi \in \mathcal{G}$. Then there is a $\chi \in \mathcal{G}^0$ and a $t \in \mathbb{R}$ such that

$$\psi = e^{-iHt} \chi.$$

Using the intertwining property (6) we obtain

$$\psi_{\text{out}} = \Omega_+^{-1} \psi = \Omega_+^{-1} e^{-iHt} \chi = e^{-iH_0 t} \Omega_+^{-1} \chi = e^{-iH_0 t} \chi_{\text{out}}. \quad (123)$$

Since \mathcal{G}^+ is invariant under time shifts it suffices to show that $\widehat{\chi}_{\text{out}}(\mathbf{k})$ is in \mathcal{G}^+ . Since $\langle x \rangle^2 H^n \chi(\mathbf{x}) \in L_2(\mathbb{R}^3)$, $0 \leq n \leq 8$, and $\langle x \rangle^4 H^n \chi(\mathbf{x}) \in L_2(\mathbb{R}^3)$, $0 \leq n \leq 3$, we have

$$\begin{aligned} H^n \chi(\mathbf{x}) &\in L_1(\mathbb{R}^3) \cap L_2(\mathbb{R}^3), \quad 0 \leq n \leq 8, \\ \langle x \rangle^j H^n \chi(\mathbf{x}) &\in L_1(\mathbb{R}^3) \cap L_2(\mathbb{R}^3), \quad 0 \leq n \leq 3, \quad j = \{1, 2\}. \end{aligned} \quad (124)$$

Using Proposition 1 (ii), (iii) we have for $f \in L^2(\mathbb{R}^3)$:

$$\mathcal{F}_+ \Omega_+ f = \mathcal{F} f, \quad (125)$$

and hence for $\chi = \Omega_+ \chi_{\text{out}}$ we have that

$$\widehat{\chi}_{\text{out}}(\mathbf{k}) = \mathcal{F}_+ \chi(\mathbf{k}) = (2\pi)^{-\frac{3}{2}} \int \varphi_+^*(\mathbf{x}, \mathbf{k}) \chi(\mathbf{x}) d^3 x. \quad (126)$$

Using the intertwining property (6) we thus have:

$$\begin{aligned} \frac{k^2}{2} \widehat{\chi}_{\text{out}}(\mathbf{k}) &= \widehat{H_0 \chi_{\text{out}}}(\mathbf{k}) = \mathcal{F}(H_0 \Omega_+^{-1} \chi)(\mathbf{k}) = \mathcal{F}(\Omega_+^{-1} H \chi)(\mathbf{k}) = \mathcal{F}_+(H \chi)(\mathbf{k}) \\ &= (2\pi)^{-\frac{3}{2}} \int \varphi_+^*(\mathbf{x}, \mathbf{k}) (H \chi)(\mathbf{x}) d^3 x. \end{aligned} \quad (127)$$

Similarly, applying H_0^n to $\widehat{\chi}_{\text{out}}(\mathbf{k})$ ($0 \leq n \leq 8$) we obtain

$$\frac{k^{2n}}{2^n} \widehat{\chi}_{\text{out}}(\mathbf{k}) = (2\pi)^{-\frac{3}{2}} \int \varphi_+^*(\mathbf{x}, \mathbf{k}) (H^n \chi)(\mathbf{x}) d^3 x. \quad (128)$$

Since the generalized eigenfunctions are bounded (Proposition 2 (ii)) and $H^n \chi \in L_1(\mathbb{R}^3)$, $0 \leq n \leq 8$, we obtain

$$|\widehat{\chi}_{\text{out}}(\mathbf{k})| \leq c(1+k)^{-16} \leq c(1+k)^{-15}. \quad (129)$$

Because of Proposition 2 (iii) and (124) we can differentiate (126) with respect to k_i and get

$$|\partial_{k_i} \widehat{\chi}_{\text{out}}(\mathbf{k})| = \left| (2\pi)^{-\frac{3}{2}} \int (\partial_{k_i} \varphi_+^*(\mathbf{x}, \mathbf{k})) \chi(\mathbf{x}) d^3 x \right| \leq c, \quad \forall \mathbf{k} \in \mathbb{R}^3 \setminus \{0\}. \quad (130)$$

Differentiating (128) with $n = 3$ with respect to k_i we obtain

$$k^6 \partial_{k_i} \widehat{\chi}_{\text{out}}(\mathbf{k}) = 8(2\pi)^{-\frac{3}{2}} \int (\partial_{k_i} \varphi_+^*(\mathbf{x}, \mathbf{k})) (H^3 \chi)(\mathbf{x}) d^3 x - 6k^5 \widehat{\chi}_{\text{out}}(\mathbf{k}) \frac{k_i}{k}. \quad (131)$$

Again the right-hand side is bounded because of Lemma 2 (iii), (124) and (129). Hence, we obtain with (130):

$$|\partial_{k_i} \widehat{\chi}_{\text{out}}(\mathbf{k})| \leq c(1+k)^{-6}, \quad \forall \mathbf{k} \in \mathbb{R}^3 \setminus \{0\}. \quad (132)$$

Using Proposition 2 (iii) and (126) we may control κ times a second derivative of $\widehat{\chi}_{\text{out}}(\mathbf{k})$, obtaining

$$|\kappa \partial_{k_j} \partial_{k_i} \widehat{\chi}_{\text{out}}(\mathbf{k})| = \left| (2\pi)^{-\frac{3}{2}} \int (\kappa \partial_{k_j} \partial_{k_i} \varphi_+^*(\mathbf{x}, \mathbf{k})) \chi(\mathbf{x}) d^3 x \right| \leq c, \quad \forall \mathbf{k} \in \mathbb{R}^3 \setminus \{0\}. \quad (133)$$

For the last inequality we have also used (124) with $j = 2$ and $n = 0$. Similarly, using (131) we obtain

$$\begin{aligned} k^6 \kappa \partial_{k_j} \partial_{k_i} \widehat{\chi}_{\text{out}}(\mathbf{k}) &= 8(2\pi)^{-\frac{3}{2}} \int (\kappa \partial_{k_j} \partial_{k_i} \varphi_+^*(\mathbf{x}, \mathbf{k})) (H^3 \chi)(\mathbf{x}) d^3 x \\ &\quad - 30k^4 \frac{k_j}{k} \frac{k_i}{k} \kappa \widehat{\chi}_{\text{out}}(\mathbf{k}) - 6k^5 \frac{k_i}{k} \kappa \partial_{k_j} \widehat{\chi}_{\text{out}}(\mathbf{k}) \\ &\quad - 6k^5 \widehat{\chi}_{\text{out}}(\mathbf{k}) \kappa \frac{k \delta_{ij} k - k_i k_j}{k^3} - 6k^5 \frac{k_j}{k} \kappa \partial_{k_i} \widehat{\chi}_{\text{out}}(\mathbf{k}), \end{aligned} \quad (134)$$

with right-hand side that is bounded because of Proposition 2 (iii), (124), (129) and (132). Hence, using (133),

$$|\kappa \partial_{\mathbf{k}}^\alpha \widehat{\chi}_{\text{out}}(\mathbf{k})| \leq c(1+k)^{-6} \leq c(1+k)^{-5}, \quad |\alpha| = 2, \quad \forall \mathbf{k} \in \mathbb{R}^3 \setminus \{0\}. \quad (135)$$

Equation (132) implies also that

$$|\partial_k \widehat{\chi}_{\text{out}}(\mathbf{k})| \leq c(1+k)^{-6}, \quad \forall \mathbf{k} \in \mathbb{R}^3 \setminus \{0\}. \quad (136)$$

Similarly, twice differentiating (126) with respect to k we obtain that

$$\left| \partial_k^2 \widehat{\chi}_{\text{out}}(\mathbf{k}) \right| \leq c, \quad \forall \mathbf{k} \in \mathbb{R}^3 \setminus \{0\}, \quad (137)$$

and then twice differentiating (128) for $n = 2$ with respect to k we obtain

$$\left| \partial_k^2 \widehat{\chi}_{\text{out}}(\mathbf{k}) \right| \leq c(1+k)^{-4} \leq c(1+k)^{-3}, \quad \forall \mathbf{k} \in \mathbb{R}^3 \setminus \{0\}, \quad (138)$$

using Proposition 2 (iv), (124), (129), (136) and (137).

With (129), (132), (135) and (138) we see that $\widehat{\chi}_{\text{out}}(\mathbf{k}) \in \mathcal{G}^+$. \square

Proof of Lemma 2. In the proof of Proposition 3 in [11] the absolute value of the flux integrated over time and the surface RS^2 with $R > R_0$ (with some $R_0 > 0$ depending on the potential) is shown to be bounded (uniformly in R) by linear combinations of integrals involving $\widehat{\psi}_{\text{out}}(\mathbf{k})$ and its derivatives, namely integrals over expressions corresponding to the left hand side of the inequalities in Definition 2. Thus these bounds are finite if $\widehat{\psi}_{\text{out}}(\mathbf{k}) \in \mathcal{G}^+$. To bound the integrated flux uniformly for all ψ_y^ϵ , $y \in A^\epsilon$ (and ϵ small enough and fixed), $\mathcal{F}(\psi_{y,\text{out}}^\epsilon)(\mathbf{k}) = \mathcal{F}(\Omega_+^{-1} \psi_y^\epsilon)(\mathbf{k})$ (note that $\psi_y^\epsilon \in \mathcal{H}_{\text{a.c.}}(H)$, for all $y \in A^\epsilon$, cf. (i) in Definition 3 or 4) must be bounded as in Definition 2 with

constants uniform in $\mathbf{y} \in A^\epsilon$. These constants depend, according to the proof of Lemma 1, on the norms of

$$\|H^n \psi_y^\epsilon\|_1, \quad 0 \leq n \leq 8 \text{ and } \|(x)^j H^n \psi_y^\epsilon\|_1, \quad 0 \leq n \leq 3, \quad j \in \{1, 2\}. \quad (139)$$

We will show that for ϵ small enough there exists a constant $C > 0$ such that

$$|H^n \psi_y^\epsilon(\mathbf{x})| \leq C(1+x)^{-6}, \quad 0 \leq n \leq 8, \quad \forall \mathbf{y} \in A^\epsilon. \quad (140)$$

Thus the norms in (139) are bounded uniformly in $\mathbf{y} \in A^\epsilon$ and Lemma 2 follows.

It remains to establish (140). We start with $n = 0$. Since $\psi \in \mathcal{S}(\mathbb{R}^3)$ and $\mathbf{y} \in A^\epsilon$, A^ϵ compact, we obtain

$$|\psi_y^\epsilon(\mathbf{x})| = \epsilon^{\frac{3}{2}} |\psi(\epsilon(\mathbf{x} - \mathbf{y}))| \leq c(1 + |\mathbf{x} - \mathbf{y}|)^{-6} \leq c(1+x)^{-6}, \quad \forall \mathbf{y} \in A^\epsilon. \quad (141)$$

For $n = 1$ we have with $\psi_y^\epsilon \equiv T_y \psi^\epsilon$ (T_y is the translation operator) and $[T_y, H_0]_- = 0$,

$$|H \psi_y^\epsilon(\mathbf{x})| = |(H_0 + V)T_y \psi^\epsilon(\mathbf{x})| = |T_y H_0 \psi^\epsilon(\mathbf{x})| + \epsilon^{\frac{3}{2}} |V(\mathbf{x}) \psi(\epsilon(\mathbf{x} - \mathbf{y}))|. \quad (142)$$

Using now $|V(\mathbf{x})| < M < \infty$ for $V \in \mathcal{V}$ or $\sup_{\mathbf{x} \in \text{supp } \psi_y^\epsilon} |V(\mathbf{x})| < M < \infty$ for $\psi \in C_0^\infty(\mathbb{R}^3)$, $V \in \mathcal{V}'$, $\mathbf{y} \in A^\epsilon$ and ϵ small enough, we obtain together with (141),

$$|H \psi_y^\epsilon(\mathbf{x})| \leq |T_y H_0 \psi^\epsilon(\mathbf{x})| + c(1+x)^{-6}. \quad (143)$$

Since $\psi^\epsilon \in \mathcal{S}(\mathbb{R}^3)$ we have that also $H_0 \psi^\epsilon \in \mathcal{S}(\mathbb{R}^3)$ so that analogously to (141), there is the bound

$$|T_y H_0 \psi^\epsilon(\mathbf{x})| \leq c(1+x)^{-6}, \quad \forall \mathbf{y} \in A^\epsilon. \quad (144)$$

Equations (143) and (144) yield (140) for $n = 1$. Analogously, we obtain (140) for $2 \leq n \leq 8$ by using the fact that $\psi \in \mathcal{S}(\mathbb{R}^3)$ and $|\partial_x^\alpha V(\mathbf{x})| < M < \infty$, $\forall |\alpha| \leq 14$, if $V \in \mathcal{V}$ or $\sup_{\mathbf{x} \in \text{supp } \psi_y^\epsilon} |\partial_x^\alpha V(\mathbf{x})| < M < \infty$, $\forall |\alpha| \leq 14$, for all $\mathbf{y} \in A^\epsilon$ and ϵ small enough if $\psi \in C_0^\infty(\mathbb{R}^3)$ and $V \in \mathcal{V}'$. \square

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