On the Common Structure of Bohmian Mechanics and the Ghirardi–Rimini–Weber Theory

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Abstract

Bohmian mechanics and the Ghirardi–Rimini–Weber theory provide opposite resolutions of the quantum measurement problem: the former postulates additional variables (the particle positions) besides the wave function, whereas the latter implements spontaneous collapses of the wave function by a nonlinear and stochastic modification of Schrödinger's equation. Still, both theories, when understood appropriately, share the following structure: They are ultimately not about wave functions but about 'matter' moving in space, represented by either particle trajectories, fields on space-time, or a discrete set of space-time points. The role of the wave function then is to govern the motion of the matter.

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Dedicated to GianCarlo Ghirardi on the occasion of his 70th birthday

Contents

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1   Introduction

Bohmian mechanics (BM) and the Ghirardi–Rimini–Weber (GRW) theory are two quantum theories without observers, and thus provide two possible solutions of the measurement problem of quantum mechanics. However, they would seem to have little in common beyond achieving the goal of describing a possible reality in which observers would find, for the outcomes of their experiments, the probabilities prescribed by the quantum formalism. They are two precise, unambiguous fundamental physical theories that describe and explain the world around us, but they appear to do this by employing opposite strategies. In Bohmian mechanics (Bohm, 1952; Bell, 1966; Dürr et al., 1992; Berndl et al., 1995) the wave function evolves according to the Schrödinger equation but is not the complete description of the state at a given time; this description involves further variables, traditionally called 'hidden variables,' namely the particle positions.

In the GRW theory (Pearle, 1976; Ghirardi et al., 1986; Bell, 1987a; Bassi and Ghirardi, 2003), in contrast, the wave function ψ describes the state of any physical system completely, but ψ collapses spontaneously, thus departing from the Schrödinger evolution. That is, the two theories choose different horns of the alternative that Bell formulated as his conclusion from the measurement problem (Bell, 1987a): 'Either the wave function, as given by the Schrödinger equation, is not everything, or it is not right.'

The two theories are always presented almost as dichotomical, as in the recent paper by Putnam (2005). Our suggestion here is instead that BM and GRW theory have much more in common than one would expect at first sight. So much, indeed, that they should be regarded as being close to each other, rather than opposite. The differences are less profound than the similarities, provided that the GRW theory is understood appropriately, as involving variables describing matter in space-time. These variables we call the *primitive ontology* (PO) of the theory, and they form the common structure of BM and GRW. The gain from the comparison with BM is the insight that the GRW theory can, and should, be understood in terms of the PO. We think this view in terms of the PO provides a deeper understanding of the GRW theory in particular, and of quantum theories without observer in general. To formulate more clearly and advertise this view is our goal.

After recalling what Bohmian mechanics is in Section 2, we introduce two concrete examples of GRW theories in Section 3. These examples involve rather different choices of crucial variables, describing matter in space-time, and give us a sense of the range of possibilities for such variables. We discuss in Section 4 the notion of the *primitive ontology* (PO) of a theory (a notion introduced in (Dürr et al., 1992)) and connect it to Bell's notion of 'local beables' (Bell, 1976). In Section 4.1 we relate the primitive ontology of a theory to the notion of physical equivalence between theories. We stress in Section 4.2 the connection, first discussed in (Goldstein, 1998), between the primitive ontology and symmetry properties, with particular concern for the generalization of such theories to a relativistically invariant quantum theory without observers. In Section 4.3 we argue that a theory without a primitive ontology is at best profoundly problematical. We proceed in Sections 5 to an analysis of the differences between GRW (with primitive ontology) and BM, and in Section 6 we discuss a variety of possible theories. We consider in Section 7.1 a 'no-collapse' reformulation of one of the GRW theories and in Section 7.2 a 'collapse' interpretation of BM. These formulations enable us to better appreciate the common structure of BM and the GRW theories, as well as the differences, as we discuss in Section 7.3. We conclude in Section 8 with a summary of this common structure.

2 Bohmian Mechanics

Bohmian mechanics is a theory of (nonrelativistic) particles in motion. The motion of a system of N particles is provided by their world lines $t \mapsto Q_i(t)$, $i = 1, \ldots, N$, where $Q_i(t)$ denotes the position in $\mathbb{R}^3$ of the i-th particle at time t. These world lines are determined by Bohm's law of motion (Bohm, 1952; Bell, 1966; Dürr et al., 1992; Berndl et al., 1995),

$$\frac{dQ_i}{dt} = v_i^\psi(Q_1, \ldots, Q_N) = \frac{\hbar}{m_i} \text{Im} \frac{\psi^* \nabla_i \psi}{\psi^* \psi}(Q_1 \ldots, Q_N), \tag{1}$$

where m_i, $i = 1, \ldots, N$, are the masses of the particles; the wave function ψ evolves according to Schrödinger's equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi, \tag{2}$$

where H is the usual nonrelativistic Schrödinger Hamiltonian; for spinless particles it is of the form

$$H = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \nabla_k^2 + V, \tag{3}$$

containing as parameters the masses of the particles as well as the potential energy function V of the system.

In the usual yet unfortunate terminology, the actual positions $Q_1, \ldots, Q_N$ of the particles are the *hidden variables* of the theory: the variables which, together with the wave function, provide a complete description of the system, the wave function alone providing only a partial, incomplete, description. From the point of view of BM, however, this is a strange terminology since it suggests that the main object of the theory is the wave function, with the additional information provided by the particles' positions playing a secondary role. The situation is rather much the opposite: BM is a theory of particles; their positions are the primary variables, and the description in terms of them must be completed by specifying the wave function to define the dynamics (1).

As a consequence of Schrödinger's equation and of Bohm's law of motion, the quantum equilibrium distribution $|\psi(q)|^2$ is equivariant. This means that if the configuration $Q(t) = (Q_1(t), \ldots, Q_N(t))$ of a system is random with distribution $|\psi_t|^2$ at some time t, then this will be true also for any other time t. Thus, the *quantum equilibrium hypothesis*, which asserts that whenever a system has wave function ψ_t, its configuration $Q(t)$ is random with distribution $|\psi_t|^2$, can consistently be assumed. This hypothesis is not as hypothetical as its name may suggest: the quantum equilibrium hypothesis follows, in fact, by the law of large numbers from the assumption that the (initial) configuration of the universe is typical (i.e., not-too-special) for the $|\Psi|^2$ distribution, with Ψ the (initial) wave function of the universe (Dürr et al., 1992). The situation resembles the way Maxwell's distribution for velocities in a classical gas follows from the assumption that the phase point of the gas is typical for the uniform distribution on the energy surface.

As a consequence of the quantum equilibrium hypothesis, a Bohmian universe, even if deterministic, appears random to its inhabitants. In fact, the probability distributions observed by the inhabitants agree exactly with those of the quantum formalism. To begin to understand why, note that any measurement apparatus must also consist of Bohmian particles. Calling Q_S the configuration of the particles of the system to be measured and Q_A the configuration of the particles of the apparatus, we can write for the configuration of the big Bohmian system relevant to the analysis of the measurement $Q = (Q_S, Q_A)$. Let us suppose that the initial wave function ψ of the big system is a product state $\Psi(q) = \Psi(q_S, q_A) = \psi(q_S)\,\phi(q_A)$.

During the measurement, this Ψ evolves according to the Schrödinger equation, and in the case of an ideal measurement it evolves to $\Psi_t = \sum_\alpha \psi_\alpha\,\phi_\alpha$, where α runs through the eigenvalues of the observable measured, ϕ_α is a state of the apparatus in which the pointer points to the value α, and ψ_α is the projection of ψ to the appropriate eigenspace of the observable. By the quantum equilibrium hypothesis, the probability for the random apparatus configuration $Q_A(t)$ to be such as to correspond to the pointer pointing to the value α is $\|\psi_\alpha\|^2$. For a more detailed discussion see (Dürr et al., 1992, 2004b).

3   Ghirardi, Rimini, and Weber

The theory proposed by Ghirardi, Rimini and Weber (1986) is in agreement with the predictions of nonrelativistic quantum mechanics as far as all present experiments are concerned (Bassi and Ghirardi, 2003); for a discussion of future experiments that may distinguish this theory from quantum mechanics, see Section V of (Bassi and Ghirardi, 2003). According to the way in which this theory is usually presented, the evolution of the wave function follows, instead of Schrödinger's equation, a stochastic jump process in Hilbert space. We shall succinctly summarize this process as follows.

Consider a quantum system described (in the standard language) by an N-'particle'[1] wave function $\psi = \psi(q_1, ..., q_N)$, $q_i \in \mathbb{R}^3$, $i = 1, \ldots, N$; for any point x in $\mathbb{R}^3$ (the 'center' of the collapse that will be defined next), define on the Hilbert space of the system the *collapse operator*

$$\Lambda_i(x) = \frac{1}{(2\pi\sigma^2)^{3/2}}\, e^{-\frac{(\widehat{Q}_i - x)^2}{2\sigma^2}}\,,\tag{4}$$

where $\widehat{Q}_i$ is the position operator of 'particle' i. Here σ is a new constant of nature of order of 10^{-7}m.

Let ψ_{t_0} be the initial wave function, i.e., the normalized wave function at some time t_0 arbitrarily chosen as initial time. Then ψ evolves in the following way:

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[1]We wish to emphasize here that there are no particles in this theory: the word 'particle' is used only for convenience in order to be able to use the standard notation and terminology.

1. It evolves unitarily, according to Schrödinger's equation, until a random time $T_1 = t_0 + \Delta T_1$, so that
$$\psi_{T_1} = U_{\Delta T_1} \psi_{t_0}, \tag{5}$$
where U_t is the unitary operator $U_t = e^{-\frac{i}{\hbar}Ht}$ corresponding to the standard Hamiltonian H governing the system, e.g., given by (3) for N spinless particles, and ΔT_1 is a random time distributed according to the exponential distribution with rate $N\lambda$ (where the quantity λ is another constant of nature of the theory,[2] of order of 10^{-15} s^{-1}).

2. At time T_1 it undergoes an instantaneous collapse with random center X_1 and random label I_1 according to

$$\psi_{T_1} \mapsto \psi_{T_1+} = \frac{\Lambda_{I_1}(X_1)^{1/2}\psi_{T_1}}{\|\Lambda_{I_1}(X_1)^{1/2}\psi_{T_1}\|}. \tag{6}$$

I_1 is chosen at random in the set $\{1,\ldots,N\}$ with uniform distribution. The center of the collapse X_1 is chosen randomly with probability distribution[3]

$$\mathbb{P}(X_1 \in dx_1|\psi_{T_1}, I_1 = i_1) = \langle\psi_{T_1}|\Lambda_{i_1}(x_1)\psi_{T_1}\rangle\, dx_1 = \|\Lambda_{i_1}(x_1)^{1/2}\psi_{T_1}\|^2 dx_1. \tag{7}$$

3. Then the algorithm is iterated: ψ_{T_1+} evolves unitarily until a random time $T_2 = T_1 + \Delta T_2$, where ΔT_2 is a random time (independent of ΔT_1) distributed according to the exponential distribution with rate $N\lambda$, and so on.

In other words, the evolution of the wave function is the Schrödinger evolution interrupted by collapses. When the wave function is ψ a collapse with center x and label i occurs at rate
$$r(x, i|\psi) = \lambda\, \langle\psi\,|\,\Lambda_i(x)\psi\rangle \tag{8}$$
and when this happens, the wave function changes to $\Lambda_i(x)^{1/2}\psi/\|\Lambda_i(x)^{1/2}\psi\|$.

Thus, if between time t_0 and any time $t > t_0$, n collapses have occurred at the times $t_0 < T_1 < T_2 < \ldots < T_n < t$, with centers $X_1,\ldots,X_n$ and labels $I_1,\ldots,I_n$, the wave function at time t will be
$$\psi_t = \frac{L_{t,t_0}^{F_n}\psi_{t_0}}{\|L_{t,t_0}^{F_n}\psi_{t_0}\|} \tag{9}$$
where $F_n = \{(X_1, T_1, I_1), \ldots, (X_n, T_n, I_n)\}$ and

$$L_{t,t_0}^{F_n} = U_{t-T_n}\Lambda_{I_n}(X_n)^{1/2}\, U_{T_n-T_{n-1}}\Lambda_{I_{n-1}}(X_{n-1})^{1/2}\, U_{T_{n-1}-T_{n-2}}\cdots\Lambda_{I_1}(X_1)^{1/2}\, U_{T_1-t_0}. \tag{10}$$

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[2]Pearle and Squires (1994) have argued that λ should be chosen differently for every 'particle,' with λ_i proportional to the mass m_i.

[3]Hereafter, when no ambiguity could arise, we use the standard notations of probability theory, according to which a capital letter, such as X, is used to denote a random variable, while the the values taken by it are denoted by small letters; $X \in dx$ is a shorthand for $X \in [x, x+dx]$, etc.

Since T_i, X_i, I_i and n are random, ψ_t is also random.

It should be observed that—unless t_0 is the initial time of the universe—also ψ_{t_0} should be regarded as random, being determined by the collapses that occurred at times earlier that t_0. However, *given* ψ_{t_0}, the statistics of the future evolution of the wave function is completely determined; for example, the joint distribution of the first n collapses after t_0, with particle labels $I_1, \ldots, I_n \in \{1, \ldots, N\}$, is

$$\mathbb{P}\big(X_1 \in dx_1, T_1 \in dt_1, I_1 = i_1, \ldots, X_n \in dx_n, T_n \in dt_n, I_n = i_n | \psi_{t_0}\big) =$$
$$\lambda^n e^{-N\lambda(t_n - t_0)} \| L_{t_n,t_0}^{f_n} \psi_{t_0} \|^2 \, dx_1 dt_1 \cdots dx_n dt_n \,, \quad (11)$$

with $f_n = \{(x_1, t_1, i_1), \ldots, (x_n, t_n, i_n)\}$ and $L_{t_n,t_0}^{f_n}$ given, *mutatis mutandis*, by (10).

This is, more or less, all there is to say about the formulation of the GRW theory according to most theorists. In contrast, GianCarlo Ghirardi believes that the description provided above is not the whole story, and we agree with him. We believe that, depending on the choice of what we call the *primitive ontology* (PO) of the theory, there are correspondingly different versions of the theory. We will discuss the notion of primitive ontology in detail in Section 4. In the subsections below we present two versions of the GRW theory, based on two different choices of the PO, namely the *matter density ontology* (in Section 3.1) and the *flash ontology* (in Section 3.2).

3.1 GRWm

In the first version of the GRW theory, denoted by GRWm, the PO is given by a field: We have a variable $m(x,t)$ for every point $x \in \mathbb{R}^3$ in space and every time t, defined by

$$m(x,t) = \sum_{i=1}^{N} m_i \int_{\mathbb{R}^{3N}} dq_1 \cdots dq_N \, \delta(q_i - x) \left| \psi(q_1, \ldots, q_N, t) \right|^2. \quad (12)$$

In words, one starts with the $|\psi|^2$–distribution in configuration space $\mathbb{R}^{3N}$, then obtains the marginal distribution of the i-th degree of freedom $q_i \in \mathbb{R}^3$ by integrating out all other variables q_j, $j \neq i$, multiplies by the mass associated with q_i, and sums over i. GRWm was essentially proposed by Ghirardi and co–workers in (Benatti et al., 1995);[4] see also (Goldstein, 1998).

The field $m(\cdot, t)$ is supposed to be understood as the density of matter in space at time t. Since these variables are functionals of the wave function ψ, they are not 'hidden variables' since, unlike the positions in BM, they need not be specified in addition to the

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[4]They first proposed (for a model slightly more complicated than the one considered here) that the matter density be given by an expression similar to (12) but this difference is not relevant for our purposes.

wave function, but rather are determined by it. Nonetheless, they are additional elements of the GRW theory that need to be posited in order to have a complete description of the world in the framework of that theory.

GRWm is a theory about the behavior of a field $m(\cdot, t)$ on three-dimensional space. The microscopic description of reality provided by the matter density field $m(\cdot, t)$ is not particle-like but instead continuous, in contrast to the particle ontology of BM. This is reminiscent of Schrödinger's early view of the wave function as representing a continuous matter field. But while Schrödinger was obliged to abandon his early view because of the tendency of the wave function to spread, the spontaneous wave function collapses built into the GRW theory tend to localize the wave function, thus counteracting this tendency and overcoming the problem.

A parallel with BM begins to emerge: they both essentially involve more than the wave function. In one the matter is spread out continuously, while in the other it is concentrated in finitely many particles; however, both theories are concerned with matter in three-dimensional space, and in some regions of space there is more than in others.

You may find GRWm a surprising proposal. You may ask, was it not the point of GRW — perhaps even its main advantage over BM — that it can do without objects beyond the wave function, such as particle trajectories or matter density? Is not the dualism present in GRWm unnecessary? That is, what is wrong with the version of the GRW theory, which we call GRW0, which involves just the wave function and nothing else? We will return to these questions in Section 4.3. To be sure, it seems that if there was nothing wrong with GRW0, then, by simplicity, it should be preferable to GRWm. We stress, however, that Ghirardi must regard GRW0 as seriously deficient; otherwise he would not have proposed anything like GRWm. We will indicate in Section 4.3 why we think Ghirardi is correct. To establish the inadequacy of GRW0 is not, however, the main point of this paper.

3.2   GRWf

According to another version of the GRW theory, which was first suggested by Bell (1987a, 1989), then adopted in (Kent, 1989; Goldstein, 1998; Tumulka, 2006a,b; Allori et al., 2005; Maudlin, forthcoming), and here denoted GRWf, the PO is given by 'events' in space-time called flashes, mathematically described by points in space-time. This is, admittedly, an unusual PO, but it is a possible one nonetheless. In GRWf matter is neither made of particles following world lines, such as in classical or Bohmian mechanics, nor of a continuous distribution of matter such as in GRWm, but rather of discrete points in space-time, in fact finitely many points in every bounded space-time region, see Figure 1.

In the GRWf theory, the space-time locations of the flashes can be read off from the
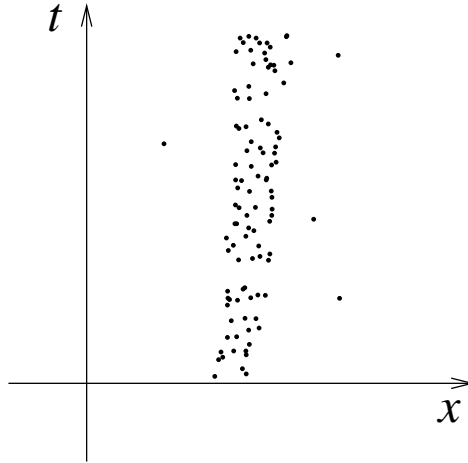
Figure 1: A typical pattern of flashes in space-time, and thus a possible world according to the GRWf theory

history of the wave function given by (9) and (10): every flash corresponds to one of the spontaneous collapses of the wave function, and its space-time location is just the space-time location of that collapse. Accordingly, equation (11) gives the joint distribution of the first n flashes, after some initial time t_0. The flashes form the set

$$F = \{(X_1, T_1), \dots, (X_k, T_k), \dots\}$$

(with $T_1 < T_2 < \dots$).

In Bell's words:

> [...] the GRW jumps (which are part of the wave function, not something else) are well localized in ordinary space. Indeed each is centered on a particular spacetime point (x, t). So we can propose these events as the basis of the 'local beables' of the theory. These are the mathematical counterparts in the theory to real events at definite places and times in the real world (as distinct from the many purely mathematical constructions that occur in the working out of physical theories, as distinct from things which may be real but not localized, and distinct from the 'observables' of other formulations of quantum mechanics, for which we have no use here). A piece of matter then is a galaxy of such events. (Bell, 1987a)

That is, Bell's idea is that GRW can account for objective reality in three-dimensional space in terms of space-time points (X_k, T_k) that correspond to the localization events (collapses) of the wave function. Note that if the number N of the degrees of freedom in the wave function is large, as in the case of a macroscopic object, the number of flashes is

also large (if $\lambda = 10^{-15}$ s^{-1} and $N = 10^{23}$, we obtain 10^8 flashes per second). Therefore, for a reasonable choice of the parameters of the GRWf theory, a cubic centimeter of solid matter contains more than 10^8 flashes per second. That is to say that large numbers of flashes can form macroscopic shapes, such as tables and chairs. That is how we find an image of our world in GRWf.

Note however that at almost every time space is in fact empty, containing no flashes and thus no matter. Thus, while the atomic theory of matter entails that space is not everywhere continuously filled with matter but rather is largely void, GRWf entails that at most times space is entirely void.

According to this theory, the world is made of flashes and the wave function serves as the tool to generate the 'law of evolution' for the flashes: equation (8) gives the rate of the flash process —the probability per unit time of the flash of label i occurring at the point x. For this reason, we prefer the word 'flash' to 'hitting' or 'collapse center': the latter words suggest that the role of these events is to affect the wave function, or that they are not more than certain facts about the wave function, whereas 'flash' suggests rather something like an elementary event. Since the wave function ψ evolves in a random way, $F = \{(X_k, T_k) : k \in \mathbb{N}\}$ is a random subset of space-time, a point process in space-time, as probabilists would say. GRWf is thus a theory whose output is a point process in space-time.[5]

3.3 Empirical Equivalence Between GRWm and GRWf

We remark that GRWm and GRWf are empirically equivalent, i.e., they make always and exactly the same predictions for the outcomes of experiments. In other words, there is

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[5]An anonymous referee has remarked that according to GRWf with the original parameters, in a single living cell there might occur as few as one flash per hour, so that the cell is empty of matter for surprisingly long periods, quite against our intuition of a cell as a rather classical object. We make a few remarks to this objection. First, one should of course be careful with the language: there is presumably no cell in GRWf, though the structure of the wave function (on configuration space—even though there are no configurations) might suggest otherwise. Second, it all depends on the choice of the parameters λ and σ, and, as long as experiments have not fixed their values, this cell argument may indeed be an argument for a choice different from GRW's original one (say, with larger λ and larger σ). We do not wish to argue here for any particular choice. Third, while most people might expect a cell to be real in much the same way as (say) a cat, one would not necessarily expect this of a single atom. Thus, it seems quite conceivable that, at some critical scale between that of atoms and that of cats, the ontological character of objects changes—as indeed it does in GRWf because of the limited resolution of matter given by the space-time density of flashes (e.g., in water approximately one flash every 20 micrometers every second). The cell example shows that the critical scale in GRWf is larger than one might have expected, and thus that GRWf is a mildly quirky picture of the world. But this mild quirkiness should be seen in perspective. In comparison, many other views about quantum reality are heavily eccentric, as they propose that reality is radically different from what we normally think it is like: e.g., that there exist parallel worlds, or that there exists no matter at all, or that reality is contradictory in itself.

no experiment we could possibly perform that would tell us whether we are in a **GRWm** world or in a **GRWf** world, assuming we are in one of the two. This should be contrasted with the fact that there are possible experiments (though we cannot perform any with the present technology) that decide whether we are in a Bohmian world or in a **GRW** world.

The reason is simple. Consider any experiment, which is finished at time t. Consider the same realization of the wave function on the time interval $[0, t]$, but associated with different primitive ontologies in the two worlds. At time t, the result gets written down, encoded in the shape of the ink; more abstractly, the result gets encoded in the position of some macroscopic amount of matter. If in the **GRWf** ontology, this matter is in position 1, then the flashes must be located in position 1; thus, the collapses are centered at position 1; thus, the wave function is near zero at position 2; thus, by (12) the density of matter is low at position 2 and high at position 1; thus, in **GRWm** the matter is also in position 1, displaying the same result as in the **GRWf** world.

We will discuss empirical equivalence again in Section 7.3.

4   Primitive Ontology

The matter density field in **GRWm**, the flashes in **GRWf**, and the particle trajectories in **BM** have something in common: they form (what we have called) the primitive ontology of these theories. The PO of a theory—and its behavior— is what the theory is fundamentally about. It is closely connected with what Bell called the 'local beables':

> [I]n the words of Bohr, 'it is decisive to recognize that, however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms'. It is the ambition of the theory of local beables to bring these 'classical terms' into the equations, and not relegate them entirely to the surrounding talk. (Bell, 1976)

The elements of the primitive ontology are the stuff that things are made of. The wave function also belongs to the ontology of **GRWm**, **GRWf** and **BM**, but not to the PO: according to these theories physical objects are not made of wave functions[6]. Instead, the role of the wave function in these theories is quite different, as we will see in the following.

In each of these theories, the only reason the wave function is of any interest at all is that it is relevant to the behavior of the PO. Roughly speaking, the wave function tells the matter how to move. In **BM** the wave function determines the motion of the

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[6]We would not go so far as Dowker and Herbauts (2005) and Nelson (1985), who have suggested that, physically, the wave function does not exist at all, and only the PO exists. But we have to admit that this view is a possibility, in fact a more serious one than the widespread view that no PO exists.

particles via equation (1), in GRWm the wave function determines the distribution of matter in the most immediate way via equation (12), and in GRWf the wave function determines the probability distribution of the future flashes via equation (11).

We now see a clear parallel between BM and the GRW theory, at least in its versions GRWm and GRWf. Each of these theories is about matter in space-time, what might be called a decoration of space-time. Each involves a dual structure $(\mathscr{X}, \psi)$: the PO $\mathscr{X}$ providing the decoration, and the wave function ψ governing the PO. The wave function in each of these theories, which has the role of generating the dynamics for the PO, has a nomological character utterly absent in the PO. This difference is crucial for understanding the symmetry properties of these theories and therefore is vital for the construction of a Lorentz invariant quantum theory without observers, as we will discuss in Section 4.2.

Even the Copenhagen interpretation (orthodox quantum theory, OQT) involves a dual structure: what might be regarded as its PO is the classical description of macroscopic objects which Bohr insisted was indispensable — including in particular pointer orientations conveying the outcomes of experiments — with the wave function serving to determine the probability relations between the successive states of these objects. In this way, ψ governs a PO, even for OQT. An important difference, however, between OQT on the one hand and BM, GRWm, and GRWf on the other is that the latter are fully precise about what belongs to the PO (particle trajectories, respectively continuous matter density or flashes) whereas the Copenhagen interpretation is rather vague, even noncommittal, on this point, since the notion of 'macroscopic' is an intrinsically vague one: of how many atoms need an object consist in order to be macroscopic? And, what exactly constitutes a 'classical description' of a macroscopic object?

Therefore, as the example of the Copenhagen interpretation of quantum mechanics makes vivid, an adequate fundamental physical theory, one with any pretension to precision, must involve a PO defined on the microscopic scale.

4.1 Primitive Ontology and Physical Equivalence

To appreciate the concept of PO, it might be useful to regard the positions of particles, the mass density and the flashes, respectively, as the *output* of BM, GRWm and GRWf, with the wave function, in contrast, serving as part of an *algorithm that generates this output*. Suppose we want to write a computer program for simulating a system (or a universe) according to a certain theory. For writing the program, we have to face the question: Which among the many variables to compute should be the output of the program? All other variables are internal variables of the program: they may be necessary for doing the computation, but they are not what the user is interested in. In the way we propose to understand BM, GRWm, and GRWf, the output of the program, the result of the simulation, should be the particle world lines, the $m(\cdot, t)$

field, respectively the flashes; the output should look like Figure 1. The wave function, in contrast, is one of the internal variables and its role is to implement the evolution for the output, the PO of the theory.

Moreover, note that there might be different ways of producing the same output, using different internal variables. For example, two wave functions that differ by a gauge transformation generate the same law for the PO. In more detail, when (external) magnetic fields are incorporated into BM by replacing all derivatives ∇_k in (1) and (2) by $\nabla_k - ie_k A(q_k)$, where A is the vector potential and e_k is the electric charge of particle k, then the gauge transformation

$$\psi \mapsto e^{i\sum_k e_k f(q_k)}\psi, \quad A \mapsto A + \nabla f \tag{13}$$

does not change the trajectories nor the quantum equilibrium distribution. As another example, one can write the law for the PO in either the Schrödinger or the Heisenberg picture. As a consequence, the same law for the PO is generated by either an evolving wave function and static operators or a static wave function and evolving operators. In more detail, BM can be reformulated in the Heisenberg picture by rewriting the law of motion as follows:

$$\frac{dQ_i}{dt} = -\frac{1}{\hbar}\mathrm{Im}\frac{\langle\psi|P(dq,t)[H,\widehat{Q}_i(t)]|\psi\rangle}{\langle\psi|P(dq,t)|\psi\rangle}(q = Q(t)), \tag{14}$$

where H is the Hamiltonian (e.g., for N spinless particles given by (3)), $\widehat{Q}_i(t)$ is the (Heisenberg-evolved) position operator (or, more precisely, triple of operators corresponding to the three dimensions of physical space) for particle i and $P(\cdot,t)$ is the projection–valued measure (PVM) defined by the joint spectral decomposition of all (Heisenberg-evolved) position operators (Dürr et al., 2005b).

We suggest that two theories be regarded as *physically equivalent* when they lead to the same history of the PO. Conversely, one could define the notion of PO in terms of physical equivalence: The PO is described by those variables which remain invariant under all physical equivalences. We discuss this issue in more detail in Sections 7.1 and 7.2, when presenting some examples.

What is 'primitive' about the primitive ontology? That becomes clear when we realize in what way the other objects in the theory (such as the wave function, or the magnetic field in classical physics) are non–primitive: One can explain what they are by explaining how they govern the behavior of the PO, while it is the entities of the PO that make direct contact with the world of our experience. We can directly compare the motion of matter in our world with the motion of matter in the theory, at least on the macroscopic scale. The other objects in the theory can be compared to our world only indirectly, by the way they affect the PO.[7]

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[7]While the notion of PO is similar to Bell's notion of local beables, it should be observed that not all

4.2 Primitive Ontology and Symmetry

The peculiar flash ontology was invented by Bell in 1987 as a step toward a relativistic GRW theory. He wrote in (Bell, 1987a) about GRWf:

> I am particularly struck by the fact that the model is as Lorentz invariant as it could be in the nonrelativistic version. It takes away the ground of my fear that any exact formulation of quantum mechanics must conflict with fundamental Lorentz invariance.

What Bell refers to in the above quotation is the following. An analogue of the relativity of simultaneity, i.e. of the invariance of the dynamics under boosts, in the framework of a nonrelativistic theory is the invariance under relative time translations for two very distant systems. Bell (1987a, 1989) verified by direct calculation that GRWf has this symmetry. However, it it is important here to appreciate what this invariance means. To say that a theory has a given symmetry is to say that

> *The possible histories of the PO, those that are allowed by the theory, when transformed according to the symmetry, will again be possible histories for the theory, and the possible probability distributions on the histories, those that are allowed by the theory, when transformed according to the symmetry, will again be possible probability distributions for the theory.*

Let us explain.

- '*The possible histories of the PO, those that are allowed by the theory...*' We give some examples, involving Galilean invariance. In classical mechanics the meaning is straightforward: the PO is that of particles, described by their positions in physical space, a history of this PO corresponds to a collection of particle trajectories—the trajectories $Q_i(t)$, $i = 1, \ldots, N$, in a universe of N particles— and a history is allowed if the particles obey Newton's law, i.e., if $m_i \ddot{Q}_i(t) = F_i(Q_1(t), ..., Q_N(t))$, where F_i is the Newtonian force acting on the ith particle. The theory is defined once the form of F_i is specified (for example, that the force is the Newtonian gravitational force).

  Consider now BM: also here the PO is that of particles and a possible history of the PO—one that is allowed by BM—is a history described by the particle trajectories

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local beables, such as the electric and magnetic fields in classical electrodynamics, need to be regarded as part of the PO. Moreover, the very conception that the PO must involve only local beables (i.e., be represented by mathematical objects grounded in familiar three-dimensional space) could in principle be questioned; this is, however, a rather delicate and difficult question that will be briefly addressed in Section 4.3 but that deserves a thorough analysis that will be undertaken in a separate work (Allori et al., unpublished,b).

$Q_i(t)$, $i = 1, \dots N$, which satisfy equation (1) *for some wave function ψ satisfying* equation (2). The theory is defined once the Hamiltonian H in (2) is specified (for example, as given by (3), for a choice of the potential V).

- '*...when transformed according to the symmetry...*' Since the PO is represented by a geometrical entity in physical space (a decoration of space-time, as we have said earlier), space-time symmetries naturally act on it, for example transforming trajectories $Q_i(t)$ to trajectories $\tilde{Q}_i(t)$. For example, under a Galilean boost (by a relative velocity v), in classical mechanics as well as in BM, the trajectories $Q_i(t)$ transform into the boosted trajectories $\tilde{Q}_i(t) = Q_i(t) + vt$.

- '*...will again be possible histories for the theory...*' Notice that $Q_i(t)$ and $\tilde{Q}_i(t)$ may arise in BM from different wave functions. In other words, the wave function must also be transformed when transforming the history of the PO. However, while there is a natural transformation of the history of the PO, there is not necessarily a corresponding natural change of the wave function. The latter is allowed to change in any way, solely determined by its relationship to the PO. For example, consider again a Galilean boost (by a relative velocity v) in BM: the boosted trajectories $\tilde{Q}_i(t) = Q_i(t) + vt$ form again a solution of (1) and (2) with ψ replaced by the transformed wave function[8]

$$\tilde{\psi}_t(q_1, \dots, q_N) = \exp\left(\frac{i}{\hbar} \sum_{i=1}^{N} m_i(q_i \cdot v - \tfrac{1}{2}v^2 t)\right) \psi_t(q_1 - vt, \dots, q_N - vt). \quad (15)$$

Since the trajectories of the PO transformed according to the symmetry are still solutions, BM is symmetric under Galilean transformation, even though the corresponding wave function has to undergo more than a simple change of variables in order to make this possible.

- '*...and the possible probability distributions on the histories, those that are allowed by the theory...*' In a deterministic theory, a probability distribution on the histories arises from a probability distribution on the initial conditions. In BM, a probability distribution on histories is possible if there exists a wave function ψ such that the given distribution is the one induced on solutions to (1) by the probability distribution $|\psi(q_1, \dots, q_N)|^2$ at some initial time.

More interesting is the case of nondeterministic theories. For these theories, i.e., for theories involving stochasticity at the fundamental level, the law for the PO amounts to a specification of possible probability distributions, for example by

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[8]Under this transformation $V = V(q_1, \dots, q_N)$ in (2) must be replaced by $\tilde{V} = V(q_1 - vt, \dots, q_N - vt)$. For V arising from the standard two-body interactions, we have that $V = \tilde{V}$, and hence the theory is invariant.

specifying the generator, or transition probabilities, of a Markov process. For example, in GRWm the history of the PO is the mass density field $m(\cdot, \cdot)$, and a probability distribution on the histories of this PO is possible if it is the distribution induced on $m(\cdot, \cdot)$, according to equation (12), by some wave function ψ with probability law given, say, by (11) (and (9)). The case of GRWf is analogous: a probability distribution for the flashes $F = \{(X_k, T_k) : k \in \mathbb{N}\}$ is possible if induced by (11) for some wave function ψ.

- '...*when transformed according to the symmetry, will again be possible probability distributions for the theory.*' The probability distribution on the histories, when transformed according to the symmetry, is the distribution of the transformed histories. In other words, the action of a transformation on every history determines the transformation of a probability distribution on the space of histories. As in the deterministic case, the wave function is allowed to change in any way compatible with its relationship to the PO. For example, consider the Galilean invariance of GRWf: let ψ and $\tilde{\psi}$ be two initial wave functions related as in (15), that is, by the usual formula for Galilean transformations in quantum mechanics. Let G_t denote the transformation operator in (15) at time t, such that $\tilde{\psi}_t = G_t \psi_t$. A simple calculation shows that

$$\Lambda_i(x + vt)^{1/2} G_t = G_t \Lambda_i(x)^{1/2}.$$

As a consequence, the distribution (7) of the (spatial location of the) first flash arising from $\tilde{\psi}_{T_1}$ is that arising from ψ_{T_1} shifted by vT_1, and the post-collapse wave functions (6) are still related by the appropriate G_t operator, i.e.,

$$\tilde{\psi}_{T_1+} = G_{T_1} \psi_{T_1+}.$$

Thus, the joint distribution of flashes arising from $\tilde{\psi}$ is just the one arising from ψ shifted by vt for every t.

Going back to the work of Bell mentioned in the beginning of this section (Bell, 1987a), what Bell had to do for GRWf, and what he did, was to confirm the invariance under relative time translations of the stochastic law for $F = \{(X_k, T_k) : k \in \mathbb{N}\}$, the galaxy of flashes. And more generally the invariance of GRWf directly concerns the stochastic law for the PO; it concerns the invariance of the law for the wave function only indirectly, contrary to what is often, erroneously, believed. Under a space-time symmetry the PO must be transformed in accord with its intrinsic geometrical nature, while wave functions (and other elements of the non-primitive ontology, if any) should be transformed in a manner dictated by their relationship to the PO.

Moreover, note that there is no reason to believe that when changing the PO of a theory the symmetry properties of the theory will remain unchanged. Actually, the

opposite is true. This fact was pointed out in (Goldstein, 1998) and has recently been emphasized also in (Tumulka, 2006a), in which it has been shown that GRWf, without interaction, can be modified so as to become a relativistic quantum theory without observers.[9] In that paper the stochastic law for the galaxy of the flashes in space-time, the PO of GRWf, with suitably modified, Lorentz-invariant equations, has been shown explicitly to be relativistically invariant (see also Tumulka, 2006c). Hence, GRWf is Lorentz invariant, but GRWm is not. Thus, one should not ask whether GRW as such is Lorentz invariant, since the answer to this question depends on the choice of PO for GRW. For details see (Maudlin, forthcoming). Similar results to those of (Tumulka, 2006a) have been obtained also by Dowker and Henson (2004) for a relativistic collapse theory on the lattice (see also Dowker and Herbauts, 2004, 2005).

We conclude with some remarks on OQT. Here the relevant PO consists, rather vaguely of course, of the 'pointer variables' registering the results of experiments that are spoken of as measurements of quantum observables. Though OQT provides neither detailed histories of the PO nor probability distributions thereof, it does provide probability distributions for the results of measurements registered by the PO, which are given by the appropriate spectral measures for the self-adjoint operators representing the observables. In particular, the mean value of the result of the measurement $\mathcal{E}$ of the quantum observable represented by the self-adjoint operator A for a system in the state ψ is

$$< A >_\psi = \frac{\langle \psi \,|\, A\psi \rangle}{\langle \psi \,|\, \psi \rangle} \,. \tag{16}$$

Now consider the action of a symmetry on the experiment $\mathcal{E}$: it transforms $\mathcal{E}$ to the experiment $\tilde{\mathcal{E}}$ arising from the natural action of the symmetry on the physical processes defining $\mathcal{E}$. If $\mathcal{E}$ is a measurement of the operator A—that is, if the probability distribution of the results of $\mathcal{E}$ are given by the spectral measures for A—then $\tilde{\mathcal{E}}$ will be a measurement of the operator $\tilde{A}$ arising from A under the symmetry. While $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are of course (usually) different experiments, it is obvious from their relationship that the distribution of the results of $\mathcal{E}$ when the system is in the state ψ is the same as the distribution of the results of $\tilde{\mathcal{E}}$ when the transformed system is in the transformed state $\tilde{\psi}$. In particular, where $\mathcal{E}$ is a measurement of A, we have that

$$< A >_\psi = \frac{\langle \psi \,|\, A\psi \rangle}{\langle \psi \,|\, \psi \rangle} = \frac{\langle \tilde{\psi} \,|\, \tilde{A}\tilde{\psi} \rangle}{\langle \tilde{\psi} \,|\, \tilde{\psi} \rangle} = < \tilde{A} >_{\tilde{\psi}} \,. \tag{17}$$

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[9]To put this result into perspective, note that the absence of interaction does not make the problem trivial. On the contrary, the main difficulty with devising a relativistic quantum theory without observers arises already in the non-interacting case: To specify a law for the PO that is relativistic but nonlocal. Note further that it would not have sufficed to specify a Lorentz-invariant evolution law for ψ (entailing suitable collapse) while leaving open the law for the PO. Finally, note that for GRWm and BM it is not known how to make them "seriously" relativistic, i.e. without the incorporation of additional structure that yields a foliation of space-time.

According to the analysis of Wigner (1939) and Bargmann (1954), these transformations on wave functions and operators are given by unitary or anti-unitary operators U, i.e., $\tilde{\psi} = U\psi$, $\tilde{A} = UAU^{-1}$, where U is an element of a unitary-projective representation of the symmetry group.

Note that while the distribution of the *result* of the experiment is, for trivial reasons, unaffected by the symmetry transformation, the macroscopic PO is in fact transformed. For example, a rotated experiment will involve a rotated 'pointer position,' or a rotated computer printout. But what the pointer is pointing to, and what the printout says, will not change. In other words, the numerical result Z of an experiment $\mathcal{E}$ should not be confused with the macroscopic configuration M of the pointer variables, the PO of OQT, the former being indeed a function of the latter, i.e., $Z = f(M)$, with the function f expressing the 'calibration' of the experiment. In $\tilde{\mathcal{E}}$, the rotated experiment, the PO (the pointer orientation) changes together with the calibration: the pointer points in a different direction $\tilde{M}$ and the scale f is rotated into $\tilde{f}$ such that $\tilde{f}(\tilde{M}) = f(M)$.

Thus, when all is said and done, although the PO of OQT is rather vague and imprecise, insofar as symmetry is concerned the situation is indeed analogous to that of theories, such as BM or GRWf, having a clear and exactly specified PO: also for OQT the possible probability distributions on the PO, those that are allowed by the theory, when transformed according to the symmetry, will again be possible probability distributions for the theory.

4.3 Without Primitive Ontology

Now let us turn to the question: What is wrong with GRW0, the bare version of GRW, which involves just the wave function and nothing else? Why does one need a PO at all? Our answer is that we do not see how the existence and behavior of tables and chairs and the like could be accounted for without positing a primitive ontology—a description of matter in space and time.

The aim of a fundamental physical theory is, we believe, to describe the world around us, and in so doing to explain our experiences to the extent of providing an account of their macroscopic counterparts, an account of the behavior of objects in 3-space. Thus it seems that for a fundamental physical theory to be satisfactory, it must involve, and fundamentally be about, 'local beables,' and not just a beable such as the wave function, which is non-local: In the words of Bell (1987a)

> [...] the wave function as a whole lives in a much bigger space, of $3N$ dimensions. It makes no sense to ask for the amplitude or phase or whatever of the wave function at a point in ordinary space. It has neither amplitude nor phase nor anything else until a multitude of points in ordinary three-space are specified.

In contrast, if a law is, like the **GRW** process in Hilbert space, about a mathematical object, like ψ, living in some abstract space, like $\mathbb{R}^{3N}$, it seems necessary to have or to add something more in order to make contact with a description in 3-space. For example, formulations of classical mechanics utilizing configuration space $\mathbb{R}^{3N}$ or phase space $\mathbb{R}^{6N}$ (such as Euler–Lagrange's or Hamilton's) are connected to a PO in 3-space (particles with trajectories) by the definitions of configuration space and phase space.

If, as we believe, a PO given by local beables is so crucial for a theory to make sense as a fundamental physical theory, one might wonder how **GRW0** could be taken seriously by so many serious people (see, e.g., Albert, 1992, 1996; Nicrosini and Rimini, 2003; Lewis, 2005). One reason, perhaps, is that if the wave function ψ is suitably collapsed, i.e., concentrated on a subset S of configuration space such that all configurations in S look macroscopically the same, all corresponding for example to a pointer pointing in the same way, then we can easily imagine what a world in the state ψ is macroscopically like: namely like the macrostate defined by configurations from S. For example, when in **GRW0** the wave function is concentrated near q, where q is a configuration describing a pointer pointing to the value a, it is easy to feel justified in concluding that there is a pointer that is pointing to the value a, and to forget that we are dealing with a theory for which there exists no arrangement of stuff in physical three-dimensional space at all.

Since the macroscopic description does not depend on whether the PO consists of world lines, flashes, or a continuous distribution of matter, and since the reasoning does not even mention the PO, it is easy to overlook the fact that a further law needs to be invoked, one which prescribes how the wave function is related to the PO, and implies that for wave functions such as described, the PO is such that its macroscopic appearance coincides (very probably) with the macroscopic appearance of configurations in S. To overlook this step is even easier when focusing very much on the measurement problem, whose central difficulty is that the wave function of object plus apparatus, if it evolves linearly, typically becomes a superposition of macroscopically distinct wave functions like ψ which thus contains no hint of the actual outcome of the experiment.

Interestingly enough, after having underlined the importance of local beables for a fundamental physical theory, Bell proposed **GRW** to be about 'stuff' in configuration ($3N$-dimensional) space. In his celebrated analysis of the quantum measurement problem (Bell, 1990), he wrote:

> The GRW-type theories have nothing in their kinematics but the wavefunction. It gives the density (in a multidimensional configuration space!) of *stuff*. To account for the narrowness of that stuff in macroscopic dimensions, the linear Schrödinger equation has to be modified, in the GRW picture by a mathematically prescribed spontaneous collapse mechanism. [Emphasis in the original.]

He made a similar remark to Ghirardi (quoted by the latter in (Bassi and Ghirardi,

2003, p. 345)) in a letter dated October 3, 1989:

> As regards ψ and the density of stuff, I think it is important that this density is in the $3N$-dimensional configuration space. So I have not thought of relating it to ordinary matter or charge density in 3-space. Even for one particle I think one would have problems with the latter. So I am inclined to the view you mention 'as it is sufficient for an objective interpretation' ... And it has to be stressed that the 'stuff' is in $3N$-space—or whatever corresponds in field theory.

As we have indicated already, we don't understand this proposal, which clearly suffers from the difficulties discussed above. Whoever suggests that matter exists not in 3-space but in $3N$-space must bridge the gap between an ontology in $3N$-space and the behavior of objects in 3-space. Strategies for doing so have in fact been proposed; see (Albert, 1996) for a proposal and (Monton, 2002) for a critique. For the reasons mentioned above, we do not believe that they can succeed.

4.4 Primitive Ontology and Quantum State

It is well known that in OQT the quantum state is naturally projective. That is, quantum states are best regarded as mathematically represented by *rays* in the system's Hilbert space $\mathscr{H}$, i.e. by the elements of the projective space $\mathbb{P}(\mathscr{H})$, consisting of equivalence classes of wave functions $\psi \in \mathscr{H}$ differing by a multiplicative constant. This follows from the rule (16) for the mean value of an observable represented by a self-adjoint operator A for a system in the state ψ. Wave functions ψ differing by a multiplicative constant give the same mean value to all observables A.

Similarly, in BM the quantum state is naturally projective: it follows from (1) that wave functions differing by a multiplicative constant are associated with the same vector field, and thus generate the same dynamics for the PO.[10]

In GRWf the quantum state is also naturally projective. Of course, for general ψ (not necessarily normalized), instead of (8) the rate for the flashes should be given by

$$r(x, i|\psi) = \lambda \frac{\langle \psi \,|\, \Lambda_i(x)\psi \rangle}{\langle \psi \,|\, \psi \rangle} \,. \tag{18}$$

In GRWm wave functions differing by a multiplicative constant of modulus 1 define the same evolution of the mass density field (12). If the wave function is multiplied by a more general constant, in order to ensure the same evolution of the mass density the right hand side of (12) could be divided by $\langle \psi | \psi \rangle$. But this is perhaps unnecessary,

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[10]And insofar as probabilities are concerned, if ψ is not normalized, these are given by $|\psi(q)|^2 / \langle \psi | \psi \rangle$, which is projective.

since universal mass densities that differ only by a multiplicative constant are arguably physically equivalent.

GRW0, involving only wave functions, does not allow us to make the same kind of argument; it is thus not clear for GRW0 why ψ should be regarded as projective, though the structure of GRW0 is compatible with doing so.

To sum up, the projective nature of the quantum state can be regarded as a consequence of the axioms of OQT, BM, GRWm and GRWf, but not of GRW0.

5   Differences between BM and GRW

We have stressed the similarity between BM and GRW. There are, of course, also significant differences. Perhaps the most obvious is that in BM the Schrödinger evolution is exact, but not in GRW. However, this difference is not so crucial. In fact we will present in Section 7.1 a reformulation of GRWf in which the Schrödinger evolution is exact.

A related important difference is that the empirical predictions of BM agree *exactly and always* with those of the quantum formalism (whenever the latter is unambiguous) while the predictions of the GRW theory don't. (The latter agree only approximately and in most cases.) In particular, one can empirically distinguish BM from the GRW theory. (However, no decisive test could as yet be performed; see (Bassi and Ghirardi, 2003) for details.) The empirical disagreement between the two theories is usually explained by appealing to the fact that in one theory the wave function obeys the Schrödinger evolution while in the other it does not. However, especially in light of the reformulation of GRWf we shall describe in Section 7.1, the empirical inequivalence between the two theories should be better regarded as having a different origin. Though we shall elaborate on this issue in Section 7.3, we shall anticipate the mathematical roots of such a difference in Section 5.2 (which however may be skipped on a first reading of this paper).

A difference in the mathematical structure of GRWf (and OQT) on the one hand and BM (but also GRWm) on the other concerns the probability distribution that each of the these theories defines on its space of histories of the PO. This probability distribution is a quadratic functional of the initial ψ for GRWf and OQT, but not for BM and GRWm. This feature is at the origin of why GRWf can be modified so as to become a fully relativistically invariant theory (see the end of Section 4.2). It will be discussed in the following subsection, which, however, will not be needed for understanding the rest of the paper.

5.1 Primitive Ontology and Quadratic Functionals

It is worth noting a feature of the mathematical structure of GRWf that it shares with OQT, but that is absent in, for example, BM and GRWm. It concerns the dependence on the (initial) wave function ψ of the probability distribution $\mathbb{P}^\psi$ that the theory defines on its space Ω of histories of the PO. In BM, Ω is the space of continuous paths in configuration space $\mathbb{R}^{3N}$, and the measure $\mathbb{P}^\psi$ corresponds to the quantum equilibrium measure, and is concentrated on a $3N$-dimensional submanifold of Ω, namely the solutions of Bohm's equation (1). In GRWf, Ω is the space of discrete subsets of space-time (possibly with labels $1, \ldots, N$), and the measure $\mathbb{P}^\psi$ is given by (11). In GRWm, Ω is a space of fields on space-time, and $\mathbb{P}^\psi$ the image under the mapping $\psi \mapsto m$ given by (12) of the distribution of the Markov process $(\psi_t)_{t \geq 0}$.

In GRWf and OQT, but not in BM or GRWm, $\mathbb{P}^\psi$ is a quadratic functional of ψ. More precisely, in GRWf and OQT it is of the form

$$\mathbb{P}^\psi(\cdot) = \langle \psi | E(\cdot) \psi \rangle \tag{19}$$

where $E(\cdot)$ is the positive-operator-valued measure (POVM) on Ω that can be read off from (11) for GRWf, and is the POVM associated with the results of a sequence of measurements for OQT (see, e.g., Dürr et al., 2004b). Neither GRWm nor BM share this property. The easiest way of seeing this begins with noting that (19) entails that any two ensembles of wave functions (corresponding to probability measures μ, μ' on the unit sphere S of Hilbert space) with the same density matrix,

$$\hat{\rho}_\mu = \int_S \mu(d\psi) \, |\psi\rangle\langle\psi| = \hat{\rho}_{\mu'} \,, \tag{20}$$

lead to the same distribution

$$\mathbb{P}_\mu(\cdot) = \int_S \mu(d\psi) \, \mathbb{P}^\psi(\cdot) = \mathrm{tr}(E(\cdot)\hat{\rho}_\mu) = \mathbb{P}_{\mu'}(\cdot) \tag{21}$$

on Ω. This is notoriously not true in BM (Bell, 1980). It is not true in GRWm either, as one easily checks, for example by considering, at just one single time, the following two ensembles of wave functions for Schrödinger's cat: μ gives probability $\frac{1}{2}$ to $2^{-1/2}(|\text{dead}\rangle + |\text{alive}\rangle)$ and $\frac{1}{2}$ to $2^{-1/2}(|\text{dead}\rangle - |\text{alive}\rangle)$, while μ' gives $\frac{1}{2}$ to $|\text{dead}\rangle$ and $\frac{1}{2}$ to $|\text{alive}\rangle$.

One can say that the essence of this difference between these theories lies in different choices of which quantity is given by a simple, namely quadratic, expression in ψ:

- the probability distribution $\mathbb{P}^\psi$ of the history of the PO both in GRWf and OQT, see (19)

- the probability distribution ρ^ψ of the PO at time t in BM,

$$\rho^\psi(q, t) = |\psi(q, t)|^2 \tag{22}$$

- the PO itself at time t in GRWm,

$$m(x,t) = \langle \psi_t | \tilde{\Lambda}(x) \, \psi_t \rangle \text{ with } \tilde{\Lambda}(x) = \sum_{i=1}^{N} m_i \, \delta(x - \widehat{Q}_i). \tag{23}$$

Note in particular the rather different roles that '$|\psi|^2$' can play for different quantum theories and different choices of the PO.

5.2 Primitive Ontology and Equivariance

In Section 2 we have recalled the notion of the equivariance of the probability distribution $|\psi|^2$ and indicated how it is the key notion for establishing the empirical agreement between BM and the predictions of the quantum formalism (whenever the latter are unambiguous). The equivariance of $|\psi|^2$ expresses the mutual compatibility, with respect to $|\psi|^2$, of the Schrödinger evolution of the wave function and the Bohmian motion of the configuration.

It would seem natural to expect that for GRWf we also have equivariance, but relative to the (stochastic) GRW evolution of the wave function instead of the Schrödinger evolution. However, the concept of the equivariance of the distribution $|\psi|^2$ is not directly applicable in this case: in fact, for GRWf there is no random variable $Q(t)$ whose distribution could agree or disagree with a $|\psi_t|^2$ distribution (or any other quantum mechanical distribution), since GRWf is a theory of flashes, not particles, and as such yields no nontrivial random variable that can be regarded as associated with a fixed time t. In this framework it seems natural to consider the notion of a time–translation equivariant distribution, in terms of which we may provide a generalized notion of equivariance as follows: Let Ω_t be the space of possible histories of the PO for times greater than or equal to t. In trajectory theories like BM, Ω_t is the space of continuous paths $[t, \infty) \to \mathcal{Q}$, where $\mathcal{Q}$ is the configuration space; in flash theories like GRWf it is the space of finite–or–countable subsets of the half space-time $[t, \infty) \times \mathbb{R}^3$. Consider an association $\psi \mapsto \mathbb{P}^\psi$ where $\mathbb{P}^\psi$ is a probability measure on Ω_0 that is compatible with the dynamics of the theory. We say that this association is *equivariant* relative to a deterministic evolution $\psi \mapsto \psi_t$ if $S_t^\star \mathbb{P}^\psi = \mathbb{P}^{\psi_t}$, where $\star$ denotes the action of the mapping on measures and S_t is a suitably defined time shift.[11] More generally, for an evolution that may be stochastic, we say that the association is *equivariant* relative to the evolution if

$$S_t^\star \mathbb{P}^\psi = \mathbb{E} \mathbb{P}^{\psi_t}, \tag{24}$$

where $\mathbb{E}$ denotes the average over the random ψ_t. With this definition, BM is equivariant relative to the Schrödinger evolution, and GRWf and GRWm are equivariant relative to the GRW evolution.

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[11]In order to define S_t properly, let R_t, $t > 0$, be the restriction mapping $\Omega_0 \to \Omega_t$, and T_τ the time translation mapping $\Omega_t \to \Omega_{t+\tau}$. Then $S_t = T_{-t} \circ R_t : \Omega_0 \to \Omega_0$ is the time shift.

	Particles	Fields	Flashes
Deterministic	BM	BQFT, Sm	
Indeterministic	SM, BTQFT, BMW, GRWp	GRWm	GRWf, Sf, Sf'

Table 1: Different possibilities for the PO of a theory are presented: particles, fields and flashes. These different primitive ontologies can evolve according to either deterministic or stochastic laws. Corresponding to these possibilities we have a variety of physical theories: Bohmian mechanics (BM), Bohmian quantum field theory (BQFT), a mass density field theory with Schrödinger evolving wave function (Sm), stochastic mechanics (SM), Bell-type quantum field theory (BTQFT), Bell's version of many-worlds (BMW), a particle GRW theory (GRWp), GRW theory with mass density (GRWm), GRW theory with flashes (GRWf), and two theories with flashes governed by Schrödinger (or Dirac) wave functions (Sf and Sf'). For a detailed description of these theories, see the text.

6   A Plethora of Theories

One may wonder whether some primitive ontologies (flashes and continuous matter density) work only with GRW-type theories while others (particle trajectories) work only with Bohm-type theories. This is not the case, as we shall explain in this section.

6.1   Particles, Fields, and Flashes

Let us analyze, with the aid of Table 1, several possibilities: there can be at least three different kinds of primitive ontologies for a fundamental physical theory, namely particles, fields, and flashes. Those primitive ontologies can evolve either according to a deterministic or to a stochastic law and this law can be implemented with the aid of a wave function evolving either stochastically or deterministically.

BM is the prototype of a theory in which we have a particle ontology that evolves deterministically according to a law specified by a wave function that also evolves deterministically. The natural analog for a theory with particle ontology with indeterministic evolution is stochastic mechanics (SM), in which the law of evolution of the particles is given by a diffusion process while the evolution of the wave function, the usual Schrödinger evolution, remains deterministic (see Nelson, 1985; Goldstein, 1987, for details). Another example involving stochastically evolving particles with a deterministically evolving wave function is provided by a Bell-type quantum field theory (BTQFT) in which, despite the name, the PO is given by particles evolving indeterministically to allow for creation and annihilation (for a description, see Dürr et al., 2004a, 2005b; Bell, 1986). Another possibility for a stochastic theory of particles is a theory GRWp in which the particle motion is governed by (1) but with a wave function that obeys a GRW-like evolution in which the collapses occur exactly as in GRW except that,

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once the time and label for the collapse has been chosen, the collapse is centered at the actual position of the particle with the chosen label, rather than at random according to equation (7). (A garbled formulation of this theory is presented in (Bohm and Hiley, 1993, p. 346).)

What in Table 1 we call a Bohmian quantum field theory (BQFT) involves only fields, evolving deterministically (Bohm, 1952; Struyve and Westman, 2006). Another example is provided by the theory Sm in which the PO is given by the mass density field (12) but evolving with a Schrödinger wave function — always evolving according to Schrödinger's equation, with no collapses. GRWm provides an example of a theory of fields that evolve stochastically.

Concerning theories with flashes, these are inevitably stochastic, and GRWf, in which the flashes track the collapses of the wave function, is the prototype. However, there are also theories with flashes in which the wave function never collapses. Such theories are thus arguably closer to BM than to GRWf. We consider two examples.

In the first example, denoted by Sf,[12] the PO consists of flashes with their distribution determined by a Schrödinger wave function $\psi = \psi(q_1, \ldots, q_N)$, that evolves always unitarily, as in BM, according to the N–'particle' Schrödinger evolution (2). The flashes are generated by the wave function exactly as in GRWf. Thus, the algorithm, whose output is the flashes, is the same as the one described in Section 3, with steps 1., 2. and 3., with the following difference: the first sentence in step 2. is dropped, since no collapse takes place. In other words, in Sf flashes occur with rate (8) but are accompanied by no changes in the wave function.[13] (This flash process defines, in fact, a Poisson process in space-time—more precisely, a Poisson system of points in $\mathbb{R}^4 \times \{1, \ldots, , N\}$—with intensity measure $r((x,t),i) = r(x,i|\psi_t)$ given by (8).) Note that, in contrast to the case of GRWf, one obtains a well defined theory by taking the limit $\sigma \to 0$ in (4), that is by replacing $\Lambda_i(x)$ in (8) with $\tilde{\Lambda}_i(x)$ given by $\tilde{\Lambda}_i(x) = \delta(\widehat{Q}_i - x)$, where $\widehat{Q}_i$ is the position operator of the i-th 'particle.'

Our last example (Sf') is the following. Consider a nonrelativistic system of N noninteracting quantum particles with wave function satisfying the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\sum_{i=1}^{N} \frac{\hbar^2}{2m_i} \nabla_i^2 \psi + \sum_{i=1}^{N} V_i(q_i)\,\psi\,, \tag{25}$$

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[12]Here S stands for Schrödinger (evolution). Using this notation we have that BM = Sp.

[13]Accordingly, equation (11) is replaced by

$$\mathbb{P}\big(X_1 \in dx_1, T_1 \in dt_1, I_1 = i_1, \ldots, X_n \in dx_n, T_n \in dt_n, I_n = i_n | \psi_{t_0}\big)$$
$$= \lambda^n e^{-N\lambda(t_n - t_0)} \prod_{k=1}^{n} \langle \psi_{t_k} | \Lambda_{i_k}(x_k) \psi_{t_k} \rangle \, dx_1 dt_1 \cdots dx_n dt_n\,,$$

where $\Lambda_i(x)$ is the collapse operator given by (4).

and suppose that, as in GRWf, each of the flashes is associated with one of the particle labels $1, \ldots, N$. Given the flashes up to the present, the next flash occurs with rate $N\lambda$, and has a label $I \in \{1, \ldots, N\}$ that is randomly chosen with uniform distribution. If this flash occurs at time T_I, its location X is random with probability distribution

$$\mathbb{P}(X \in dX_I | I, T_I, \{X_k, T_k\}_{k \neq I}) = \mathcal{N} \left| \psi(X_1, T_1, \ldots, X_N, T_N) \right|^2 dX_I, \qquad (26)$$

where $\mathcal{N}$ is a normalizing factor, $\psi = \psi(q_1, t_1, \ldots, q_N, t_N)$ is a multi-time wave function evolving according to the set of N equations

$$i\hbar \frac{\partial \psi}{\partial t_i} = -\frac{\hbar^2}{2m_i} \nabla_i^2 \psi + V_i(q_i)\, \psi \qquad (27)$$

for every $i \in \{1, \ldots, N\}$, and T_k and X_k are, for $k \neq I$, the time and location of the last flash with label k. The reason that this model is assumed to be noninteracting is precisely to guarantee the existence of the multi-time wave function in (26). Sf′ is an example of a theory with a flash ontology that arguably is empirically equivalent to OQT (unlike GRWf)—at least, it would be if it were extended to incorporate interactions between particles—and avoids the many-worlds character of Sf (see Section 6.2 below).

A provisional moral that emerges is that relativistic invariance might be connected with a flash ontology, since GRWf is the only theory in Table 1 (except for Sm and Sf, which have a rather extraordinary character that we discuss in Section 6.2 below) of which we know how it can be made relativistically invariant without postulating a preferred foliation of space-time (or any other equivalent additional structure). Finally, note that all the theories in Table 1 are empirically equivalent (suitably understood) to OQT except GRWm, GRWf, and GRWp.

6.2 Schrödinger Wave Functions and Many-Worlds

A rather peculiar theory representing the world as if it were, at any given time, a collection of particles with classical configuration $Q = (Q_1, \ldots, Q_N)$ is Bell's version of many-worlds (BMW) (Bell, 1981). In BMW the wave function ψ evolves according to Schrödinger's equation and (Bell, 1981)

> instantaneous classical configurations . . . are supposed to exist, and to be distributed . . . with probability $|\psi|^2$. But no pairing of configurations at different times, as would be effected by the existence of trajectories, is supposed.

This can be understood as suggesting that the configurations at different times are not connected by any law. It could also be regarded as suggesting that configurations at different times are (statistically) independent, and that is how we shall understand it here. The world described by BMW is so radically different from what we are accustomed

to that it is hard to take BMW seriously. In fact, for example, at some time during the past second, according to BMW, there were on the earth dinosaurs instead of humans, because of the independence and the fact that, in any no-collapse version of quantum theory, there are parts of the wave function of the universe in which the dinosaurs have never become extinct. In this theory, the actual past will typically entirely disagree with what is suggested by our memories, by history books, by photographs and by other records of (what we call) the past.

Also Sf and Sm, though they are simple mathematical modifications of GRWf and GRWm respectively, provide very different pictures of reality, so different indeed from what we usually believe reality should be like that it would seem hard to take these theories seriously. In Sf and Sm, apparatus pointers never point in a specific direction (except when a certain direction in OQT would have probability more or less one), but rather all directions are, so to speak, realized at once. As a consequence, one is led to conclude that their predictions don't agree with those of the quantum formalism. Still, it can be argued that these theories do not predict any *observable* deviation from the quantum formalism: there is, arguably, no conceivable experiment that could help us decide whether our world is governed by Sf or Sm on the one hand or by the quantum formalism on the other. The reason for this surprising claim is that Sf and Sm can be regarded as many-worlds formulations of quantum mechanics. Let us explain.

At first glance, in an Sf or Sm world, the after-measurement state of the apparatus seems only to suggest that matter is very spread out. However, if one considers the flashes, governed by the rate (8), or the mass density (12), that correspond to macroscopic superpositions, one sees that they form independent families of correlated flashes or mass density associated with the terms of the superposition, with no interaction between the families. The families can indeed be regarded as comprising many worlds, superimposed on a single space-time. Metaphorically speaking, the universe according to Sf or Sm resembles the situation of a TV set that is not correctly tuned, so that one always sees a mixture of two channels. In principle, one might watch two movies at the same time in this way, with each movie conveying its own story composed of temporally and spatially correlated events.

Thus Sf and Sm are analogous to Everett's many-worlds (EMW) formulation of quantum mechanics (Everett, 1957), but with the 'worlds' explicitly realized in the same space-time. Since the different worlds do not interact among themselves—they are, so to speak, reciprocally transparent—this difference should not be regarded as crucial. Thus, to the extent that one is willing to grant that EMW entails no observable deviation from the quantum formalism, the same should be granted to Sf and Sm. Moreover, contrarily to EMW, but similarly to BMW, Sf and Sm have a clear PO upon which the existence and behavior of the macroscopic counterparts of our experience can be grounded.

This ontological clarity notwithstanding, in Sf and Sm reality is of course very different from what we usually believe it to be like. It is populated with ghosts we do not perceive, or rather, with what are like ghosts from our perspective, because the ghosts are as real as we are, and from their perspective we are the ghosts. We plan to give a more complete discussion of Sf and Sm in a future work.

We note that the theory Sm is closely related to—if not precisely the same as—the version of quantum mechanics proposed by Schrödinger (1926). After all, Schrödinger originally regarded his theory as describing a continuous distribution of matter (or charge) spread out in *physical space* in accord with the wave function on *configuration space* (Schrödinger, 1926). He soon rejected this theory because he thought that it rather clearly conflicted with experiment. Schrödinger's rejection of this theory was perhaps a bit hasty. Be that as it may, according to what we have said above, Schrödinger did in fact create the first many-worlds theory, though he probably was not aware that he had done so. (We wonder whether he would have been pleased if he had been).[14]

7   The Flexible Wave Function

In this section we elaborate on the notion of physical equivalence by considering physically equivalent formulations of GRWf and BM for which the laws of evolution of the wave function are very different from the standard ones. We conclude with some remarks on the notion of empirical equivalence.

7.1   GRWf Without Collapse

As a consequence of the view that the GRW theory is ultimately not about wave functions but about either flashes or matter density, the process ψ_t in Hilbert space (representing the collapsing wave function) should no longer be regarded as playing the central role in the GRW theory. Instead, the central role is played by the random set F of flashes for GRWf, respectively by the random matter density function $m(\cdot, t)$ for GRWm. From this understanding of GRWf as being fundamentally about flashes, we obtain a lot of flexibility as to how we should regard the wave function and prescribe its behavior. As we point out in this section, it is not necessary to regard the wave function in GRWf as undergoing collapse; instead, one can formulate GRWf in such a way that it involves a wave function ψ that evolves linearly (i.e., following the usual Schrödinger evolution).

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[14]However, Schrödinger did write that (Schrödinger, 1927, p. 120) '$\psi\bar{\psi}$ is a kind of weight-function in the system's configuration space. The wave-mechanical configuration of the system is a superposition of many, strictly speaking of all, point-mechanical configurations kinematically possible. Thus, each point-mechanical configuration contributes to the true wave-mechanical configuration with a certain weight, which is given precisely by $\psi\bar{\psi}$. If we like paradoxes, we may say that the system exists, as it were, simultaneously in all the positions kinematically imaginable, but not 'equally strongly' in all.'

Suppose the wave function at time t is ψ_t. Then according to equation (8), for GRWf the rate for the next flash is given by

$$r(x, i | \psi_t) = \lambda \| \Lambda_i(x)^{1/2} \psi_t \|^2. \tag{28}$$

Observe that ψ_t, given by equation (9), is determined by ψ_{t_0} and the flashes (X_k, T_k) that occur between the times t_0 and t; it can be rewritten as follows:

$$\psi_t = \frac{\Lambda_{I_n}(X_n, T_n; t)^{1/2} \cdots \Lambda_{I_1}(X_1, T_1; t)^{1/2} \psi_t^L}{\| \Lambda_{I_n}(X_n, T_n; t)^{1/2} \cdots \Lambda_{I_1}(X_1, T_1; t)^{1/2} \psi_t^L \|} \tag{29}$$

where we have introduced the Heisenberg-evolved operators (with respect to time t)

$$\Lambda_{I_k}(X_k, T_k; t)^{1/2} = U_{t-T_k} \Lambda_{I_k}(X_k)^{1/2} U_{T_k - t} = U_{t-T_k} \Lambda_{I_k}(X_k)^{1/2} U_{t-T_k}^{-1} \tag{30}$$

and the linearly evolved wave function

$$\psi_t^L = U_{t-t_0} \psi_{t_0}, \tag{31}$$

where t_0 is the initial (universal) time. By inserting ψ_t given by equation (29) in (28) one obtains that

$$r(x, i | \psi_t) = \lambda \frac{\| \Lambda_i(x)^{1/2} \Lambda_{I_n}(X_n, T_n; t)^{1/2} \cdots \Lambda_{I_1}(X_1, T_1; t)^{1/2} \psi_t^L \|^2}{\| \Lambda_{I_n}(X_n, T_n; t)^{1/2} \cdots \Lambda_{I_1}(X_1, T_1; t)^{1/2} \psi_t^L \|^2}. \tag{32}$$

Suppose that the initial wave function is ψ_{t_0}, i.e., that the linearly evolved wave function at time t is ψ_t^L. Then the right hand side of equation (32) defines the conditional rate for the next flash after time t, given the flashes in the past of t. Note that this conditional rate thus defines precisely the same flash process as GRWf. In particular, we have that

$$\mathbb{P}_{\psi_t^L}(\text{future flashes} | \text{past flashes}) = \mathbb{P}(\text{future flashes} | \psi_t). \tag{33}$$

The collapsed wave function ψ_t provides precisely the same information as the linearly evolving wave function ψ_t^L together with all the flashes. Thus, one arrives at the surprising conclusion that the Schrödinger wave function can be regarded as governing the evolution of the space-time point process of GRWf, so that GRWf can indeed be regarded as a *no-collapse* theory involving flashes. We say 'no-collapse' to underline that the dynamics of the PO is then governed by a wave function evolving according to the standard, linear Schrödinger equation (2). However, while the probability distribution of the future flashes, given the collapsing wave function ψ_t, does not depend on the past flashes, given only ψ_t^L it does.

The two versions of GRWf, one using the collapsing wave function ψ_t and the other using the non–collapsing wave function ψ_t^L, should be regarded not as two different

theories but rather as two formulations of the same theory, GRWf, because they lead to the same distribution of the flashes and thus are physically equivalent. We conclude from this discussion that what many have considered to be the crucial, irreducible difference between BM and GRWf, namely that the wave function collapses in GRWf but does not in BM, is not in fact an objective difference at all, but rather a matter of how GRWf is presented.

We close this section with a remark. A notable difference between the two presentations of GRWf is that while the GRW collapse process ψ_t is a Markov process,[15] the point-process F of flashes is generically non–Markovian. In more detail, we regard a point process in space-time as Markovian if for all $t_1 < t_2$,

$$\mathbb{P}\big(\text{future of } t_2 \big| \text{past of } t_2\big) = \mathbb{P}\big(\text{future of } t_2 \big| \text{strip between } t_1 \text{ and } t_2\big), \qquad (34)$$

where 'future of t_2' refers to the configuration of points after time t_2, etc.. To see that F is non–Markovian, note that the distribution of the flashes in the future of t_2 depends on what happened between time 0 and time t_2, while the strip in space-time between t_1 and t_2 may provide little or no useful information, as it may, for small duration $t_2 - t_1$, contain no flashes at all.[16]

For a Markovian flash process events in a time interval $[t_1, t_2]$ are independent of those in a disjoint time interval $[t_3, t_4]$, which, as discussed in Section 6, would be rather unreasonable for a model of our world. In passing, we note that Sf can indeed be regarded as a sort of Markovian approximation of (the linear version of) GRWf for which, at any time, the past is completely ignored in the computation of the conditional probability of future flashes.

7.2   Bohmian Mechanics With Collapse

In Section 7.1 we showed that GRWf can be reformulated in terms of a linearly evolving wave function. Conversely, BM can be reformulated so that it involves a 'collapsed' wave function. In this formulation the evolution of the wave function depends on the actual configuration. The state at time t is described by the pair (Q_t, ψ_t^C), where $Q = (Q_1, \ldots, Q_N)$ is the (usual) configuration but $\psi_t^C : \mathbb{R}^{3N} \to \mathbb{C}$ is a different wave function than usual, a collapsed wave function. Instead of equations (1) and (2), the

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[15]This means that $\mathbb{P}\big(\text{future}\big|\text{past \& present}\big) = \mathbb{P}\big(\text{future}\big|\text{present}\big)$. In more detail, the distribution of the ψ_t for all $t > t_0$ conditional on the ψ_t for all $t \leq t_0$ coincides with the distribution of the future conditional on ψ_{t_0}.

[16]The matter density field $m(\cdot, t)$ is generically Markovian, but rather by coincidence: Given the initial wave function, different patterns of collapse centers between time 0 and time t_2 should be expected to lead to different fields $m(\cdot, t_2)$, so that the past (or equivalently ψ_{t_2}) may be mathematically determined from $m(\cdot, t_2)$.

state evolves according to

$$\frac{dQ_i}{dt} = \frac{\hbar}{m_i} \mathrm{Im} \frac{\psi^{C*}\nabla_i\psi^C}{\psi^{C*}\psi^C}(Q_1,\ldots,Q_N),\tag{35}$$

which is the same as (1) with ψ replaced by ψ^C, and

$$i\hbar\frac{\partial\psi^C}{\partial t} = -\sum_{i=1}^N \frac{\hbar^2}{2m_i}(\nabla_i - i\tilde{A}_i)^2\psi^C + (V + \tilde{V})\psi^C\tag{36}$$

which is the same as Schrödinger's equation except for the imaginary pseudo-potentials ($\sigma \approx 10^{-7}$ m is the same constant as in GRW)

$$\tilde{A}_i = \frac{i}{\sigma^2}(q_i - Q_i),\quad \tilde{V} = -\frac{i}{\sigma^2}\sum_{i=1}^N \frac{\hbar^2}{m_i}(q_i - Q_i)\cdot\mathrm{Im}\frac{\psi^{C*}\nabla_i\psi^C}{\psi^{C*}\psi^C}\tag{37}$$

making equation (36) nonlinear and Q-dependent. A solution $t \mapsto (Q_t, \psi_t^C)$ of equations (35) and (36) can be obtained from a solution $t \mapsto (Q_t, \psi_t)$ of equations (1) and (2) by setting

$$\psi^C(q_1,\ldots,q_N) = \exp\left(-\sum_{i=1}^N \frac{(q_i - Q_i)^2}{2\sigma^2}\right)\psi(q_1,\ldots,q_N).\tag{38}$$

This is readily checked by inserting (38) into equations (35) and (36). The ensemble of trajectories with distribution $|\psi|^2$ cannot be expressed in a simple way in terms of ψ^C. Nonetheless, for given initial configuration Q_0, we obtain from equations (35) and (36), with given initial ψ_0^C, the same trajectory $t \mapsto Q_t$ as from equations (1) and (2) with the corresponding ψ_0. This may be enough to speak of physical equivalence.

One can read off from (38) that ψ^C is a *collapsed* wave function: Whenever ψ is a superposition (such as for Schrödinger's cat) of macroscopically different states with disjoint supports in configuration space, then in ψ^C all contributions except the one containing the actual configuration Q_t are damped down to near zero. (Still, the evolution is such that when two disjoint packets again overlap, the trajectories display an interference pattern.)

Of course, the unitarily-evolving ψ_t is much more natural than ψ_t^C as a mathematical tool for defining the trajectory $t \mapsto Q_t$; (2) is a simpler equation than (36). Still, the example shows that we have the choice in BM between using a collapsed wave function ψ^C or a spread out wave function ψ.

7.3 Empirical Equivalence and Equivariance

The facts that GRWf can be reformulated so that the wave function evolves linearly, in the usual manner according to Schrödinger's equation, and that BM can be reformulated in terms of a collapsed wave function indicate that the disagreement between the

predictions of the two theories should not be regarded as arising merely from the fact that they involve different wave function evolutions. It is our contention that the source of the empirical disagreement between BM and GRWf can be regarded as lying, neither in their having different evolutions for the wave function, nor in their having different ontologies, but rather in the presence or absence of equivariance with respect to the Schrödinger evolution. More explicitly, we claim that a theory is empirically equivalent to the quantum formalism (i.e., that its predictions agree with those of the quantum formalism) if it yields an equivariant distribution (defining typicality) relative to the Schrödinger evolution that can be regarded as 'effectively $|\psi|^2$.' Let us explain.

The view we have proposed about the PO of a theory and the corresponding role of the wave function has immediate consequences for the criteria for the *empirical equivalence* of two theories, i.e., the statement that they make (exactly and always) the same predictions for the outcomes of experiments.

Before discussing these consequences, let us note a couple of remarkable aspects of the notion of empirical equivalence. One is that, despite the difficulty of formulating the *empirical content* of a theory precisely (a difficulty mainly owed to the vagueness of the notion 'macroscopic'), one can sometimes establish the empirical equivalence of theories; for example, that of BM and SM or that of GRWm and GRWf; for further examples see (Goldstein et al., 2005). Another remarkable aspect is that empirical equivalence occurs at all. One might have expected instead that different theories typically make different predictions, and indeed the theories of classical physics would provide plenty examples. But in quantum mechanics empirical equivalence is a widespread phenomenon; see (Goldstein et al., 2005) for discussion of this point.

Let us turn to the criteria for empirical equivalence. Since the empirical equivalence of two theories basically amounts to the assertion that the two worlds, governed by the two theories, share the same macroscopic appearance, we have to focus on how to read off the macroscopic appearance of a possible world according to a theory. And according to our view about PO, the macroscopic appearance is a function of the PO— but not directly a function of the wave function. In cases in which one can deduce the macroscopic appearance of a system from its wave function, this is so only by virtue of a law of the theory implying that this wave function is accompanied by a PO with a certain macroscopic appearance. In short, empirical equivalence amounts to a statement about the PO. This view is exemplified by our proof of empirical equivalence between GRWm and GRWf in Section 3.3. In more detail, the position Z_t of, say, a pointer at time (circa) t is a function of the PO: In BM and GRWm it can be regarded as a function $Z_t = Z(Q_t)$ of the configuration, respectively as a function $Z_t = Z(m(\cdot, t))$ of the m field, at time t, whereas in GRWf it is best regarded as a function of the history of flashes over the past millisecond or so.

Concerning the empirical equivalence between a theory and OQT, we need to ask

whether the probability of the event $Z_t = z$ agrees with the distribution predicted by standard quantum mechanics. The latter can be obtained from the Schrödinger wave function ψ_t for a sufficiently big system containing the pointer by integrating $|\psi_t|^2$ over all configurations in which the pointer points to z. Thus, regardless of what the PO of a theory is, all that is required for the empirical equivalence between the theory and OQT is that it provide the correct $|\psi_t|^2$ probability distributions for the relevant variables Z_t. When this is so we may speak of an 'effective $|\psi_t|^2$–distribution,' or of *macroscopic* $|\psi|^2$ *Schrödinger equivariance*. Thus, empirical equivalence to OQT amounts to having macroscopic $|\psi|^2$ Schrödinger equivariance. (This applies to 'normal' theories in which pointers point; the situation is different for theories with a many-worlds character as discussed in Section 6.2.)

GRWf (or GRWm) predicts (approximately) the quantum mechanical distribution only under certain circumstances, including, e.g., that the experimental control over decoherence is limited, and that the universe is young on the timescale of the 'universal warming' predicted by GRWf/GRWm (see Bassi and Ghirardi, 2003, for details). Moreover, we know that GRWf, roughly speaking, makes the same predictions as does the quantum formalism for short times, i.e., before too many collapses have occurred. Thus, GRWf yields an effective $|\psi|^2$–distribution for times near the initial time t_0. Now, if GRWf were 'effectively $|\psi|^2$–equivariant,' its predictions would be the same as those of quantum theory for all times. It is the absence of this macroscopic $|\psi|^2$ Schrödinger equivariance that renders GRWf empirically inequivalent to quantum theory and to BM. We shall elaborate on this in a future work (Allori et al., unpublished,a).

The most succinct expression of the source of the empirical disagreement between BM and GRWf is thus the assertion that BM is effectively $|\psi|^2$-equivariant relative to the Schrödinger evolution while GRWf is not. The macroscopic Schrödinger equivariance of BM follows, of course, from its microscopic $|\psi|^2$ Schrödinger equivariance, while the lack of macroscopic $|\psi|^2$ Schrödinger equivariance for GRWf follows from the warming associated with the GRW evolution and the fact that GRWf, as discussed in Section 5.2, is microscopically equivariant relative to that evolution. In fact, it follows from the GRW warming that there is, for GRWf, no equivariant association $\psi \mapsto \mathbb{P}^\psi$ with ψ a Schrödinger-evolving wave function.[17]

8 What is a Quantum Theory without Observers?

To conclude, we delineate the common structure of GRWm, GRWf, and BM:

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[17]Since the GRWf flash process is non–Markovian, the formulation of the notion of equivariant association given in Section 5.2 is not appropriate here; instead, $\mathbb{P}^\psi$ should now be understood to be a probability measure on the space Ω of possible histories of the PO for all times, but one whose conditional probabilities for the future of any time given its past are as prescribed, here by the formula (33). The association is *equivariant* if $T^\star_{-t}\mathbb{P}^\psi = \mathbb{P}^{\psi_t}$, with T_τ now the time translation mapping on Ω.

(i) There is a clear primitive ontology, and it describes matter in space and time.

(ii) There is a state vector ψ in Hilbert space that evolves either unitarily or, at least, for microscopic systems very probably for a long time approximately unitarily.

(iii) The state vector ψ governs the behavior of the PO by means of (possibly stochastic) laws.

(iv) The theory provides a notion of a *typical* history of the PO (of the universe), for example by a probability distribution on the space of all possible histories; from this notion of typicality the probabilistic predictions emerge.

(v) The predicted probability distribution of the macroscopic configuration at time t determined by the PO (usually) agrees (at least approximately) with that of the quantum formalism.

The features (i)–(v) are common to these three theories, but they are also desiderata, presumably even necessary conditions, for any satisfactory quantum theory without observers.[18]

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[18]A certain generalization of (i)–(v) is supported by the considerations in (Dürr et al., 2005a), where it is argued that some systems in a Bohmian universe should be regarded as being governed, or guided, not by a vector ψ in Hilbert space but by a density matrix ρ on Hilbert space, the so-called conditional density matrix. But this does not amount to a big conceptual difference.

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