Approach to Hyperuniformity in the One-Dimensional Facilitated Exclusion Process

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Abstract

For the one-dimensional Facilitated Exclusion Process with initial state a product measure of density $\rho = 1/2 - \delta$, $\delta \geq 0$, there exists an infinite-time limiting state ν_ρ in which all particles are isolated and hence cannot move. We study the variance $V(L)$, under ν_ρ, of the number of particles in an interval of L sites. Under $\nu_{1/2}$ either all odd or all even sites are occupied, so that $V(L) = 0$ for L even and $V(L) = 1/4$ for L odd: the state is *hyperuniform* [21], since $V(L)$ grows more slowly than L. We prove that for densities approaching $1/2$ from below there exist three regimes in L, in which the variance grows at different rates: for $L \gg \delta^{-2}$, $V(L) \simeq \rho(1-\rho)L$, just as in the initial state; for $A(\delta) \ll L \ll \delta^{-2}$, with $A(\delta) = \delta^{-2/3}$ for L odd and $A(\delta) = 1$ for L even, $V(L) \simeq CL^{3/2}$ with $C = 2\sqrt{2/\pi}/3$; and for $L \ll \delta^{-2/3}$ with L odd, $V(L) \simeq 1/4$. The analysis is based on a careful study of a renewal process with a long tail. Our study is motivated by simulation results showing similar behavior in higher dimensions; we discuss this background briefly.

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1 Introduction

In the Facilitated Exclusion Process (FEP) on $\mathbb{Z}^d$, also known as the Conserved Lattice Gas process, each site of the lattice can be occupied by at most one particle, so that a configuration η is an element of the configuration space $X = \{0,1\}^{\mathbb{Z}^d}$. At Poisson-distributed times a particle chooses one of its nearest-neighbor sites at random and attempts to jump to it, succeeding only if the target site is unoccupied and the original site has at least one occupied (*facilitating*) neighbor. (Variations of this dynamics, with simultaneous updating or with some other rule for choosing the target site, have also been considered [13].) Note that when $d = 1$ a chosen particle can jump in at most one direction. We will always assume that the system is started in a Bernoulli initial state $\mu_0^{(\rho)}$ of density $\rho < 1$, that is, a product measure in which the η_i are independent and take value 1 with probability ρ. The evolved state at time t will then be denoted $\mu_t^{(\rho)}$; it is clearly translation invariant (TI).

The evolution of this system, or of minor variations of it, has been investigated for $d = 1$ [1–4, 7, 8, 10–13, 17, 22], primarily theoretically, and for $d \geq 2$ [13, 14, 16, 19], primarily via simulation in a cubical box with periodic boundary conditions. These investigations strongly suggest the existence of a TI limiting state

$$\nu_\rho := \lim_{t\to\infty} \mu_t^{(\rho)}. \tag{1}$$

This existence was proved for $d = 1$ and $\rho \leq 1/2$ in [22] (and the limiting state was first described there). Moreover, there appears to be a *critical density* ρ_c such that if $\rho \leq \rho_c$ then ν_ρ is a *frozen* state in which all particles are isolated and hence unable to move, while if $\rho > \rho_c$ then ν_ρ is an *active* stationary state in which there is a nonzero density of particles with an occupied neighboring site. Necessarily $\rho_c \leq 1/2$, since for $\rho > 1/2$ it is geometrically impossible for all the particles to be isolated, and indeed equality holds for $d = 1$. But for $d \geq 2$, simulations suggest values of ρ_c which are much smaller, for example, $\rho_c \approx 0.3308$ [13] for $d = 2$.

Our main interest here will be in *fluctuations* in the measure ν_ρ, that is, in the variance $V_\rho(L) := \mathrm{Var}_{\nu_\rho}(N(L))$ of the number $N(L)$ of particles in a cubical box of side L. In general, if such a variance computed from a TI measure μ grows as L^d when $L \nearrow \infty$, we say that μ has *normal fluctuations*. Hexner and Levine [14] observe that, in 2d and 3d, ν_{ρ_c} is not of this character but rather is *hyperuniform* [21]: $V_{\rho_c}(L)$ grows more slowly than L^d, specifically, $V_{\rho_c}(L) \sim L^{\lambda_1}$, with $\lambda_1 \approx 1.57$ in 2d and $\lambda_1 \approx 2.76$ in 3d [14]. (We will relate two quantities by "$\sim$" or, respectively, by "$\simeq$", to express the fact that asymptotically their ratio is bounded away from both 0 and ∞ or, respectively, is equal to 1. The symbol "$\simeq$", used extensively in this paper to express and prove our results, will be more precisely defined in Definition 1.)

Hexner and Levine also discuss the behavior of $V_\rho(L)$ as $\rho \nearrow \rho_c$. Further consideration of this behavior has led us [13] to the following conjecture, which we state for $d \geq 2$; the $d = 1$ version is Theorem 1 below. We introduce the notation $\delta = \rho_c - \rho > 0$.

Fluctuation Conjecture. For the FEP with $d \geq 2$ a critical density ρ_c as described above exists, and ν_{ρ_c} is hyperuniform. For ρ less than but close to ρ_c, three regimes in L may be identified. For *small* L (but still with $L \gg 1$) the variances grow approximately as in the hyperuniform state at ρ_c: $V_\rho(L) \simeq C_1 L^{\lambda_1}$. At some (approximately defined) scale $L_1(\delta)$ the variances enter the regime of *intermediate* L, in which they grow as $V_\rho(L) \simeq C_2(\delta) L^{\lambda_2}$ with $\lambda_2 > d > \lambda_1$ and $C_2(\delta) > 0$. Then above an (approximate) scale $L_2 = L_2(\delta)$ the growth is as $V_\rho(L) \simeq \rho(1-\rho)L^d$, that is, exactly as in the initial Bernoulli measure $\mu_0^{(\rho)}$. Finally, as $\rho \nearrow \rho_c$, $L_1(\delta)$ and $L_2(\delta)$ increase as $L_i \sim \delta^{-\gamma_i}$ for some exponents γ_1, γ_2 satisfying $\gamma_2 > \gamma_1 > 0$.

In the remainder of the paper we restrict our consideration to the $d = 1$ model. In Section 2 we state in Theorem 1 our main result, the one-dimensional version of the conjecture, and in Section 3 we describe the 1d limiting measure ν_ρ. The proof of Theorem 1 is given in Section 4.

2 Statement of the result

The key to the rigorous establishment of a version of the Fluctuation Conjecture in dimension $d = 1$ is that there the existence and exact value of the critical density are known—$\rho_c = 1/2$—and that we also have a complete description of the limiting measure ν_ρ for $\rho \leq 1/2$ [1,10,11,22]. (This measure was first identified and discussed in [22], a reference which has just come to our attention. We regret that we did not properly credit this work in earlier papers.) We will discuss ν_ρ for $\rho < 1/2$ in Section 3; for the moment let us note that the measure at the critical density is particularly simple:

$$\nu_{1/2} = \frac{1}{2}(\delta_{\eta^*} + \delta_{\eta^\dagger}), \tag{2}$$

where η^* and $\eta^\dagger$ are the two configurations in $X = \{0,1\}^{\mathbb{Z}}$ in which holes and particles strictly alternate.

The Fluctuation Conjecture concerns the asymptotic behavior of the quantity $V_\rho(L)$ with L "small," "intermediate," or "large," yet in each case also sufficiently large. To give a precise result in 1d we introduce some notation for the description of this behavior.

Definition 1. Assume that L is a positive integer and δ a positive real number (in applications we will have $\delta = \rho_c - \rho$), that $A(\delta, L)$ and $B(\delta, L)$ are real valued functions whose asymptotic behavior in L we wish to compare, and that $L_1(\delta)$ and $L_2(\delta)$ are positive functions (these play the role of setting the scales of the various regions). Then we write respectively

$$A(\delta, L) \simeq B(\delta, L) \quad \text{for} \quad \begin{cases} L \ll L_1(\delta), \\ L_1(\delta) \ll L \ll L_2(\delta), \\ L \gg L_2(\delta), \end{cases} \tag{3}$$

if for any $\epsilon > 0$ there exists a $\delta_0 > 0$, together with a (small) number $s > 0$ and/or a (large) number $l > 0$, such that for $\delta < \delta_0$ we have, respectively,

$$1 - \epsilon < \frac{A(\delta, L)}{B(\delta, L)} < 1 + \epsilon \qquad \begin{cases} \text{for } L < sL_1(\delta), \\ \text{for } lL_1(\delta) < L < sL_2(\delta), \\ \text{for } L > lL_2(\delta). \end{cases} \tag{4}$$

If A and B depend also on some additional parameter(s) α we say that (3) holds *uniformly* for α in some (possibly L- or δ-dependent) set if s and/or l, as well as δ_0, may be chosen so that (4) holds for all such α.

With this notation established we may state our main result; we assume that $0 < \rho < 1/2$, that $\delta = 1/2 - \rho$, and that ν_ρ is the measure (1) of the 1d FEP. In contrast to the $d \geq 2$ behavior described in the the Fluctuation Conjecture, for $d = 1$ the variances $V_\rho(L)$ behave differently for L odd and L even.

Theorem 1. *Let $N^{(\delta)}(L)$ be the number of particles on the sites $1, 2, \ldots, L$, with distribution determined by ν_ρ. Then:*
(a) For L odd,

$$V_\rho(L) = \mathrm{Var}(N^{(\delta)}(L)) \simeq \begin{cases} \frac{1}{4}, & \text{for } L \ll \delta^{-2/3}, \\ \frac{2}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2}, & \text{for } \delta^{-2/3} \ll L \ll \delta^{-2}, \\ \frac{1}{4}L, & \text{for } L \gg \delta^{-2}. \end{cases} \tag{5}$$

(b) For L even,

$$V_\rho(L) = \mathrm{Var}(N^{(\delta)}(L)) \simeq \begin{cases} \frac{2}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2}, & \textit{for } 1 \ll L \ll \delta^{-2}, \\ \frac{1}{4}L, & \textit{for } L \gg \delta^{-2}. \end{cases} \tag{6}$$

We actually have a stronger result for the asymptotics in the large-L region. The "right" estimate for $\mathrm{Var}(N^{(\delta)}(L))$ there is $\rho(1-\rho)L$, as stated in the Fluctuation Conjecture and as we discuss further in Remark 1 below, and this is true for all, not just small, δ:

Corollary 2 (to the proof of Theorem 1)**.** *For any $\epsilon > 0$ there exists an $l > 0$ such that for any $\delta \in (0, 1/2)$,*

$$1 - \epsilon < \frac{\mathrm{Var}(N^{(\delta)}(L))}{\rho(1-\rho)L} < 1 + \epsilon \qquad \textit{for } L > l\delta^{-2}. \tag{7}$$

In fact, it follows from (82) below that (7) holds with $l = (2\epsilon)^{-1}$.

A comparison of Theorem 1(a) with the Fluctuation Conjecture shows that for L odd the behavior of $V_\rho(L)$ in one dimension corresponds directly to the conjectured behavior in higher dimension (but without the condition $L \gg 1$). In particular it follows from (2) that

$$V_{1/2}(L) = \begin{cases} \frac{1}{4}, & \text{if } L \text{ is odd}, \\ 0, & \text{if } L \text{ is even}, \end{cases} \tag{8}$$

which implies that $\nu_{1/2}$ is hyperuniform and also explains the $V_\rho(L) \simeq 1/4$ behavior in Theorem 1 for small odd L. The variables introduced in the conjecture become $\lambda_1 = 0$, $C_1 = 1/4$, $\lambda_2 = 3/2$, $C_2(\delta) = \sqrt{8/\pi}\,\delta/3$, $\gamma_1 = 2/3$, and $\gamma_2 = 2$. On the other hand, for even L the "small" growth region is absent in one dimension: for small and moderate values of L the variances grow as $C_2(\delta)L^{3/2}$. This odd/even distinction may be regarded as a legacy of (8) when δ is perturbed away from 0.

Remark 1. Corollary 2 certainly implies that $\lim_{L\to\infty} V_\rho(L)/L = \rho(1-\rho)$ for all $\rho < 1/2$, and this part of the result, although not the scale δ^{-2} at which the limit is achieved, may be obtained by an elementary argument [11]. For with probability 1 each particle will move only a finite distance during the evolution, so that for L sufficiently large $N(L)$ will, to high relative accuracy, be the same at the end of the evolution as it was at the beginning, and $\mathrm{Var}(N(L))$ will be the same as for the original Bernoulli measure.

Remark 2. There are several one-dimensional models with exclusion and facilitation, closely related to the FEP, for which also $\rho_c = 1/2$ and for which $\lim_{t\to\infty} \mu_t^{(\rho)}$ is for $\rho \le 1/2$ the measure ν_ρ that we are considering here, and hence for which the fluctuations $V_\rho(L)$ satisfy Theorem 1. In particular, this is true of the totally asymmetric, discrete-time (parallel) updating in which all particles attempt to jump at the same time, and only to the right [11]. It is also true of an asymmetric version of the continuous-time model of Section 1 in which particles attempt to jump to the left or right at different rates [1].

3 The limiting measure ν_ρ for $\rho < 1/2$

A key ingredient for understanding the behavior described in Theorem 1, especially the behavior in the intermediate regime, is the renewal structure of the stationary state ν_ρ defined in (1), $\rho < 1/2$; we now describe this structure [11, 22]. Since this state is frozen, the occurrence of adjacent 1's has probability zero, so that the state is supported on configurations of the form

$$\cdots 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ \hat{0}\ 1\ 0\ 1\ 0 \cdots 1\ 0\ 0\ \hat{0}\ \hat{0}\ \hat{0}\ 1\ 0\ 1\ 0\ 1\ 0 \cdots 1\ 0\ 0\ \hat{0} \cdots \tag{9}$$

$$= \cdots 0\ (1\ 0\ )^{X_{-1}}\ 0\ (1\ 0)^{X_0}\ 0\ (1\ 0)^{X_1}\ 0\ (1\ 0)^{X_2} 0 \cdots . \tag{10}$$

We focus particular attention on the 00's in (9); the *second* 0 of each such pair, marked as $\hat{0}$ in (9) and corresponding to a 0 outside the parentheses in (10), will be called a *renewal event*. (Note that adjacent renewal events correspond to a zero value for the corresponding X_i.) If we let $\hat{\nu}_\rho$ be the measure ν_ρ conditioned on the occurrence of a renewal event at the origin, then under $\hat{\nu}_\rho$ the X_i's in (10) are independent random variables that are identically distributed, with the distribution described in (11).

The measure ν_ρ may now be described as the unique ergodic TI measure such that conditioning on the occurrence of a renewal event at the origin yields the measure $\hat{\nu}_\rho$ as just described. Explicitly, if $\hat{\nu}_\rho^{(k)}$, $k \ge 1$, denotes the restriction of the measure $\hat{\nu}_\rho$ to configurations in which the first renewal event to the right of the origin lies at site k, and $\hat{\nu}_\rho^{(k,i)}$, $0 \le i < k$, its translation by i sites to the left, then $\nu_\rho = Z^{-1} \sum_{k\ge1} \sum_{i=1}^{k-1} \hat{\nu}_\rho^{(k,i)}$, with Z a normalizing factor.

Remark 3. The density of the renewal events, i.e., the probability under ν_ρ of finding such an event at, say, site 1 (or equivalently of finding adjacent 0's at sites 0 and 1), is $1 - 2\rho = 2\delta$, since under ν_ρ the probability of adjacent 1's is zero.

It is shown in [11, 22] that the distribution of the X_i's under $\hat{\nu}_\rho$ is that of a random variable $\widehat{X}^{(\delta)}$ for which

$$
\begin{aligned}
P(\widehat{X}^{(\delta)} = n) &= C_n \rho^n (1-\rho)^{n+1} \\
&= \frac{1+2\delta}{2\cdot 4^n} C_n (1-4\delta^2)^n,
\end{aligned}
\qquad n = 0, 1, 2, \ldots, \tag{11}
$$

with C_n the n^{th} Catalan number [20]:

$$
C_n := \frac{1}{n+1}\binom{2n}{n} = \frac{4^n}{n^{3/2}\sqrt{\pi}}\left(1 + O\left(\frac{1}{n}\right)\right). \tag{12}
$$

Here we have used Stirling's formula with error bounds. Thus for $n \gg 1$,

$$
P(\widehat{X}^{(\delta)} = n) \simeq \frac{1+2\delta}{2n^{3/2}\sqrt{\pi}}(1-4\delta^2)^n \simeq \frac{1}{2n^{3/2}\sqrt{\pi}} e^{-4\delta^2 n}, \tag{13}
$$

where the first approximation holds for all δ, $0 \le \delta < 1/2$, and the second for $\delta^2 n \le 1$. (The Catalan number C_n arises here as the number of random walks of length $2n$, with steps ± 1, which begin and end at the origin and take only nonnegative values.)

It follows immediately from (2) and Remark 3 that

$$
\nu_{1/2} = \lim_{\delta \to 0} \nu_\rho \tag{14}
$$

in the sense of weak convergence, i.e., that $\nu_{1/2}(A) = \lim_{\delta\to 0} \nu_\rho(A)$ for every $A \subset X$ defined in terms of the configuration on a finite set of sites. In particular, the limiting measure contains no renewal events. On the other hand, the limit $\hat{\nu}_{1/2} = \lim_{\delta \to 0} \hat{\nu}_\rho$ is not so trivial: $\hat{\nu}_{1/2}$ is the probability distribution on configurations of the form (10) for which there is a renewal event at the origin and the i.i.d. random variables X_i have the distribution of $\widehat{X}^{(0)}$:

$$
P(\widehat{X}^{(0)} = n) = \lim_{\delta \to 0} P(\widehat{X}^{(\delta)} = n) = \frac{C_n}{2\cdot 4^n} \simeq \frac{1}{2\sqrt{\pi} n^{3/2}}. \tag{15}
$$

In particular, there exist $c_1, c_2 > 0$ such that

$$
c_1 n^{-3/2} < P(\widehat{X}^{(0)} = n) < c_2 n^{-3/2}, \qquad n \ge 1. \tag{16}
$$

Note that $\widehat{X}^{(0)} = \lim_{\delta\to 0} \widehat{X}^{(\delta)}$ (limit in distribution); note also that although $\hat{\nu}_\rho$ for $\rho < 1/2$ was obtained from ν_ρ by conditioning on a renewal event at the origin, $\hat{\nu}_{1/2}$ cannot be so obtained from $\nu_{1/2}$.

$\widehat{X}^{(0)}$ has a 3/2 power-law tail, and this 3/2 is, as we shall show, the origin of the 3/2 in the $L^{3/2}$ behavior of the variance in the intermediate regime. (If 3/2 were replaced by γ, with $1 < \gamma \leq 2$, we would have had L^γ behavior there [13]). Further, the fact that the exponential decay in (13) becomes significant when n is of order δ^{-2} is the origin of the fact that the transition to the large L regime occurs for L of order δ^{-2}.

Notation: Here, for the reader's convenience, we summarize our notation, reviewing some that was introduced earlier and also defining some new notation that will be used in the sequel. We write $\rho = 1/2 - \delta$, with $0 \leq \delta < 1/2$, define $J_L = \{1, 2, \ldots, L\}$, and call the second of a pair of consecutive empty sites a renewal event. ν_ρ denotes the infinite-time limit state (1) for the one-dimensional FEP at density ρ, and $\hat{\nu}_\rho$ the state defined for $\delta > 0$ by conditioning ν_ρ on the occurrence of a renewal event at the origin, and for $\delta = 0$ as $\lim_{\delta\searrow 0} \hat{\nu}_\rho$. We will use the following random variables:

- $N^{(\delta)}(L)$, the number of particles in J_L under ν_ρ;
- $N^{(\delta)}_{\rm ren}(L)$ and $\widehat{N}^{(\delta)}_{\rm ren}(L)$, the number of renewal events in J_L under ν_ρ and $\hat{\nu}_\rho$, respectively (and, by convention, $\widehat{N}^{(\delta)}_{\rm ren}(0) = 0$);
- $\widehat{X}^{(\delta)}$, a random variable with the distribution under $\hat{\nu}_\rho$ of the X_i in (10);
- $\widehat{Y}^{(\delta)} = 2\widehat{X}^{(\delta)} + 1$, the distance between renewal events under $\hat{\nu}_\rho$;
- $F^{(\delta)}$, the location of the first renewal event to the right of the origin under ν_ρ.

We make a further remark and a general notational convention. First, the translation invariance of ν_ρ implies that the number of renewal events in any interval of L sites, conditioned on the occurrence of a renewal event at the immediately preceding site, has the same distribution as $\widehat{N}^{(\delta)}_{\rm ren}(L)$. Second, we will let P denote probabilities for random variables such as those above, whether defined using ν_ρ or $\hat{\nu}_\rho$; thus, for example, $P(N^{(\delta)}_{\rm ren}(L) = n) = \nu_\rho(N^{(\delta)}_{\rm ren}(L) = n)$ while $P(\widehat{N}^{(\delta)}_{\rm ren}(L) = n) = \hat{\nu}_\rho(\widehat{N}^{(\delta)}_{\rm ren}(L) = n)$.

Note that from Remark 3,

$$E\bigl(N^{(\delta)}_{\rm ren}(L)\bigr) = 2\delta L. \tag{17}$$

Note also that while, from (14), $N^{(0)}_{\rm ren}(L) = \lim_{\delta\to 0} N^{(\delta)}_{\rm ren}(L)$ (limit in distribution) is the zero random variable, since in $\nu_{1/2}$ there are no renewal events,

$\widehat{N}^{(0)}_{\rm ren}(L) = \lim_{\delta\to 0} \widehat{N}^{(\delta)}_{\rm ren}(L)$ is non-trivial. We stress that the key to the $\delta \searrow 0$ asymptotics described in Theorem 1 lies not in $\nu_{1/2}$ but in $\hat{\nu}_{1/2}$.

4 Proof of Theorem 1

The proof of Theorem 1 is broken into nine steps, as follows:

Step 1: Express $N^{(\delta)}(L)$ in terms of $N^{(\delta)}_{\rm ren}(L)$.

Step 2: Express the distribution and second moment of $N^{(\delta)}_{\rm ren}(L)$ in terms of those of $\widehat{N}^{(\delta)}_{\rm ren}(L)$.

Step 3: Approximate the expressions found in Step 2 by replacing $\widehat{N}^{(\delta)}_{\rm ren}(L)$ by $\widehat{N}^{(0)}_{\rm ren}(L)$.

Step 4: The expressions obtained in Step 3 involve the quantity p_δ defined in (26). In Step 4 we express this quantity in terms of $\widehat{Y}^{(\delta)}$, then replace the occurrences of $\widehat{Y}^{(\delta)}$ by $\widehat{Y}^{(0)}$.

Step 5: Obtain the large-L asymptotics of the distribution of $\widehat{N}^{(0)}_{\rm ren}(L)$ and of its second moment, and insert these into the expressions found in Step 4.

Step 6: Use the asymptotics of the distribution of $\widehat{Y}^{(0)}$ to further approximate the expressions found in Step 5.

Step 7: Obtain from the expressions found in Step 6 the asymptotics of $\mathrm{Var}(N^{(\delta)}_{\rm ren}(L))$.

Step 8: Use Step 1 to obtain the results of Theorem 1 for $L \ll \delta^{-2}$ from the expression found in Step 7 for $\mathrm{Var}(N^{(\delta)}_{\rm ren}(L))$.

Step 9: Use some facts about the truncated two-point correlation function for ν_ρ to obtain the results of Theorem 1 for $L \gg \delta^{-2}$.

We now consider these steps in order.

Step 1: In this step we again use the notation introduced in (9), so that the values which may be taken by a configuration η_i are $\hat{0}$, 0, and 1, where $\hat{0}$ denotes a renewal event, 0 an empty site preceded by a 1, and 1 an occupied site. Now we observe that $L - N_{\rm ren}^{(\delta)}(L)$ is odd if and only if the pair (η_1, η_L) has value $(0,0)$, $(0,\hat{0})$, $(\hat{0},1)$, or $(1,1)$; moreover,

$$N^{(\delta)}(L) = \frac{1}{2}\big[L - (N_{\rm ren}^{(\delta)}(L) + \sigma^{(\delta)}(L))\big], \tag{18}$$

where

$$\sigma^{(\delta)}(L) = \begin{cases} 0, & \text{if } L - N_{\rm ren}^{(\delta)}(L) \text{ is even,} \\ 1, & \text{if } L - N_{\rm ren}^{(\delta)}(L) \text{ is odd and } (\eta_1,\eta_L) \text{ is } (0,0) \text{ or } (0,\hat{0}), \\ -1, & \text{if } L - N_{\rm ren}^{(\delta)}(L) \text{ is odd and } (\eta_1,\eta_L) \text{ is } (\hat{0},1) \text{ or } (1,1). \end{cases} \tag{19}$$

One checks this by induction on $N_{\rm ren}^{(\delta)}(L)$; the case $N_{\rm ren}^{(\delta)}(L) = 0$ is easy. For the induction step one passes from a configuration η to another η' by removing a $\hat{0}$ from some site i with $1 \le i \le L$ and setting $\eta'_j = \eta_j$ if $j < i$ and $\eta'_j = \eta_{j+1}$ if $j \ge i$; one then applies the induction assumption to η' on J_{L-1}, noting that then L and $N_{\rm ren}^{(\delta)}(L)$ both decrease by 1, and observing that $(\eta'_1, \eta'_{L-1}) \ne (\eta_1, \eta_L)$ only if $i = 1$ and $\eta'_1 = 1$ or $i = L$ and $\eta'_{L-1} = 0$, and that in each of these cases σ is unchanged and (18) remains valid.

From (18) we have that

$$\mathrm{Var}(N^{(\delta)}(L) = \frac{1}{4}\,\mathrm{Var}\big(N_{\rm ren}^{(\delta)}(L) + \sigma^{(\delta)}(L)\big). \tag{20}$$

To simplify this expression further we note that, writing E for expectation, we have

$$E\big(\sigma^{(\delta)}(L) \,\big|\, N_{\rm ren}^{(\delta)}(L) = n\big) = 0, \quad \text{for any } n \ge 0, \tag{21}$$

as we will argue shortly. But then

$$E(\sigma^{(\delta)}(L)) = 0 \tag{22}$$

and

$$E(N_{\rm ren}^{(\delta)}(L)\sigma^{(\delta)}(L)) = \mathrm{Cov}(N_{\rm ren}^{(\delta)}(L), \sigma^{(\delta)}(L)) = 0, \tag{23}$$

so that

$$\mathrm{Var}\big(N_{\rm ren}^{(\delta)}(L) + \sigma^{(\delta)}(L)\big) = \mathrm{Var}\big(N_{\rm ren}^{(\delta)}(L)\big) + \mathrm{Var}\big(\sigma^{(\delta)}(L)\big). \tag{24}$$

Thus from(20) and (22),

$$\begin{aligned} \mathrm{Var}(N^{(\delta)}(L)) &= \frac{1}{4}\big(\,\mathrm{Var}(N_{\rm ren}^{(\delta)}(L)) + \mathrm{Var}(\sigma^{(\delta)}(L))\big) \\ &= \frac{1}{4}\big(\,\mathrm{Var}(N_{\rm ren}^{(\delta)}(L)) + P(\sigma^{(\delta)}(L) \ne 0)\big). \end{aligned} \tag{25}$$

To verify (21) we first note that, from (1) and the reflection invariance of the Bernoulli measure and of the dynamics, ν_ρ is invariant under reflection about any (integer or half-integer) point. (21) then follows from the observation that reflection about $L/2$, the midpoint of the interval $\{0, 1, \ldots, L\}$ (*not* of J_L), leaves $N_{\rm ren}^{(\delta)}(L)$ unchanged and, when $L - N_{\rm ren}^{(\delta)}(L)$ is odd, changes the sign of $\sigma^{(\delta)}(L)$, as one sees by checking separately for the four possible values of (η_1, η_L) which can then occur.

Step 2: With $F^{(\delta)}$ as defined towards the end of Section 3, let

$$p_\delta(l) = P(F^{(\delta)} = l). \tag{26}$$

The critical observation for Step 2 is that the number of renewal events in J_L, if there are any, is one more than the number of such events to the right of $F^{(\delta)}$. Thus for $n \geq 1$,

$$\begin{aligned} P\big(N_{\rm ren}^{(\delta)}(L) = n\big) &= \sum_{l=1}^{L} p_\delta(l) P\big(N_{\rm ren}^{(\delta)}(L) = n \,\big|\, F^{(\delta)} = l)\big) \\ &= \sum_{l=1}^{L} p_\delta(l) P\big(\widehat{N}_{\rm ren}^{(\delta)}(L-l) = n-1\big), \end{aligned} \tag{27}$$

and so

$$E\big(N_{\rm ren}^{(\delta)}(L)^2\big) = \sum_{l=1}^{L} p_\delta(l) E\big((\widehat{N}_{\rm ren}^{(\delta)}(L-l)+1)^2\big). \tag{28}$$

Step 3: As indicated earlier, the next step is to control the approximation arising from the replacement of $\widehat{N}_{\rm ren}^{(\delta)}$ by $\widehat{N}_{\rm ren}^{(0)}$ in (27) and (28). Specifically, we will show that for $n \geq 1$,

$$P\big(N_{\rm ren}^{(\delta)}(L) = n\big) \simeq \sum_{l=1}^{L} p_\delta(l) P\big(\widehat{N}_{\rm ren}^{(0)}(L-l) = n-1\big) \quad \text{for } L \ll \delta^{-2}, \tag{29}$$

uniformly in $n \leq k\sqrt{L}$ for k any fixed positive integer (see Definition 1), and also that

$$E\big(N_{\rm ren}^{(\delta)}(L)^2\big) \simeq \sum_{l=1}^{L} p_\delta(l) E\big((\widehat{N}_{\rm ren}^{(0)}(L-l)+1)^2\big) \quad \text{for } L \ll \delta^{-2}. \tag{30}$$

Note that the right hand sides of (29) and (30) both mix quantities defined for $\delta > 0$ with those defined for $\delta = 0$ (p_δ and $\widehat{N}^{(0)}_{\rm ren}$ respectively). These equations are more delicate than they may appear because they demand that we control the errors in these approximations by requiring merely that, for small δ, the quantity $L\delta^2$ be sufficiently small regardless of the size of L itself.

Let us fix $\widetilde{L} < L$ ($\widetilde{L}$ plays the role of $L - l$ in (29) and (30)) and let $\eta^{(\widetilde{L})}$ and $\hat{\eta}^{(\widetilde{L},\delta)}$ denote respectively a fixed and a random configuration on $J_{\widetilde{L}}$, such that (i) $\eta^{(\widetilde{L})}$ contains $m = n - 1$ renewal events, with the convention that if $\eta^{(\widetilde{L})}_1 = 0$ then this is a renewal event, and (ii) $\hat{\eta}^{(\widetilde{L},\delta)}$ is the restriction to $J_{\widetilde{L}}$ of a configuration distributed according to $\hat{\nu}_\rho$. Then for $\delta \geq 0$,

$$P\big(\widehat{N}^{(\delta)}_{\rm ren}(\widetilde{L}) = n - 1\big) = \sum_{\eta^{(\widetilde{L})}} P(\hat{\eta}^{(\widetilde{L},\delta)} = \eta^{(\widetilde{L})}). \tag{31}$$

(29) will now follow from (31) once we show that for all $\eta^{(\widetilde{L})}$, uniformly in $m \leq k\sqrt{L}$,

$$P(\hat{\eta}^{(\widetilde{L},\delta)} = \eta^{(\widetilde{L})}) \simeq P(\hat{\eta}^{(\widetilde{L},0)} = \eta^{(\widetilde{L})}) \quad \text{for } L \ll \delta^{-2}. \tag{32}$$

To verify (32) we let $2x_i + 1$, $i = 1, \ldots, m$, be the distances between the renewal events in $\eta^{(\widetilde{L})}$, with $2x_1 + 1$ the distance from the origin to the first renewal event. Further, we define

$$q_\delta(L) = P\big(\widehat{Y}^{(\delta)} > L\big) = P\left(\widehat{X}^{(\delta)} > \left\lfloor \frac{L-1}{2} \right\rfloor\right) \tag{33}$$

with $\widehat{Y}^{(\delta)} = 2\widehat{X}^{(\delta)} + 1$ as defined at the end of Section 3, and

$$r_\delta(L) = \frac{q_\delta(L)}{q_0(L)}. \tag{34}$$

Now from (11) and (15),

$$P(\widehat{X}^{(\delta)} = l) = (1 + 2\delta)(1 - 4\delta^2)^l P(\widehat{X}^{(0)} = l). \tag{35}$$

Then, with L' the distance from the last renewal event in $\eta^{(\widetilde{L})}$ to the right boundary $\widetilde{L}$ of $J_{\widetilde{L}}$, we have

$$\begin{aligned} P(\hat{\eta}^{(\widetilde{L},\delta)} = \eta^{(\widetilde{L})}) &= \prod\nolimits_{i=1}^{m} P(\widehat{X}^{(\delta)} = x_i)\; q_\delta(L') \\ &= (1+2\delta)^m (1 - 4\delta^2)^{\sum_{i=1}^m x_i} r_\delta(L') P(\hat{\eta}^{(\widetilde{L},0)} = \eta^{(\widetilde{L})}). \end{aligned} \tag{36}$$

We complete the argument for (32) by showing that each of the first three factors on the right hand side of (36) is asymptotic to 1 for $L \ll \delta^{-2}$, uniformly in $n \leq k\sqrt{L}$ for any fixed positive k. First, under this condition,

$$1 \leq (1+2\delta)^m \leq (1+2\delta)^{k\sqrt{L}} \leq e^{2k\delta\sqrt{L}}. \tag{37}$$

Next, from $1-x \geq e^{-2x}$ for $0 \leq x \leq 1/2$ we have for any $M \geq 0$,

$$1 \geq (1-4\delta^2)^M \geq e^{-8M\delta^2} \geq 1-8M\delta^2 \quad \text{for } 0 \leq \delta \leq \frac{1}{2\sqrt{2}}, \tag{38}$$

and with $M = \sum_{i=1}^m x_i \leq L/2$ this gives the desired asymptotics of the second factor. We will finally show that

$$r_\delta(L) \simeq 1 \quad \text{for } L \ll \delta^{-2}; \tag{39}$$

this, with L replaced by L', gives the needed control of the right hand side of (36).

We first observe that (16) implies that there exists a $c > 0$, independent of L, such that if for $\epsilon > 0$ we set $K = c\epsilon^{-2}L$ then the tail of the series for $q_0(L)$ beyond K is relatively small, i.e., $q_0(K) < \epsilon q_0(L)$, which implies

$$q_0(L) < \frac{1}{1-\epsilon} \sum_{L<l\leq K} P(\widehat{Y}^{(0)} = l). \tag{40}$$

Further, for $L\delta^2 < \epsilon^3/(8c)$ it follows from (38) that

$$\sum_{L<l\leq K} P(\widehat{Y}^{(\delta)} = l) > (1-\epsilon) \sum_{L<l\leq K} P(\widehat{Y}^{(0)} = l). \tag{41}$$

Then from (11), (40), and (41),

$$1+2\delta \geq \frac{q_\delta(L)}{q_0(L)} > (1-\epsilon)\frac{\sum_{L<l\leq K} P(\widehat{Y}^{(\delta)} = l)}{\sum_{L<l\leq K} P(\widehat{Y}^{(0)} = l)} > (1-\epsilon)^2. \tag{42}$$

This completes the verification of (39). (32) then follows from (36)–(39).

We remark that in fact the first inequality in (42) can be strengthened:

$$\frac{q_\delta(L)}{q_0(L)} \leq 1 \qquad \text{for all } L \geq 1. \tag{43}$$

For from (11) the difference $q_0(L) - q_\delta(L)$ is increasing for $L \leq L^*$ and decreasing for $L > L^*$, where $L^* = -\log(1+2\delta)/\log(1-4\delta^2)$, and vanishes at $L = 0$ and as $L \to \infty$.

We now turn to (30). For any positive k we write $\widehat{N}^{(\delta)}_{\rm ren}(L) = \widehat{N}^{(\delta)}_{\leq k\sqrt{L}}(L) + \widehat{N}^{(\delta)}_{>k\sqrt{L}}(L)$, where

$$\widehat{N}^{(\delta)}_{\leq x}(L) := \widehat{N}^{(\delta)}_{\rm ren}(L) I_{\left\{\widehat{N}^{(\delta)}_{\rm ren}(L)\leq x\right\}} \tag{44}$$

and

$$\widehat{N}^{(\delta)}_{>x}(L) := \widehat{N}^{(\delta)}_{\rm ren}(L) I_{\left\{\widehat{N}^{(\delta)}_{\rm ren}(L)> x\right\}}, \tag{45}$$

with $I_{\{\cdot\}}$ denoting the indicator function of the set $\{\cdot\}$. Then from (32),

$$E(\widehat{N}^{(\delta)}_{\leq k\sqrt{L}}(L)^2) \simeq E(\widehat{N}^{(0)}_{\leq k\sqrt{L}}(L)^2) \quad \text{for } L \ll \delta^{-2}. \tag{46}$$

(30) will follow easily once we strengthen (46) to

$$E(\widehat{N}^{(\delta)}_{\rm ren}(L)^2) \simeq E(\widehat{N}^{(0)}_{\rm ren}(L)^2) \quad \text{for } L \ll \delta^{-2}. \tag{47}$$

There are two crucial facts for obtaining (47) from (46), via the approximations expressed in (54) below. The first, to be proved shortly, is that for any $\epsilon > 0$ there exists a $k > 0$ such that

$$E\big(\widehat{N}^{(\delta)}_{>k\sqrt{L}}(L)^2\big) \leq \epsilon L \quad \text{for } L \ll \delta^{-2} \tag{48}$$

(where $L \ll \delta^{-2}$ holds for all L if $\delta = 0$). The second, to be proved in Step 5, is that $\lim_{L\to\infty} E(\widehat{N}^{(0)}_{\rm ren}(L)^2)/L = 1$, so that for some constant $C > 0$,

$$E(\widehat{N}^{(0)}_{\rm ren}(L)^2) \geq CL \quad \text{for all } L \geq 1. \tag{49}$$

(47) follows from (46), (48) (for both $\delta = 0$ and $\delta > 0$) and (49).

To see this, fix $\epsilon > 0$ and take k so that (48) holds. Then from (48) for $\delta = 0$ and (49) we have (for all L)

$$\left| \frac{E\big(\widehat{N}^{(0)}_{\leq k\sqrt{L}}(L)^2\big)}{E\big(\widehat{N}^{(0)}_{\rm ren}(L)^2\big)} - 1 \right| = \frac{E\big(\widehat{N}^{(0)}_{>k\sqrt{L}}(L)^2\big)}{E\big(\widehat{N}^{(0)}_{\rm ren}(L)^2\big)} < \frac{\epsilon}{C}. \tag{50}$$

Now by (46) we may take $L\delta^2$ so small that

$$\left| \frac{E\big(\widehat{N}^{(\delta)}_{\leq k\sqrt{L}}(L)^2\big)}{E\big(\widehat{N}^{(0)}_{\leq k\sqrt{L}}(L)^2\big)} - 1 \right| < \epsilon. \tag{51}$$

From this,

$$\begin{aligned} E\big(\widehat{N}^{(\delta)}_{\le k\sqrt{L}}(L)^2\big) &> E\big(\widehat{N}^{(0)}_{\le k\sqrt{L}}(L)^2\big)(1-\epsilon) \\ &= \big[E\big(\widehat{N}^{(0)}_{\mathrm{ren}}(L)^2\big) - E\big(\widehat{N}^{(0)}_{>k\sqrt{L}}(L)^2\big)\big](1-\epsilon) \\ &> (C-\epsilon)(1-\epsilon)L, \end{aligned} \tag{52}$$

and then we have from (48) (possibly with a further restriction on $L\delta^2$) that

$$\left| \frac{E\big(\widehat{N}^{(\delta)}_{\le k\sqrt{L}}(L)^2\big)}{E\big(\widehat{N}^{(\delta)}_{\mathrm{ren}}(L)^2\big)} - 1 \right| \le \frac{E\big(\widehat{N}^{(\delta)}_{>k\sqrt{L}}(L)^2\big)}{E\big(\widehat{N}^{(\delta)}_{\le k\sqrt{L}}(L)^2\big)} < \frac{\epsilon}{(C-\epsilon)(1-\epsilon)}. \tag{53}$$

Since ϵ here is arbitrary, (50) and (53) imply respectively that

$$E\big(\widehat{N}^{(0)}_{\mathrm{ren}}(L)^2\big) \simeq E\big(\widehat{N}^{(0)}_{\le k\sqrt{L}}(L)^2\big) \quad \text{and} \quad E\big(\widehat{N}^{(\delta)}_{\mathrm{ren}}(L)^2\big) \simeq E\big(\widehat{N}^{(\delta)}_{\le k\sqrt{L}}(L)^2\big) \tag{54}$$

for $L \ll \delta^{-2}$, and with (46), (47) follows.

To conclude Step 3 we must establish (48). To do so we first note that for any integer $n \ge 1$,

$$P(\widehat{N}^{(\delta)}_{\mathrm{ren}}(L) \ge n) \le P\left(\widehat{Y}^{(\delta)} \le L\right)^n = (1 - q_\delta(L))^n, \tag{55}$$

since the distances between the successive renewal events in J_L (including the distance of the first such event from the origin), which are independent, must each be no greater than L.

In the remainder of this section we write $q = q_\delta(L)$. For any integer-valued random variable N, and any integer $n_c \ge 1$, we have (as a consequence of summation by parts) that

$$E(N^2 I_{\{N \ge n_c\}}) = n_c^2 P(N \ge n_c) + \sum_{n=n_c+1}^{\infty} (2n-1) P(N \ge n), \tag{56}$$

provided that $n^2 P(N \ge n) \to 0$ as $n \to \infty$. Thus from (55),

$$\begin{aligned} E(\widehat{N}^{(\delta)}_{>n_c}(L)^2) &\le n_c^2 (1-q)^{n_c} + \sum_{n=n_c+1}^{\infty} (2n-1)(1-q)^n \\ &\le \left(n_c^2 + \frac{2n_c}{q} + \frac{2}{q^2}\right)(1-q)^{n_c} \\ &\le 2\left(n_c + \frac{1}{q}\right)^2 e^{-qn_c}. \end{aligned} \tag{57}$$

If $\delta > 0$ then it follows from (13) and (33) that for $L\delta^2$ sufficiently small there is an $A > 0$ such that

$$q \geq \frac{A}{\sqrt{L}}. \tag{58}$$

Moreover, (15) implies the same conclusion, for any L, when $\delta = 0$. If now for any $k > 0$ we set $n_c = \left\lfloor k\sqrt{L} \right\rfloor$ then (57) and (58) yield

$$E(\widehat{N}^{(\delta)}_{>k\sqrt{L}}(L)^2) = E(\widehat{N}^{(\delta)}_{>n_c}(L)^2) \leq \left(k + \frac{1}{A}\right)^2 e^A e^{-Ak}\, L, \tag{59}$$

and (48) will hold for sufficiently large k.

Step 4: Equation (30), the starting point for our future investigations, involves $p_\delta(l) = P(F^{(\delta)} = l)$, the probability under ν_ρ that the first renewal event to the right of the origin occurs at site $l > 0$. This happens precisely when there is a renewal event at some site $-l' \leq 0$, an event with probability 2δ (see Remark 3), and the next renewal event to its right is at l, so that

$$p_\delta(l) = 2\delta \sum_{l' \geq 0} P(\widehat{Y}^{(\delta)} = l' + l) = 2\delta P(\widehat{Y}^{(\delta)} \geq l). \tag{60}$$

(Note that $p_\delta(l) = 2\delta q_\delta(l-1)$.) Now (39) yields

$$p_\delta(l) \simeq 2\delta P(\widehat{Y}^{(0)} \geq l) \quad \text{for } l \ll \delta^{-2}, \tag{61}$$

and thus from (30) we have that

$$E\big(N^{(\delta)}_{\text{ren}}(L)^2\big) \simeq 2\delta \sum_{l=1}^{L} P(\widehat{Y}^{(0)} \geq l) E\big((\widehat{N}^{(0)}_{\text{ren}}(L-l)+1)^2\big) \quad \text{for } L \ll \delta^{-2}. \tag{62}$$

Step 5: In this step we obtain the large-L asymptotics of $E\big(\widehat{N}^{(0)}_{\text{ren}}(L)^2\big)$:

$$E\big(\widehat{N}^{(0)}_{\text{ren}}(L)^2\big) \simeq L, \quad \text{for } L \gg 1. \tag{63}$$

This formula may be obtained from [9], but we give a self-contained proof, arguing from the detailed form of the distribution of the renewal random variable $\widehat{Y}^{(0)} = 2\widehat{X}^{(0)} + 1$. Recall (see (11)) that $P(\widehat{X}^{(0)} = n) = C_n 2^{-(2n+1)}$; the Catalan number C_n counts the number of paths between time 0 and time $2n$ of a random walk which starts and ends at the origin while never taking any positive value. Thus $\widehat{Y}^{(0)}$ has the same distribution as the time of first arrival at site 1 of a simple symmetric random walk W_l, $l = 0, 1, \ldots$, which starts at the origin. As a consequence, $\widehat{N}^{(0)}_{\rm ren}(L)$ has the same distribution as the maximum value $M(L)$ of W_l over the interval $[0, L]$. From [6], Section III.7, Theorem 1 we then have

$$P(\widehat{N}^{(0)}_{\rm ren}(L) = n) = \begin{cases} P(W_L = n), & \text{if } L - n \text{ is even,} \\ P(W_L = n+1), & \text{if } L - n \text{ is odd.} \end{cases} \tag{64}$$

An easy calculation from (64) gives, for L odd,

$$E\big(\widehat{N}^{(0)}_{\rm ren}(L)^2\big) = E\big(W_L^2\big) - E\big(|W_L|\big) + \frac{1}{2}\big(1 - P(W_L = 0)\big), \tag{65}$$

and this yields (63), since $E\big(W_L^2\big) = L$ and $E\big(|W_L|\big) \le \sqrt{L}$, by the Cauchy-Schwarz inequality.

Remark 4. (a) From (64) one can show easily that $\widehat{N}^{(0)}_{\rm ren}(L)/\sqrt{L}$ converges in distribution, as $L \to \infty$, to $|Z|$, with Z a standard normal random variable.
(b) In [11, 22] a random walk representation of particle configurations (there called a height function or height process) was used to obtain the distribution (11). The Catalan numbers play the same role in this derivation that they do above.

Step 6: Equation (62) provides the leading order small-δ approximation to $E\big(N^{(\delta)}_{\rm ren}(L)^2\big)$, valid for $L \ll \delta^{-2}$. In this step we use (63) and (66) below to approximate this moment to leading order in L, as well.

First, note that from (13) we have at once that

$$P\big(\widehat{Y}^{(0)} \ge l\big) = P\Big(\widehat{X}^{(0)} \ge \frac{l-1}{2}\Big) \simeq \sqrt{\frac{2}{\pi l}} \qquad \text{for } l \gg 1. \tag{66}$$

Substituting (63) and (66) into (62) yields, at least formally,

$$E\big(N^{(\delta)}_{\rm ren}(L)^2\big) \simeq 2\sqrt{\frac{2}{\pi}}\,\delta \sum_{l=1}^{L} \frac{L-l}{\sqrt{l}}, \quad \text{for } 1 \ll L \ll \delta^{-2}. \tag{67}$$

We will justify (67) shortly, but for the moment only note that the restriction $L \gg 1$, not present in (62), arises from (63) and (66). From (67) we have that for $1 \ll L \ll \delta^{-2}$,

$$E\big(N_{\mathrm{ren}}^{(\delta)}(L)^2\big) \simeq 2\sqrt{\frac{2}{\pi}}\delta \int_1^L \frac{L-x}{\sqrt{x}}\,dx$$
$$= 2\sqrt{\frac{2}{\pi}}\delta L^{3/2}\int_{1/L}^1 \frac{1-y}{\sqrt{y}}\,dy \simeq \frac{8}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2}. \tag{68}$$

This is our final approximation for $E\big(N_{\mathrm{ren}}^{(\delta)}(L)^2\big)$.

We now return to (67). We are justified (when l is not too large) in replacing $E\big((\widehat{N}_{\mathrm{ren}}^{(0)}(L-l)+1)^2\big)$ by $E\big((\widehat{N}_{\mathrm{ren}}^{(0)}(L-l))^2\big)$ in passing from (62) to (67) since, from (63) and the Cauchy-Schwarz inequality, $E\big(\widehat{N}_{\mathrm{ren}}^{(0)}(L)\big) \ll E\big(\widehat{N}_{\mathrm{ren}}^{(0)}(L)^2\big)$ for $L \gg 1$. Moreover, the substitutions in (62) suggested by (63) and (66) are valid provided that l and $L-l$ are sufficiently large. More precisely, using (63) and (66), we can conclude that there exists an integer l_*, which does not depend on L, such that

$$2\sqrt{\frac{2}{\pi}}\delta\sum_{l=l_*}^{L-l_*}\frac{L-l}{\sqrt{l}} \simeq 2\delta\sum_{l=l_*}^{L-l_*} P(\widehat{Y}^{(0)} \geq l)E\big((\widehat{N}_{\mathrm{ren}}^{(0)}(L-l)+1)^2\big). \tag{69}$$

But note that the sums in (62) and (67) over $1 \leq l < l_*$ and $L-l_* < l \leq L$ are $O(L)$ and $o(1)$, respectively, as $L \to \infty$, while the sums over $l_* \leq l \leq L-l_*$ is of order $L^{3/2}$ (see (68)). This completes the justification.

Step 7: From (17) and (68),

$$\frac{E\big(N_{\mathrm{ren}}^{(\delta)}(L)\big)^2}{E\big(N_{\mathrm{ren}}^{(\delta)}(L)^2\big)} \simeq \frac{3}{2}\sqrt{\frac{\pi}{2}}\delta L^{1/2} \ll 1 \quad \text{for } 1 \ll L \ll \delta^{-2}. \tag{70}$$

Thus, again from (68),

$$\mathrm{Var}\big(N_{\mathrm{ren}}^{(\delta)}(L)\big) \simeq E\big(N_{\mathrm{ren}}^{(\delta)}(L)^2\big) \simeq \frac{8}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2}, \quad \text{for } 1 \ll L \ll \delta^{-2}. \tag{71}$$

Step 8: We can now combine the results of Step 1 and Step 7 to obtain the parts of Theorem 1 which concern $L \ll \delta^{-2}$. For from (25) and (71) we have that

$$\mathrm{Var}(N^{(\delta)}(L)) \simeq \frac{2}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2} + \frac{1}{4}P(\sigma^{(\delta)}(L) \neq 0) \quad \text{for } 1 \ll L \ll \delta^{-2}, \tag{72}$$

with $\sigma^{(\delta)}(L)$ defined in (19).

If L is even then, from (17),

$$\begin{aligned} P(\sigma^{(\delta)}(L) \neq 0) &= P(N_{\text{ren}}^{(\delta)}(L) \text{ is odd}) \\ &\leq P(N_{\text{ren}}^{(\delta)}(L) > 0) \leq E(N_{\text{ren}}^{(\delta)}(L))) = 2\delta L. \end{aligned} \tag{73}$$

Since $L \ll L^{3/2}$ for $L \gg 1$, the $L \ll \delta^{-2}$ part of Theorem 1(b) follows.

On the other hand, if L is odd then, again from (17),

$$\begin{aligned} P(\sigma^{(\delta)}(L) \neq 0) &= P(N_{\text{ren}}^{(\delta)}(L) \text{ is even}) \\ &\geq P(N_{\text{ren}}^{(\delta)}(L) = 0) \geq 1 - 2\delta L \simeq 1 \quad \text{for } L \ll \delta^{-1}. \end{aligned} \tag{74}$$

Since $\delta L^{3/2} \gg 1$ for $L \gg \delta^{-2/3}$ and $\delta L^{3/2} \ll 1$ for $L \ll \delta^{-2/3}$, we obtain from (74) and (72) the conclusions of Theorem 1(a) for $1 \ll L \ll \delta^{-2}$. To remove the restriction that $L \gg 1$ we note that

$$\mathrm{Var}\big(N_{\text{ren}}^{(\delta)}(L)\big) \leq E(N_{\text{ren}}^{(\delta)}(L)^2) \leq E(N_{\text{ren}}^{(\delta)}(L')^2) \tag{75}$$

for $L \leq L'$. Choosing L' such that also $1 \ll L' \ll \delta^{-2/3}$, we see using (25), (68), and (74) that $\mathrm{Var}(N^{(\delta)}(L)) \simeq \frac{1}{4}$ for $1 \leq L \ll \delta^{-2/3}$.

Remark 5. Concerning the estimate $P(N_{\text{ren}}^{(\delta)}(L) > 0) \leq 2\delta L$ used in (73) and (74), note that it follows from (26), (61), and (66) that in fact for $L \gg 1$,

$$P(N_{\text{ren}}^{(\delta)}(L) > 0) = P(F^{(\delta)} \leq L) \simeq 2\delta \int_0^L \sqrt{\frac{2}{\pi l}}\, dl \simeq 4\sqrt{\frac{2}{\pi}}\delta\sqrt{L}. \tag{76}$$

Step 9: We now turn to the $L \gg \delta^{-2}$ part of Theorem 1 and to the related Corollary 2 (recall also Remark 1). We write

$$\begin{aligned} \mathrm{Var}(N^{(\delta)}(L)) &= \sum_{1 \leq k \leq L} \mathrm{Var}_{\nu_\rho}(\eta_k) + 2 \sum_{1 \leq k < i \leq L} \mathrm{Cov}_{\nu_\rho}(\eta_k, \eta_i) \\ &= \rho(1-\rho)L + 2 \sum_{k=1}^{L-1} \sum_{j=1}^{k} g_\rho^T(j), \end{aligned} \tag{77}$$

with g_ρ^T the truncated two-point correlation function for the TI state ν_ρ:

$$g_\rho^T(k) = E(\eta_j \eta_{j+k}) - \rho^2. \tag{78}$$

It is shown in [11] that for all $n \geq 0$,

$$g_\rho^T(2n+1) + g_\rho^T(2n+2) = 0, \tag{79}$$

so that (77) becomes

$$\mathrm{Var}(N^{(\delta)}(L)) = \rho(1-\rho)L + 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{L-1} g_\rho^T(k). \tag{80}$$

We will show shortly that for $j \geq 0$,

$$\left| g_\rho^T(2j+1) \right| \leq \rho^2 (1-4\delta^2)^j. \tag{81}$$

Then from (80),

$$\left| \frac{\mathrm{Var}(N^{(\delta)}(L))}{\rho(1-\rho)L} - 1 \right| \leq \frac{2\rho}{(1-\rho)L} \sum_{j=0}^{\infty} (1-4\delta^2)^j < \frac{1}{2\delta^2 L}, \tag{82}$$

and this verifies the result stated in Corollary 2. The $L \gg \delta^{-2}$ cases of Theorem 1 then follow by taking the δ_0 of Definition 1 sufficiently small.

Consider now (81). The generating function of the g_ρ^T is computed in [11]:

$$G_\rho(z) := \sum_{k=1}^{\infty} g_\rho^T(k) z^k = \frac{z\left(\sqrt{1-z^2(1-4\delta^2)} - 2\delta\right)^2}{4(z-1)(z+1)^2}. \tag{83}$$

The numerator in (83) has double zeros (on its first sheet) at $z = \pm 1$, so that G_ρ is analytic at these points with singularities at $z = \pm z_*$, where $z_* := (1-4\delta^2)^{-1/2}$. Expressing $g_\rho^T(k)$ via Cauchy's formula as an integral over a small circle around the origin, distorting this contour to obtain the sum of integrals of the discontinuity of G_ρ across cuts on the real axis from z_* to ∞ and from $-z_*$ to $-\infty$, and making the change of variable $z \to -z$ in the second of these integrals, we obtain a representation of $g_\rho^T(k)$ which for k odd is

$$g_\rho^T(k) = -\frac{2\delta}{\pi z_*} \int_{z_*}^{\infty} \frac{\sqrt{z^2 - z_*^2}}{z^k (z^2-1)^2} \, dz, \qquad k \text{ odd}. \tag{84}$$

This implies that for k odd, $|g_\rho^T(k)| \leq z_*^{-(k-1)} |g_\rho^T(1)|$, and since $g_\rho^T(1) = -\rho^2$, (81) follows.

This completes the proof of Theorem 1.

5 Concluding remarks

We note that on $\mathbb{Z}$ the approach to hyperuniformity as $\rho \searrow \rho_c$ is different from that for $\rho \nearrow \rho_c$; in the former case the unique stationary measure for the FEP on $\mathbb{Z}$ is known and [13, 15]

$$\lim_{L\to\infty} \frac{1}{L} V_\rho(L) = \rho(1-\rho)(2\rho-1) \quad \text{for } \rho > 1/2. \tag{85}$$

Thus from Theorem 1 and (8), $\lim_{L\to\infty} L^{-1}V_\rho(L)$ is continuous in ρ from above, but not from below, at $\rho = 1/2$. We expect similar behavior on $\mathbb{Z}^d$.

The case of the FEP on a *ladder*, a system consisting of two (infinite) rows of sites, was studied numerically in [17, 18]. Here again, as on $\mathbb{Z}^d$ with $d \geq 2$, $\rho_c < 1/2$ (for the continuous-time symmetric FEP on the ladder, $\rho_c \approx 0.4755$ or, for the slightly different dynamics of [16], $\rho_c \approx 0.4874$ [18]). The results of [18] also suggest that the Fluctuation Conjecture holds for this model (although the scaling behavior of the $L_i(\delta)$ is not discussed). Rigorously, one may observe that for $\rho < \rho_c$ the portions of the system to the left and right of an empty square are independent under the $t \to \infty$ limiting measure ν_ρ, implying in particular that the locations of the empty squares (when these have a nonzero density) form a renewal process and that the portions of the system between them are jointly independent. Further, we see that the critical density must satisfy $\rho_c \geq 1/4$, since at smaller densities there would always be a finite density of empty squares and the stationary state would be frozen [18]. We have no such lower bound for ρ_c on $\mathbb{Z}^d$, $d \geq 2$.

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References

[1] A. Ayyer, S. Goldstein, J. L. Lebowitz, and E. R. Speer, Stationary States of the One-dimensional Facilitated Asymmetric Exclusion Process. *Ann. I. H. Poincaré-Pr.* **59**, 728–742 (2023).

[2] J. Baik, G. Barraquand, I. Corwin, and T. Suidan, Facilitated Exclusion Process. *Computation and Combinatorics in Dynamics, Stochastics and Control*, 1–35, Abel Symp. **13**, Springer, Cham, 2018.

[3] O. Blondel, C. Erignoux, M. Sasada, and M. Simon, Hydrodynamic Limit for a Facilitated Exclusion Process. *Ann. I. H. Poincaré-Pr.* **56**, 667714 (2020).

[4] O. Blondel, C. Erignoux, and M. Simon, Stefan Problem for a Non-Ergodic Facilitated Exclusion Process. *Prob. Math. Phys.* **2**, 127–178 (2021).

[5] U. Basu and P. K. Mohanty, Active-Absorbing-State Phase Transition Beyond Directed Percolation: A Class of Exactly Solvable Models. *Phys. Rev. E* **79**, 041143 (2009).

[6] W. Feller, *An Introduction to Probability Theory and its Applications, Volume I*, Third edition. John Wiley & Sons, New York, 1968.

[7] A. Gabel, P. L. Krapivsky, and S. Redner, Facilitated Asymmetric Exclusion. *Phys. Rev. Lett.* **105**, 210603 (2010).

[8] A. Gabel and S. Redner, Cooperativity-Driven Singularities in Asymmetric Exclusion, *J. Stat. Mech.* **2011**, P06008 (2011).

[9] C. Godrèche and J. M. Luck, Statistics of Occupation Time of Renewal Processes. *J. Stat. Phys.* **104**, 489–524 (2001).

[10] S. Goldstein, J. L. Lebowitz and E. R. Speer, Exact Solution of the F-TASEP . *J. Stat. Mech.* **2019** 123202 (2019).

[11] S. Goldstein, J. L. Lebowitz and E. R. Speer, The Discrete-Time Facilitated Totally Asymmetric Simple Exclusion Process. *Pure Appl. Funct. Anal.* **6**, 177–203 (2021).

[12] S. Goldstein, J. L. Lebowitz and E. R. Speer, Stationary States of the One-Dimensional Discrete-Time Facilitated Symmetric Exclusion Process. *J. Math. Phys.* **63**, 083301 (2022).

[13] S. Goldstein, J. L. Lebowitz and E. R. Speer, Approach to Hyperuniformity of Steady States of Facilitated Exclusion Processes. *J. Phys.: Condens. Matter* **36**, 345402 (2024).

[14] D. Hexner and D. Levine, Hyperuniformity of Critical Absorbing States. *Phys. Rev. Lett.* **114**, 110602 (2015).

[15] T. D. Lee and C. N. Yang, Statistical Theory of Equations of State and Phase Relations II: Lattice Gas and Ising Model. *Phys. Rev.* **87**, 410–419 (1952).

[16] S. Lübeck, Scaling Behavior of the Absorbing Phase Transition in a Conserved Lattice Gas Around the Upper Critical Dimension. *Phys. Rev. E* **64**, 016123 (2001).

[17] M. J. Oliveira, Conserved Lattice Gas Model with Infinitely Many Absorbing States in One Dimension. *Phys. Rev. E* **71**, 016112 (2005).

[18] C. Ramirez-Ibanez, *Facilitated Exclusion Processes on a Ladder*. Doctoral dissertation, Rutgers (2023).

[19] M. Rossi, R. Pastor-Satorras, and A. Vespignani, Universality Class of Absorbing Phase Transitions with a Conserved Field. *Phys. Rev. Lett.* **85**, 1803 (2000).

[20] R. P. Stanley, *Catalan Numbers.* Cambridge University Press, Cambridge, 2015.

[21] S. Torquato, Hyperuniform States of Matter. *Phys. Rep.* **745**, 1–95 (2018).

[22] L. Zhao and D. Chen, The Invariant Measures and the Limiting Behaviors of the Facilitated TASEP. *Stat. Probabil. Lett.* **154**, 108557 (2019).