II. Martingales and risk-neutral measures.

A. Martingales: Definition

Martingales are models of gambling games that are fair to risk-neutral players.

Let $F_0 \leq F_1 \leq F_2 \leq F_3 \leq \ldots$ be a filtration; $F_k$ represents the information available to the player after the $k^{th}$ play. Let $X_1, X_2, \ldots$ represent the player's fortune; $X_k$ is what the player has after the $k^{th}$ play.

The conditional expectation

$$E[X_{k+1} | F_k]$$

is what the player expects to have after the next play given his position and knowledge at the end of play $k$.

He considers the game fair if

$$E[X_{k+1} | F_k] = X_k.$$  

This is the martingale property.

Definition \{X_n\}_n is a martingale w.r.t. the filtration \{F_n\}_n if:

(i) $X_n$ is $F_n$-measurable for all $n$;

(ii) $E|X_n| < \infty$ for all $n$;

(iii) $E[X_{n+1} | F_n] = X_n$ for all $n$.  \(12\)

(Actually (iii) implies $X_n$ is $F_n$-meas. but we include this as condition (ii) for emphasis)
B. Martingales: Examples

Example 1. Let \( \xi_1, \xi_2, \ldots \) be independent and identically distributed with \( P(\xi_i = 1) = \frac{1}{2} \), \( P(\xi_i = -1) = \frac{1}{2} \).

Let \( W_n = \sum_{i=1}^{n} \xi_i \) and let \( \mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n) \). Let \( W_0 = 0 \) \( \mathcal{F}_0 = \{\emptyset, \{\Omega\}\} \). Observe that \( E[\xi_i] = 0 \) for all \( i \).

Then \( \{W_n\} \) is a martingale w.r.t. \( \{\mathcal{F}_n\}_{n \geq 0} \) because \( W_n \) is \( \mathcal{F}_n \)-measurable for every \( n \), and
\[
E[ W_{n+1} | \mathcal{F}_n ] = E[ W_n + \xi_{n+1} | \mathcal{F}_n ] \\
= E[ W_n | \mathcal{F}_n ] + E[\xi_{n+1} | \mathcal{F}_n ] \\
= W_n + E[\xi_{n+1}] \\
= W_n , \quad (\text{since } E[\xi_{n+1}] = 0)
\]

(Think of a game in which, on every toss you lose a dollar with probability \( 1/2 \) and win a dollar with probability \( 1/2 \); \( W_n \) is then your total winnings after \( n \) plays.) This example is also called symmetric random walk.

Example 2. Betting on a martingale.

Let \( \{W_n\} \) be the martingale of Example 1. Suppose now that you can bet any amount \( \Delta_{n-1} \) on play \( n \) (\( \Delta_{n-1} \) is the amount you bet after completing play \( n-1 \)) with the restrictions that for every \( n \geq 1 \) \( \Delta_{n-1} \) must be \( \mathcal{F}_{n-1} \)-measurable (so that you can only use the information known up to time \( n-1 \) in deciding the amount \( \Delta_{n-1} \)) and that \( E[|\Delta_{n}|] < \infty \) for all \( n \geq 0 \).

The amount you win on play \( n \) is therefore \( \Delta_{n-1} \xi_n = \Delta_{n-1}(W_n - W_{n-1}) \).

Your total winnings after \( n \) plays is
\[
X_n = \sum_{k=1}^{n} \Delta_{k-1} \xi_k = \sum_{k=1}^{n} \Delta_{k-1} (W_k - W_{k-1})
\]
Set \( X_0 = 0 \).

Then \( \{X_n\}_{n \geq 0} \) is a martingale w.r.t. \( \{\mathcal{F}_n\}_{n \geq 0} \).

Since
\[
X_n = \sum_{k=1}^{n} \Delta_k \xi_k
\]
contains only terms that depend on \( \xi_1, \ldots, \xi_n \), it is \( \mathcal{F}_n \)-measurable. Also,
\[
E \left[ X_{n+1} \mid \mathcal{F}_n \right] = E \left[ X_n + \Delta_n \xi_{n+1} \mid \mathcal{F}_n \right]
\]
\[
= E[ X_n \mid \mathcal{F}_n ] + E[ \Delta_n \xi_{n+1} \mid \mathcal{F}_n ]
\]
\[
= X_n + \Delta_n E[ \xi_{n+1} \mid \mathcal{F}_n ] \quad \text{(Thm 2.3.2)}
\]
\[
= X_n + \Delta_n E[ \xi_{n+1} ] \quad \text{(since \( \xi_{n+1} \) is independent of \( \mathcal{F}_n \))}
\]
\[
= X_n,
\]
which confirms the martingale property.

Example 3 Claim 2 above, formula (5) with \( f = \theta_{k+1} \), shows that using the measure \( \bar{\mathbb{P}} \),
\[
\left\{ \frac{1}{(1+r)^{k}} S(t_k) \right\}
\]
is a martingale w.r.t. \( \{\mathcal{F}(t_k)\} \).

To emphasize that the probability measure \( \bar{\mathbb{P}} \) with \( \bar{\rho} \) and \( \bar{\theta} \) defined by
\[
\bar{\rho} = \frac{1+r-h-d}{u-d} \quad \bar{\theta} = \frac{u-(1+r)h}{u-d}
\]
is being used, we will say \( \left\{ \frac{1}{(1+r)^{k}} S(t_k) \right\} \) (the discounted price process) is a martingale with respect to \( \{\mathcal{F}(t_k)\} \) and \( \bar{\mathbb{P}} \).
Example 4. Consider again the arbitrage free multi-period model. Let \( \{C(t_n, S(t_n))\} \) be the price process of a contingent claim with payoff \( C(T, S(T)) \) at \( T=t_n \). It was shown in the proof of Theorem 2 that
\[
C(t_n, S(t_n)) = \frac{1}{1+r} E \left[ C(t_{n+1}, S(t_{n+1})) | \mathcal{F}(t_n) \right]
\]

It follows that
\[
\left\{ \frac{1}{(1+r)^k} C(t_k, S(t_k)) \right\}_{k \geq 0}
\]

is a martingale w.r.t \( \mathcal{F}(t_n) \) for all \( k \geq 0 \) and \( P \).

C. Simple properties of martingales.

Let \( \{X_n\}_{n \geq 0} \) be a random process defined on \((\Omega, \mathcal{F}, P)\). Suppose \( \{X_n\}_{n \geq 0} \) is a martingale w.r.t \( \{\mathcal{F}_n\}_{n \geq 0} \). Then
\[
E[X_n] = E[X_0] \quad \text{for all } n. \quad (13)
\]
whenever \( m \geq n \)
\[
E[X_m | \mathcal{F}_n] = X_n. \quad (14)
\]

(13) is true because
\[
E[X_n] = E[E[X_n | \mathcal{F}_{n-1}]] = E[X_{n-1}] \quad \text{for all } n \geq 1.
\]

(14) is true by iterated conditioning: if \( m > n+1 \)
\[
E[X_m | \mathcal{F}_n] = E[E[X_m | \mathcal{F}_{m-1}] | \mathcal{F}_n] = E[X_{m-1} | \mathcal{F}_n]
\]
\[
= E[E[X_{m-1} | \mathcal{F}_{m-2}] | \mathcal{F}_n] = E[X_{m-2} | \mathcal{F}_n]
\]
\[
\vdots
\]
\[
= E[X_{n+1} | \mathcal{F}_n] = X_n.
\]

continue in this fashion.
The converse to (14) is also true. Let \( \{ \mathcal{F}_n \}_{n \geq 0} \) be a filtration. Fix \( N > 1 \) and suppose \( Z \) is \( \mathcal{F}_N \)-measurable. Define

\[
X_n = E[Z | \mathcal{F}_n] \quad n = 0, 1, \ldots, N
\]

Then \( \{X_n\} \) is a martingale w.r.t \( \{ \mathcal{F}_n \}_{n \geq 0} \) and \( X_N = Z \). Indeed, since \( Z \) is \( \mathcal{F}_N \)-measurable, \( X_N = E[Z | \mathcal{F}_N] = Z \). and if \( 0 \leq k < N \)

\[
E[X_{k+1} | \mathcal{F}_k] = E[E[Z | \mathcal{F}_{k+1}] | \mathcal{F}_k] \\
= E[Z | \mathcal{F}_k] \quad \text{(iterated conditioning)} \\
= X_k \quad \text{(by definition)}.
\]

**Example 5.** Consider an arbitrage free multi-period binomial market model. We saw in problem 7 of Assignment 4 that a contingent claim that pays \( H(\omega_1, \ldots, \omega_n) \) at \( T = t_n = nh \) has price

\[
V(t_n) = \frac{1}{(1 + rh)^{n-k}} E\left[ H | \mathcal{F}(t_k) \right]
\]

Multiplying by \( \frac{1}{(1 + rh)^k} \)

\[
\frac{V(t_k)}{(1 + rh)^k} = E\left[ \frac{1}{(1 + rh)^n} H | \mathcal{F}(t_k) \right]
\]

Thus \( \left\{ \frac{V(t_k)}{(1 + rh)^k} \right\}_{k \geq 0} \) is a martingale w.r.t \( \{ \mathcal{F}(t_k) \}_{k \geq 0} \) and \( \mathbb{P} \). This result generalizes the result of Example 4 on the previous page.
III Risk Neutral Measures for multi-period models.

Look at the results of Examples 3, 4, and 5. They all say that the discounted price process, whether it be for the underlying price or for a contingent claim, is a martingale under the probability measure \( \tilde{P} \).

Notice also that the value \( B(t, k) \) of a unit of money market account is \( B(t, k) = (1 + rh)^k \), and if this is discounted

\[
\frac{B(t, k)}{(1 + rh)^k} = 1 \quad \text{for all } k
\]

A process which is just constant is trivially a martingale.

Definition A general multi-period market model (not necessarily binomial) is defined by a risk-free rate \( r \), period \( h \) and risky asset processes \( \{ S_1(t, k) \}_{0 \leq k \leq n}, \ldots, \{ S_p(t, k) \}_{0 \leq k \leq n} \) on \((\mathcal{F}, \mathbb{F})\).

Let

\[
\mathcal{F}(t, k) = \sigma (S_1(t, j), \ldots, S_p(t, j), 0 \leq j \leq k)
\]

A probability measure \( \tilde{P} \) on \((\mathcal{F}, \mathbb{F})\) is said to be risk-neutral if

\[
\left\{ \frac{1}{(1 + rh)^k} S_i(t, k) \right\}_{k \geq 0}
\]

is a martingale w.r.t. \( \{ \mathcal{F}(t, k) \} \) and \( \tilde{P} \)

for each \( i, 1 \leq i \leq p \).
Example 6. The measure \( \tilde{P} \) defined on the multi-asset, binomial model when \( 0 < 1 + r < u \) is risk-neutral. This follows from Example 3.

Theorem 3. Assume \( \Omega \) is finite. A multi-period market is arbitrage free if and only if there exists a risk-neutral probability measure \( \tilde{P} \) for the market.

In this case, any attainable contingent claim has a unique price process \( \{V(t_k)\} \) given by

\[
V(t_k) = \frac{1}{(1 + r)^{n-k}} \tilde{E}[H | \mathcal{F}(t_k)]
\]

(15)

where \( H \) is the payoff of the claim at \( T = t_n \).

The problem with this statement is that we have not yet given a definition of "attainable" for a contingent claim in a multi-period model --- we will come to this. Meanwhile, note that (15) looks exactly like the formula in Example 5, but the setting is now more general -- the model is not necessarily binomial and there may be many risky assets.

Theorem 3 is the first fundamental theorem of asset pricing for a discrete time-discrete outcome multi-period model.