Thus the quadratic variation of $W$ on $[0,T]$ is

$$[W, W] = \begin{bmatrix} [W, W] \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} [W, W] \end{bmatrix}$$

Therefore,

$$[W, W] = \begin{bmatrix} [W, W] \end{bmatrix}$$

and let $x \in \mathbb{R}$.

Also, since $W^{-1}$ and $W^{-2}$ are martingales, we have

$$\int_0^t e^{-s} \, dW_s = \begin{bmatrix} e^{-s} \end{bmatrix} \begin{bmatrix} W_s \end{bmatrix}$$

for $s \leq t$. For $x > 0$, let $W$ be a Brownian motion. Let $T > 0$. For a

$\varepsilon > 0$, we have

$$E \left[ e^{-\varepsilon W_T} - e^{-\varepsilon X_T} \right] = 0$$

where $X_t = a t + b W_t$ for $t \leq T$. If $a > 0$, then

$$E \left[ e^{-\varepsilon X_T} \right] = e^{-\varepsilon a T}$$

and

$$E \left[ e^{-\varepsilon W_T} \right] = e^{-\varepsilon b T}$$

Thus, $X_T$ and $W_T$ are independent. This is Exercise 3.3 in Shreve and is Exercise 3.4. (a) Quotable Variation of Brownian Motion

NOTES FOR LECTURE 8

8.4.4.2. Martingale Primes
Theorem: The integral of a function f(x) on an interval [a, b] is defined as the limit of a sum of areas of rectangles as the number of rectangles approaches infinity. The area under the curve of f(x) from a to b is given by:

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx
\]

Proof: Let f(x) be a non-negative, integrable function on [a, b]. Divide the interval [a, b] into n subintervals of equal length \( \Delta x = \frac{b-a}{n} \). For each subinterval, choose a point \( x_i \) and construct a rectangle with base \( \Delta x \) and height \( f(x_i) \). The area of each rectangle is given by \( f(x_i) \Delta x \).

The sum of these areas is:

\[
\sum_{i=1}^{n} f(x_i) \Delta x
\]

As \( n \to \infty \), \( \Delta x \to 0 \), and the sum approaches the area under the curve, which is the definite integral:

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx
\]

This is the definition of the definite integral of f(x) from a to b.

(1) If f(x) is continuous on [a, b], then f(x) is integrable on [a, b].

(2) If f(x) is integrable on [a, b], then \( \int_{a}^{b} f(x) \, dx \geq 0 \).

(3) If f(x) and g(x) are integrable on [a, b], then \( \int_{a}^{b} [f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \).

(4) If f(x) is integrable on [a, b] and c is a constant, then \( \int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx \).

The Riemann–Stieltjes integral is defined for an integrand that is not continuous.

II

Value of the definite integral:

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

(5) If f(x) is integrable on [a, b] and \( \alpha \) is a constant, then \( \int_{a}^{b} \alpha f(x) \, dx = \alpha \int_{a}^{b} f(x) \, dx \).

(6) If f(x) is integrable on [a, b] and g(x) is a step function, then \( \int_{a}^{b} f(x) g(x) \, dx \) exists.

Putting these results together, we have:

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]
Some applications may help to grasp what is going on here:

\[
\sum_{w=1}^{\infty} \frac{c(w)}{\sin(w)} = \int_{0}^{\infty} f(x) \, dx
\]

This may also be expressed as:

\[
\sum_{w=1}^{\infty} \frac{c(w)}{\sin(w)} = \int_{0}^{\infty} f(x) \, dx
\]

Then, for notational convenience, write:

\[
\int_{\infty}^{0} \frac{c(x) \, dx}{\sin(x)} = \int_{0}^{\infty} f(x) \, dx
\]

In the remaining sum, we have:

\[
\int_{0}^{\infty} f(x) \, dx
\]
The property at the end of Chapter 6 is extended to an n-dimensional case.

\[ \frac{\partial}{\partial t} \int_{t_0}^{t_1} e^{-t} \left( f(x(t), y(t)) \right) dt = f(x(t_1), y(t_1)) \left( 1 - e^{-t_1} \right) - f(x(t_0), y(t_0)) \left( 1 - e^{-t_0} \right) \]

where \( f \) is a function of \( x, y, \) and \( t \).

This property holds for functions \( f \) of several variables.

Example: Assume that \( f \) is continuous and differentiable.

\[ \int_{t_0}^{t_1} e^{-t} f(x(t), y(t)) dt \]

can be evaluated by integrating with respect to \( t \) for a fixed value of \( x \) and \( y \).
The integral of the function \( f(x) \) is given by:

\[ \int_a^b f(x) \, dx \]

The general form of the integral is:

\[ \int \frac{1}{x} \, dx = \ln|x| + C \]

A specific solution is:

\[ \int \frac{1}{x} \, dx = \ln|x| + C \]

The integral of a constant function is:

\[ \int C \, dx = Cx + C \]

For some sequence, \( a_n \to 0 \) as \( n \to \infty \), and constant \( c \):

\[ \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_a^b f(x) \, dx \]

The integral of a function with respect to another function is:

\[ \int f(x) \, g(x) \, dx = \int f(x) \, h(x) \, dx \]

The definite integral is:

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \]

The fundamental theorem of calculus states that:

\[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \]

Example: Assume that \( f \) is continuous on \( [a, b] \).

Theorem: If \( f \) is integrable on \( [a, b] \), then:

\[ \int_a^b f(x) \, dx = \int_a^b f(x) \, dx \]
Thus, according to the previous definition, if $\mathbf{E}(X) = 0$, then

$$\mathbb{E}[X] = \sum_{x \in X} x \mathbb{P}(X = x)$$

is the expected value of the random variable $X$. A random variable $X$ is said to be adapted to the process $\{\mathbf{E}(X)\}_{t \geq 0}$ defined on the same

Definition. A stochastic process $\{\mathbf{E}(X)\}_{t \geq 0}$ is said to be adapted to the process $\{\mathbf{E}(X)\}_{t \geq 0}$ defined on the same

A. An Information Process

If, for any $t > 0$, the amount left of $\mathbf{E}(X)$ is a simple random process, then $\mathbf{E}(X)$ is a Markov process.

In practice, for each $t > 0$, $\mathbf{E}(X)$ is a Markov process, as well as a random variable with the random variable $\mathbf{E}(X)$ at time $t$.

C. A Markov process

If $\mathbf{E}(X)$ is a Markov process, then $\mathbf{E}(X)$ is a Markov process. Recall that, in general, a Markov process $\{\mathbf{E}(X)\}_{t \geq 0}$ is a Markov process, and let $\{\mathbf{E}(X)\}_{t \geq 0}$ be a Markov process.

In the same way, $\mathbf{E}(X)$ is a Markov process. Thus, $\mathbf{E}(X)$ is a Markov process.

Finally, $\mathbf{E}(X)$ is a Markov process.
For each \( t \neq 0 \), there is a continuous function \( W_t \) such that

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-t^2 x^2} \, dx = 1.
\]

Let \( X_t = \frac{1}{\sqrt{t}} W_t \). Then the distribution of \( X_t \) is normal with

\[
\mathbb{E}[X_t] = 0, \quad \mathbb{V}[X_t] = 1.
\]

(1) \( X_t \) is a martingale with respect to the filtration \( \mathcal{F}_s \). Let \( \mathbb{E}[X_t | \mathcal{F}_s] = X_s \). Then

\[
\mathbb{V}[X_t | \mathcal{F}_s] = 0.
\]
Thus, using (10),

\[
\sum_{k=0}^{l} \mathcal{E} = \left[\mathcal{E} \left(2^{k} \text{MP}(x)\right) + \right]
\]

(20)
\[ \begin{align*}
&\left( \left( 1 - \alpha \right) \mathbf{W}_{\text{mn}} - \left( \beta \right) \mathbf{W}_{\text{m}} \right) \mathbf{x} = \mathbf{y} \\
&\left( \left( \alpha \right) \mathbf{W}_{\text{mn}} - \left( 1 - \beta \right) \mathbf{W}_{\text{m}} \right) \mathbf{x} = \mathbf{y}
\end{align*} \]
\[ \left\| \begin{array}{c} \phi \end{array} \right\| = \begin{bmatrix} \phi \end{bmatrix} \begin{bmatrix} W & V \end{bmatrix} \begin{bmatrix} \phi \end{bmatrix} \]

Theorem 1: Let \( V \) be an orthonormal set.

\[ \sum_{i=1}^{\infty} \lambda_i \phi_i \]

Where \( \lambda_i \) are the eigenvalues of \( V \).

\[ \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \phi_i \]

Define \( \Phi \) to be a stochastic process on \( \Omega \).

\[ \left\| \begin{array}{c} \Phi \end{array} \right\| = \begin{bmatrix} \Phi \end{bmatrix} \begin{bmatrix} W & V \end{bmatrix} \begin{bmatrix} \Phi \end{bmatrix} \]

If \( \Phi \) is a stochastic process, then \( \left\| \begin{array}{c} \Phi \end{array} \right\| \)

The reason for the above result is that the definition of \( \Phi \) is that of a vector \( \phi \).

\[ \left\| \begin{array}{c} \phi \end{array} \right\| = \begin{bmatrix} \phi \end{bmatrix} \begin{bmatrix} W & V \end{bmatrix} \begin{bmatrix} \phi \end{bmatrix} \]

\[ \begin{bmatrix} X \end{bmatrix} E = \left\| \begin{array}{c} X \end{array} \right\| \]

If \( X \) is a random variable, define

\[ E \]

A comment on property (ii) of Theorem 2.

Thus:
The quadratic variation \( \mathbb{Q}_X(t) \) is given by:

\[
\mathbb{Q}_X(t) = \int_0^t \langle X, X \rangle_s \, ds
\]

Moreover, \( X \) can be defined so that it is continuous in \( t \).

Theorem 9.3.1: \( \mathbb{Q}_X(t) \) is a martingale under \( \mathbb{Q}_X \).

Moreover, \( X \) can be defined so that it is continuous in \( t \).

Theorem 9.3.2: \( \mathbb{Q}_X(t) \) is a martingale under \( \mathbb{Q}_X \).

Moreover, \( X \) can be defined so that it is continuous in \( t \).

Theorem 9.3.3: \( \mathbb{Q}_X(t) \) is a martingale under \( \mathbb{Q}_X \).

Moreover, \( X \) can be defined so that it is continuous in \( t \).

Theorem 9.3.4: \( \mathbb{Q}_X(t) \) is a martingale under \( \mathbb{Q}_X \).

Moreover, \( X \) can be defined so that it is continuous in \( t \).

Theorem 9.3.5: \( \mathbb{Q}_X(t) \) is a martingale under \( \mathbb{Q}_X \).

Moreover, \( X \) can be defined so that it is continuous in \( t \).

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Theorem 9.3.5: \( \mathbb{Q}_X(t) \) is a martingale under \( \mathbb{Q}_X \).

Moreover, \( X \) can be defined so that it is continuous in \( t \).
\[
\left[\frac{1}{M} - \frac{1}{\lambda^2 M}\right] \left[\frac{1}{M} - \frac{1}{\lambda^2 M}\right] \left[\frac{1}{M} - \frac{1}{\lambda^2 M}\right] = 0
\]

This leads to:

\[
\frac{1}{M} = \frac{1}{\lambda^2 M}
\]

Hence, we have:

\[
M = \lambda^2
\]

This is the solution to the equation. For example, let \( M \) be a function of time, and for any \( t > 0 \):

\[
\int_0^t \frac{dM}{M}\int_0^{1/\lambda^2} \frac{d\lambda}{(\lambda^2 - 1)} = \int_0^t \frac{dM}{M}
\]

Therefore, \( M(t) \) is obtained.

Next, we consider the case where \( \lambda \) is constant.

\[\text{Example:}\]

Consider the equation:

\[
\frac{dM}{dt} = \frac{1}{M} - \frac{1}{\lambda^2 M}
\]

Which we already solve to find the solution (C) (\( t > 0 \)).
This is a continuation of the previous page. The text is a mathematical derivation and explanation.

\[ \frac{d}{dt} \mathbf{M} = \mathbf{F}(t) \]

where \( \mathbf{F}(t) \) represents the external forces acting on the system. The goal is to find the time evolution of the moment tensor \( \mathbf{M} \).

To do this, we start from the governing equation for \( \mathbf{M} \):

\[ \frac{d}{dt} \mathbf{M} = \mathbf{F}(t) \]

We solve this differential equation to find \( \mathbf{M}(t) \).

The solution involves integrating \( \mathbf{F}(t) \) over time.

\[ \mathbf{M}(t) = \int_0^t \mathbf{F}(\tau) d\tau + \mathbf{M}(0) \]

where \( \mathbf{M}(0) \) is the initial moment tensor.

The next step is to apply boundary conditions and initial conditions to determine the specific solution.

**The result is**

\[ \mathbf{M}(t) \]