Stability of periodic solutions of state-dependent delay-differential equations

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**Abstract**

We consider a class of autonomous delay-differential equations

\[ \dot{z}(t) = f(z_t) \]

which includes equations of the form

\[ \dot{z}(t) = g(z(t), z(t-r_1), \ldots, z(t-r_n)), \]
\[ r_i = r_i(z(t)) \quad \text{for} \quad 1 \leq i \leq n, \quad (*) \]

with state-dependent delays \( r_i(z(t)) \geq 0 \). The functions \( g \) and \( r_i \) satisfy appropriate smoothness conditions.

We assume there exists a periodic solution \( z = x(t) \) which is linearly asymptotically stable, namely with all nontrivial characteristic multipliers \( \mu \) satisfying \( |\mu| < 1 \). We prove that the appropriate nonlinear stability properties hold for \( x(t) \), namely, that this solution is asymptotically orbitally stable with asymptotic phase, and enjoys an exponential rate of attraction given in terms of the leading nontrivial characteristic multiplier.

A principal difficulty which distinguishes the analysis of equations such as (\( \ast \)) from ones with constant delays, is that even with \( g \) and \( r_i \) smooth, the associated function \( f \) is not smooth in function space. Techniques of Hartung, Krisztin, Walther, and Wu are employed to resolve these issues.

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1. Introduction

In this paper we study stability questions for a broad class of autonomous state-dependent delay-differential equations. Specifically, we prove that linearized asymptotic stability of a periodic solution \( x(t) \) implies nonlinear (Lyapunov) stability of that solution, in fact, asymptotic orbital stability with asymptotic phase, and exponential attraction at a rate determined by the leading nontrivial characteristic multiplier. This is, of course, the analog of a classic theorem in ordinary differential equations; see, for example, [1]. The corresponding result for retarded equations with constant delay also has been known for many years; see [7]. Among the equations we treat are those with pointwise state-dependent delays such as

\[
\dot{z}(t) = g(z(t), z(t - r_1), \ldots, z(t - r_n)), \quad r_i = r_i(z(t)) \quad \text{for } 1 \leq i \leq n,
\]

where

\[
g : U_g \subseteq \mathbb{R}^{m(n+1)} \to \mathbb{R}^m, \quad r_i : U_{r_i} \subseteq \mathbb{R}^m \to [0, R] \quad \text{for } 1 \leq i \leq n,
\]

for some (typically open) sets \( U_g \) and \( U_{r_i} \). In the case \( n = 1 \) this equation takes the form

\[
\dot{z}(t) = g(z(t), z(t - r)), \quad r = r(z(t)),
\]

where

\[
g : U_g \subseteq \mathbb{R}^{2m} \to \mathbb{R}^m, \quad r : U_r \subseteq \mathbb{R}^m \to [0, R].
\]

The model equation

\[
\varepsilon \dot{z}(t) = -z(t) - kz(t - r), \quad r = r(z(t)) = 1 + z(t)
\]

with \( \varepsilon > 0 \) and \( k > 1 \), considered in [5] (see also [4]), is a special case. Generally, we follow the setting of Walther [8] for state-dependent equations (see also Hartung, Krisztin, Walther, and Wu [3]), which we now outline. Consider an autonomous equation

\[
\dot{z}(t) = f(z_t)
\]

where

\[
f : U_X \subseteq X \to \mathbb{R}^m \quad \text{is continuous,} \quad X = C([-R, 0], \mathbb{R}^m),
\]

\[
z_t \in X \quad \text{is given by } z_t(\theta) = z(t + \theta) \quad \text{for } \theta \in [-R, 0],
\]

and where \( U_X \) is an open subset of \( X \). This is the classic setting of Hale, as described in the book of Hale and Verduyn Lunel [2]. Local existence of the initial value problem

\[
z_0 = \varphi
\]

for any \( \varphi \in U_X \) in forward time is guaranteed, that is, the problem (1.5), (1.7) has a solution \( z(t) \) for \( 0 \leq t < \delta \) for some \( \delta > 0 \). Note that Eq. (1.2) falls into this class by taking

\[
f(\varphi) = g(\varphi(0), \varphi(-r(0)))
\]

\[
(1.8)
\]
For this equation we assume that \( U_g \subseteq R^{2m} \) and \( U_r \subseteq R^m \) in (1.3) are open sets on which \( g \) and \( r \) are continuous, and

\[
U_X = \{ \varphi \in X \mid (\varphi(0), \varphi(-r(\varphi(0)))) \in U_g \text{ and } \varphi(0) \in U_r \}. \tag{1.9}
\]

For the model equation (1.4) one has \( g(z, \zeta) = -z - k \zeta \) and \( r(z) = 1 + z \), with \( U_g = R^2 \) and \( U_r = (-1, R - 1) \), where typically one takes \( R \geq k + 1 \).

One has the analogous facts for the more general multiple-delay equation (1.1) as for the single-delay equation (1.2).

In general the solution of the initial value problem (1.5), (1.7) is not unique, although it is unique if \( f \) is locally lipschitz. We remark that for the special case of (1.8) and its multiple-delay generalization, the map \( f \) is generally not locally lipschitz even if the functions \( g \) and \( r \) are smooth, unless \( r \) is a constant. Thus in general, there is no assurance of a unique solution to the initial value problem (1.2), (1.7). On the other hand, if the initial function \( \varphi \) is lipschitz and \( g \) and \( r \) are both locally lipschitz, uniqueness does hold for (1.2) with (1.7).

Following Walther, let us consider Eq. (1.5) with a nonlinearity \( f \) for which

\[
\text{\textbf{(H1)}} f : U_Y \subseteq Y \rightarrow R^m \text{ is } C^1\text{-smooth, where } Y = C^1([-R, 0], R^m) \text{ and } U_Y \text{ is open in } Y.
\]

Here we do not necessarily assume that \( f \) is defined in a domain \( U_X \subseteq X \) as in (1.6), although we do keep the notation \( z_t(\theta) = z(t + \theta) \) for \( \theta \in [-R, 0] \). For definiteness we shall always take

\[
\|\varphi\|_X = \sup_{\theta \in [-R, 0]} |\varphi(\theta)|, \quad \|\varphi\|_Y = \max\{\|\varphi\|_X, \|\dot{\varphi}\|_X\}, \tag{1.10}
\]

for the norm of \( \varphi \in X \) or \( \varphi \in Y \), respectively. We consider solutions \( z(t) \) of (1.5) with initial conditions (1.7) which are \( C^1 \)-smooth on their initial interval \([-R, 0]\), and which in addition satisfy the compatibility condition \( \dot{\varphi}(0) = f(\varphi) \). This condition ensures that the left- and right-hand derivatives of any such solution \( z(t) \) at \( t = 0 \) are equal. Such solutions are thus \( C^1 \) on any maximal interval \([-R, \omega)\) of existence. In this spirit Walther introduces the so-called \textbf{solution manifold} \( S \subseteq Y \), defined to be

\[
S = \{ \varphi \in U_Y \mid \dot{\varphi}(0) = f(\varphi) \},
\]

which is a relatively closed subset of \( U_Y \). It is easily seen that any solution \( z(t) \) with initial condition \( z_0 \in S \) on the solution manifold satisfies \( z_t \in S \) for all \( t \geq 0 \) in its maximal interval of existence.

In general, condition (H1) alone is not enough to guarantee local existence of solutions to (1.5), (1.7), with \( \varphi \in S \). For example, if \( f(\varphi) = \varphi(0) + \dot{\varphi}(0) \) for every \( \varphi \in U_Y = Y \), then \( S = \{ \varphi \in Y \mid \varphi(0) = 0 \} \), and so any solution \( z(t) \) to this initial value problem would have to satisfy \( z_t \in S \) for every \( t \geq 0 \) in the interval of existence, namely, \( z_0(0) = z(t) = 0 \) for such \( t \). However, if the initial condition \( \varphi \) satisfied \( \varphi(0) = 0 \) but \( \dot{\varphi}(0) \neq 0 \), we see that \( \dot{z}(t) \) would be discontinuous at \( t = 0 \). Thus \( z(t) \) could not be considered a solution, as this would mean \( z_t \notin Y \) for \( t \in (0, R) \) in the interval of existence.

We recall two other conditions on \( f \) which play a central role in [3] and in [8], and which with (H1) are sufficient to obtain unique local solutions:

\[
\textbf{(H2)} \text{ The Fréchet derivative } D_f(\varphi) \in L(Y, R^m) \text{ at any } \varphi \in U_Y \text{ has a (necessarily unique) continuous linear extension, denoted } D_e f(\varphi) \in L(X, R^m), \text{ to the larger space } X.
\]

\[
\textbf{(H3)} \text{ The map } (\varphi, \psi) \rightarrow D_e f(\varphi) \psi \text{ from } U_Y \times X \subseteq Y \times X \text{ into } R^m \text{ is jointly continuous in } \varphi \text{ and } \psi.
\]

See Proposition 2.3 below for a detailed statement of the relevant results on existence, uniqueness, and regularity.

Generally, if \( Z_1 \) and \( Z_2 \) are Banach spaces, we let \( L(Z_1, Z_2) \) denote the Banach space of bounded linear operators from \( Z_1 \) to \( Z_2 \) endowed with the operator norm. We remark that under the above
conditions, the extended map $D_x f(\varphi)$ may not vary continuously with $\varphi \in U_Y$ in its operator norm, that is, as an element of the space $\mathcal{L}(X, \mathbb{R}^m)$.

Let us also remark that conditions (H1), (H2), and (H3) together imply several useful properties of $f$, in particular that:

(H4) If $\varphi \in U_Y$ then there exists a neighborhood $\varphi \in V \subseteq U_Y$ and a constant $B > 0$ such that $\|D_x f(\varphi)\|_{\mathcal{L}(X, \mathbb{R}^m)} \leq B$ for every $\tilde{\varphi} \in V$.

(H5) If $\varphi \in U_Y$ then there exists a neighborhood $\varphi \in V \subseteq U_Y$ and a constant $B > 0$ such that $|f(\varphi) - f(\tilde{\varphi})| \leq B \|\varphi - \tilde{\varphi}\|_X$ for every $\varphi, \tilde{\varphi} \in V$.

Properties (H4) and (H5) are closely related to the property of a map being almost locally lipschitz, which was introduced in [6]. Note that while the relevant neighborhoods in these properties are in the space $X$, the relevant norms are associated with the space $Y$. The proof of (H4) is an easy exercise using the uniform boundedness principle; indeed, if there does not exist such a neighborhood $V$ for which the required bound $B$ exists, then there would be a sequence $\varphi_n \in U_Y$ with $\varphi_n \to \varphi$ in $Y$, with $\|D_x f(\varphi_n)\|_{\mathcal{L}(X, \mathbb{R}^m)} \to \infty$. But then for every $\psi \in X$ the sequence $D_x f(\varphi_n)\psi$ would be bounded, with the limit $D_x f(\varphi)\psi$, by condition (H3). Thus the uniform boundedness principle would imply that $\|D_x f(\varphi_n)\|_{\mathcal{L}(X, \mathbb{R}^m)}$ is bounded, a contradiction.

Also, (H5) follows from (H4) by a mean-value theorem. In particular, assuming that $V$ in (H4) is convex, one has for any $\tilde{\varphi}, \hat{\varphi} \in V$ that

$$f(\tilde{\varphi}) - f(\hat{\varphi}) = \int_0^1 Df(s\hat{\varphi} + (1-s)\tilde{\varphi})(\hat{\varphi} - \tilde{\varphi}) \, ds,$$

as $f$ is $C^1$ on $U_Y$. By (H4) we have the bound

$$|Df(s\hat{\varphi} + (1-s)\tilde{\varphi})(\hat{\varphi} - \tilde{\varphi})| = |D_x f(s\hat{\varphi} + (1-s)\tilde{\varphi})(\hat{\varphi} - \tilde{\varphi})| \leq B \|\tilde{\varphi} - \hat{\varphi}\|_X$$

on the above integrand, to give (H5).

With the above conditions, we now state the main results of this paper. The following theorem is the basic result on asymptotic stability with asymptotic phase, for the class of equations considered.

**Theorem 1.1.** Assume that $f$ satisfies conditions (H1), (H2), and (H3), and suppose that $z = x(t)$ is a non-constant periodic solution of Eq. (1.5) of period $p > 0$, with $x_t \in U_Y$ for every $t \in \mathbb{R}$. Assume that the trivial eigenvalue $\lambda = 1$ of the monodromy operator $M$ of this solution has simple algebraic multiplicity. Also assume that the remaining spectrum of $M$ lies strictly inside the unit circle, specifically that

$$\sup \{|\lambda| \mid \lambda \in \text{spec}(M) \setminus \{1\}\} = \lambda_0 < 1$$

for some $\lambda_0$. (In other words, $z = x(t)$ has linearized asymptotic stability.)

Then $z = x(t)$ is asymptotically orbitally stable with asymptotic phase in the following sense. There exist $K_1, K_2 > 0$ such that the following holds. Given any

$$\mu < \frac{|\log \lambda_0|}{p},$$

there exists $K_3(\mu) > 0$ such that if $\|\varphi - x_\sigma\|_Y \leq K_1$ for some $\varphi \in S$ and $\sigma \in \mathbb{R}$, then there exists $\theta \in \mathbb{R}$ such that

$$\|z_t - x_{t+\sigma+\theta}\|_Y \leq K_3(\mu)e^{-\mu t}\|\varphi - x_\sigma\|_Y, \quad |\theta| \leq K_2\|\varphi - x_\sigma\|_Y,$$

for all $t \geq 0$, where $z(t)$ is the solution of the initial value problem $z_0 = \varphi$ to Eq. (1.5).
A second result, involving the $C^0$ norm instead of the $C^1$ norm, follows more or less directly from the above theorem provided that the following stronger condition holds in place of (H1):

\textbf{(H1')} Condition (H1) holds. Moreover, there exists a set $U_X \subseteq X$, open in $X$ and with

$$U_Y = U_X \cap Y,$$


\begin{equation}
\left\| z_t - x_{t + \sigma + \theta} \right\|_X \leq K'_3(\mu)e^{-\mu t} \left\| \varphi - x_{\sigma} \right\|_X \text{ for } 0 \leq t \leq R,
\end{equation}


\begin{equation}
\left\| z_t - x_{t + \sigma + \theta} \right\|_Y \leq K'_3(\mu)e^{-\mu t} \left\| \varphi - x_{\sigma} \right\|_X \text{ for } t \geq R, \quad |\theta| \leq K'_2 \left\| \varphi - x_{\sigma} \right\|_X.
\end{equation}

for some $\theta \in \mathbb{R}$.

Concerning notation, we shall let $z(t)$ denote a general solution of Eq. (1.5), often (although not always) lying on $S \subseteq Y$, while $x(t)$ is reserved for the periodic solution in the statements of the above results. In the setting of Corollary 1.2, note that $z_t \in S$ for all $t \geq R$. Of course for the periodic solution it is the case that $x_t \in S$, and also that $\dot{x}_t \neq 0$, for every $t \in \mathbb{R}$.

Let us also remark that in both Theorem 1.1 and Corollary 1.2, it is implicit that the solution $z(t)$ starting near the periodic solution $x(t)$ exists for all $t \geq 0$, that is, it is part of the conclusion of these results that the maximal interval of existence of this solution is $[0, \infty)$.

While this paper is a companion to [5], it can also be considered a self-contained work to be read on its own. As such, we do not necessarily follow the notational conventions of [5], and in fact our notation herein adheres more closely to that of [3] and [8].

2. The functional analytic setting

We begin by recalling some basic results from [3] and [8].

**Proposition 2.1.** (See [3, Theorem 3.2.1] and [8, Proposition 1].) Assume that $f$ satisfies conditions (H1), (H2), and (H3). Then for every $\varphi \in S$ the map $\psi \rightarrow \psi(0) - Df(\varphi)\psi$ takes $Y$ onto $\mathbb{R}^m$. Thus $S$ is an embedded $C^1$ submanifold of $Y$ of codimension $m$, with tangent space

$$T_{\varphi}S = \left\{ \psi \in Y \mid \psi(0) = Df(\varphi)\psi \right\}$$

at any $\varphi \in S$.

We remark that the tangent space $T_{\varphi}S$ in the above result is dense in the space $X$. This follows easily from the fact that the operator $Df(\varphi)$ on $Y$ has a continuous extension $D_{\psi}f(\varphi)$ to $X$.

**Proposition 2.2.** Assume that $g : U_g \subseteq \mathbb{R}^{2m} \rightarrow \mathbb{R}^m$ and $r : U_r \subseteq \mathbb{R}^m \rightarrow [0, R]$ are $C^1$ functions, where $U_g$ and $U_r$ are open subsets of $\mathbb{R}^{2m}$ and $\mathbb{R}^m$. Let $U_X$ be as in (1.9) and $U_Y = U_X \cap Y$, and let $f(\varphi)$ be defined by Eq. (1.8)
for \( \varphi \in U_X \). Then \( f : U_X \to \mathbb{R} \) satisfies condition (H1’) (and thus condition (H1)), as well as conditions (H2) and (H3). The Fréchet derivative of \( f \) is given by

\[
Df(\varphi)\psi = (D_1g(\varphi(0), \varphi(-r(\varphi(0)))) - D_2g(\varphi(0), \varphi(-r(\varphi(0))))\varphi(-r(\varphi(0)))(\varphi(0)))(\psi(0)
+ (D_2g(\varphi(0), \varphi(-r(\varphi(0)))))(\varphi(-r(\varphi(0))))(\psi(-r(\varphi(0))))
\]

for any \( \varphi \in U_Y \) and \( \psi \in Y \), and with the same formula for the extension \( D_f(\varphi) \) with \( \varphi \in U_Y \) and \( \psi \in X \).

The analogous results hold for the function \( f \) associated to the multiple-delay problem (1.1).

**Proof.** For simplicity we consider only the single-delay problem, with \( f \) as in (1.8). Following [8], we write

\[
f(\varphi) = g(Ev(\varphi, 0), Ev(\varphi, -r(Ev(\varphi, 0))))
\]

for \( \varphi \in U_Y \), where \( Ev : Y \times [-R, 0] \to \mathbb{R}^m \) is the evaluation map given by

\[
Ev(\varphi, t) = \varphi(t).
\]

As noted in [8], the function \( Ev \) is \( C^1 \)-smooth with derivative

\[
D Ev(\varphi, t)(\psi, s) = \psi(t) + \varphi(t)s.
\]

so in particular the function \( f \) is \( C^1 \) on \( U_Y \). This establishes (H1). Further, using (2.2) one sees that the derivative of \( f \) is

\[
Df(\varphi)\psi = D_1g(Ev(\varphi, 0), Ev(\varphi, -r(Ev(\varphi, 0))))\psi(0)
+ D_2g(Ev(\varphi, 0), Ev(\varphi, -r(Ev(\varphi, 0))))(\psi(-r(Ev(\varphi, 0))))
- \varphi(-r(Ev(\varphi, 0)))r'(Ev(\varphi, 0))\psi(0),
\]

which is identical to Eq. (2.1). From this formula one sees easily that both (H2) and (H3) hold.

There remains to prove condition (H1’). Certainly \( f \) is continuous on the set \( U_X \) given by (1.9). Now with \( Q \subseteq S \) as in the statement of condition (H1’), note that \( \{\varphi(0)\varphi \in Q \} \) is a compact subset of the open set \( U_r \subseteq \mathbb{R}^m \). Thus there exists a quantity \( B'' > 0 \) such that \( \varphi(0) + \beta \in U_r \) whenever \( \varphi \in Q \) and \( |\beta| \leq B'' \). Further, there exists \( B_1 > 0 \) such that \( |r'(\varphi(0) + \beta)| \leq B_1 \) whenever \( \varphi \in Q \) and \( |\beta| \leq B'' \).

Next let \( B_2 > 0 \) be such that \( \|\varphi\|_Y \leq B_2 \) for every \( \varphi \in Q \), where we are using the fact that \( Q \) is compact in \( Y \). Noting that the set \( \{(\varphi(0), \varphi(-r(\varphi(0)))) \varphi \in Q \} \) is a compact subset of the open set \( U_{\tilde{g}} \subseteq \mathbb{R}^{2m} \), we see that by decreasing \( B'' \) if necessary, we may assume that \( (\varphi(0) + \beta, \varphi(-r(\varphi(0)) + \gamma) \) \( \in U_{\tilde{g}} \) whenever \( \varphi \in Q \) with \( |\beta| \leq B'' \) and \( |\gamma| \leq (B_1B_2 + 1)B'' \). Further, there exist \( B_3, B_4 > 0 \) such that

\[
|D_1g(\varphi(0) + \beta, \varphi(-r(\varphi(0)) + \gamma))| \leq B_3, \quad |D_2g(\varphi(0) + \beta, \varphi(-r(\varphi(0)) + \gamma)| \leq B_4,
\]

for any such \( \varphi, \beta, \) and \( \gamma \).

With \( B'' \) as above, assume that \( \varphi \in Q \) and \( \tilde{\varphi} \in X \) with \( \|\varphi - \tilde{\varphi}\|_X \leq B'' \). Then \( |\varphi(0) - \tilde{\varphi}(0)| \leq B'' \) and so \( \tilde{\varphi}(0) \in U_r \). Also,
and therefore \((\tilde{\varphi}(0), \tilde{\varphi}(-r(\tilde{\varphi}(0))))\) \(\in U_g\). Thus \(\tilde{\varphi} \in U_X\). Finally,

\[
|f(\varphi) - f(\tilde{\varphi})| = |g(\varphi(0), \varphi(-r(\tilde{\varphi}(0)))) - g(\tilde{\varphi}(0), \tilde{\varphi}(-r(\tilde{\varphi}(0))))| \\
\leq B_3|\varphi(0) - \tilde{\varphi}(0)| + B_4|\varphi(-r(\tilde{\varphi}(0))) - \tilde{\varphi}(-r(\tilde{\varphi}(0)))| \\
\leq (B_3 + B_4(B_1B_2 + 1))\|\varphi - \tilde{\varphi}\|_X.
\]

Upon setting \(B' = B_3 + B_4(B_1B_2 + 1)\), we obtain the desired result. \(\square\)

**Proposition 2.3.** (See [3, Theorem 3.2.1] and [8, Theorem 1.]) Assume that \(f\) satisfies conditions (H1), (H2), and (H3). Then for every \(\varphi \in S\) the initial value problem (1.5), (1.7) has a unique solution \(z(t) = z(t, \varphi)\) on a maximal interval \(0 \leq t < \omega(\varphi)\), where \(0 < \omega(\varphi) \leq \infty\). Moreover, \(z_t(\varphi) \in S\) for every such \(t\). Denote

\[
\mathcal{D} = \left\{ (t, \varphi) \in [0, \infty) \times S \mid t < \omega(\varphi) \right\},
\]

\[
\mathcal{D}_t = \left\{ \varphi \in S \mid (t, \varphi) \in \mathcal{D} \right\}, \quad \mathcal{D}_s = \left\{ (t, \varphi) \in \mathcal{D} \mid t > R \right\}.
\]

where \(t \geq 0\) in the definition of \(\mathcal{D}_t\), and denote

\[
Z(t, \varphi) = z_t(\varphi) \quad \text{for} \quad (t, \varphi) \in \mathcal{D},
\]

and so \(Z : \mathcal{D} \rightarrow S\) is the semiflow map. Then \(\mathcal{D}\) is a relatively open subset of \([0, \infty) \times S\), and \(Z\) is continuous, jointly in \(t\) and \(\varphi\), on \(\mathcal{D}\). Also, for every fixed \(t \geq 0\) the map \(\varphi \rightarrow Z(t, \varphi)\) is \(C^1\)-smooth on \(\mathcal{D}_t\). Further, the map \(Z\) is \(C^1\)-smooth, again jointly in \(t\) and \(\varphi\), on \(\mathcal{D}_s\). Finally, the derivative \(D_2Z(t, \varphi)\) of the semiflow map with respect to its initial condition is given by

\[
D_2Z(t, \varphi)\psi = y_t
\]

where \(y(t)\) is the unique solution of the linear variational equation

\[
\dot{y}(t) = Df(z_t)y_t,
\]

with initial condition \(y_0 = \psi \in T_{z_t}S\), with \(z_t = Z(t, \varphi)\) on its maximal interval, and where \(y_t \in T_{z_t}S\) holds.

Let us remark that the proof of local existence in the \(C^1\) setting of Proposition 2.3 above is not a direct application of the usual Picard iteration. Rather it is a variant of the Picard method which involves both \(C^1\) and \(C^0\) estimates in the spirit of (1.11) and (1.12), where both conditions (H4) and (H5) come into play.

As in [8], given a solution \(z(t)\) of Eq. (1.5) with \(z_t\) lying on the solution manifold \(S\), one may generally consider the linear variational equation (2.4). Proposition 2.3 asserts the existence and uniqueness of a solution to (2.4) with the initial condition \(y_0\) taken in the tangent space \(T_{z_0}S\), as variations are taken only with respect to the \(C^1\) norm in \(S\). Thus one has \(y_t \in T_{z_t}S\) for every \(t \geq 0\) in the interval...
of existence, and so $D_2 Z(t, \varphi) \in \mathcal{L}(T_{z_t} S, T_{z_t} S)$. In particular, $T_{z_t} S$ and $T_{z_t} S$ are subspaces of $Y$ and are thus endowed with the $C^1$ norm.

On the other hand, one can instead consider the related equation

$$\dot{y}(t) = D_e f(z_t) y_t,$$  \hspace{1cm} (2.5)

and with arbitrary continuous (not necessarily smooth) initial conditions $y_0 = \psi \in X$. This amounts to taking variations in the larger space $X$ and with respect to the $C^0$ norm. (We still, however, have $z_t \in S$ for the solution of the underlying nonlinear equation.) As noted earlier, the coefficient operator $D_e f(z_t)$ need not vary continuously with $t$ in the space $\mathcal{L}(X, \mathbb{R}^m)$. However, it follows easily from property (H4) that for every $t > 0$ in the maximal interval of the solution $z(t)$, there is a uniform bound

$$\| D_e f(z_t) \|_{\mathcal{L}(X, \mathbb{R}^m)} \leq B_0 = B_0(t) \quad \text{for} \ 0 \leq s \leq t$$  \hspace{1cm} (2.6)

for the norm of $D_e f(z_t)$. With this, and again using the continuity condition (H3), one easily shows the existence of a unique solution to Eq. (2.5) for any $y_0 \in X$, along with a standard bound

$$\| y_s \|_X \leq e^{B_0 s} \| y_0 \|_X \quad \text{for} \ 0 \leq s \leq t.$$  \hspace{1cm} (2.7)

In the special case that $y_0 \in T_{z_0} S$, then we have the bound $|\dot{y}(s)| \leq \| \dot{y}_0 \|_X$ for $-R \leq s \leq 0$, and $|\dot{y}(s)| \leq B_0 \| y_s \|_X \leq B_0 e^{B_0 s} \| y_0 \|_X$ for $0 \leq s \leq t$, using (2.5) and (2.7). With (2.7) this gives the bound

$$\| y_s \|_Y \leq B_0 e^{B_0 s} \| y_0 \|_Y \quad \text{for} \ 0 \leq s \leq t, \quad B_0 = \max\{B_0, 1\},$$  \hspace{1cm} (2.8)

where the definition (1.10) of the norm on $Y$ is used.

3. The monodromy operator

The monodromy operator of our periodic solution can be defined using either Eq. (2.4) or Eq. (2.5). As we shall see, for the class of equations considered here the two approaches are essentially equivalent.

Throughout this section we assume all the conditions in the statement of Theorem 1.1. Let us note that we do not require that $p > 0$ be the minimal period; unless noted otherwise, $p$ can be any integer multiple of the minimal period. Of course the monodromy operator will depend on which such period is chosen.

In defining the monodromy operator, we may, on the one hand, consider the operator $\tilde{M}$ defined by

$$\tilde{M} = D_2 Z(p, x_0)$$

as in (2.3), and so $\tilde{M} \in \mathcal{L}(T_{x_0} S, T_{x_0} S)$. By Proposition 2.3 the operator $\tilde{M}$ is given by the formula

$$\tilde{M} y_0 = y_p, \quad y_0 \in T_{x_0} S,$$

where $y(t)$ is the solution of the linear equation (2.4) with the periodic solution $x_t$ in place of $z_t$. On the other hand, a formally different operator $M$ is obtained by solving equation (2.5), again with $x_t$ in place of $z_t$, and defining

$$My_0 = y_p, \quad y_0 \in X.$
Then $M \in \mathcal{L}(X, X)$ for this operator, and $M$ agrees with $\tilde{M}$ on $T_{x_0}S_1$, which is densely contained in $X$. We observe that the Banach spaces $X$ and $T_{x_0}S$ on which $M$ and $\tilde{M}$ act are endowed with different norms, namely the $C^0$ and $C^1$ norms, respectively.

Differentiating the identity $\dot{x}(t) = f(x_t)$ with respect to $t$ gives $\dot{y}(t) = Df(x_t)y_t$ where $y(t) = \dot{x}(t)$. Then $\tilde{M}y_0 = y_p$, and as $y_p = y_0 \neq 0$, it follows that

$$M\dot{x}_0 = \tilde{M}\dot{x}_0 = \dot{x}_0,$$

which is the trivial eigenvector of the monodromy operator with trivial eigenvalue $\lambda = 1$.

In a standard fashion, $M^n$ and $\tilde{M}^n$ are the monodromy operators for $x(t)$ considered with period $np$, where $M$ and $\tilde{M}$ as above are the monodromy operators corresponding to period $p$. In the case that $np \geq R$, we have the following compactness result.

**Proposition 3.1.** Assume all the conditions in the statement of Theorem 1.1 hold, and let $M$ and $\tilde{M}$ denote the monodromy operators of $x(t)$ with period $p$ as above. Also assume that $n \geq 1$ is such that $np \geq R$. Then the monodromy operators $M^n \in \mathcal{L}(X, X)$ and $\tilde{M}^n \in \mathcal{L}(T_{x_0}S, T_{x_0}S)$ of $x(t)$ with period $np$ are compact.

**Proof.** Let $B_0$ be as in (2.6) with $t = np$, with $z_s$ replaced by $x_s$, and let $B_{00}$ be as in the equality in (2.8). Considering first the operator $M^n$, let $y(t)$ satisfy Eq. (2.5), with $z_t$ replaced by $x_t$. Then $M^n y_0 = y_{np}$, and we have

$$\|y_{np}\|_x \leq e^{B_{0np}}\|y_0\|_x \quad (3.1)$$

by (2.7). Further, as $np \geq R$ we have that $[np - R, np] \subseteq [0, np]$, and so $|\dot{y}(s)| \leq B_0e^{B_0s}\|y_0\|_x$ for $s \in [np - R, np]$, from the differential equation (2.5) and (2.7). Thus

$$\|\dot{y}_{np}\|_x \leq B_0e^{B_{0np}}\|y_0\|_x. \quad (3.2)$$

Thus the elements $y_{np}$ are uniformly bounded and equicontinuous for $y_0$ in bounded set of $X$, and so $M^n$ is compact.

Now consider $\tilde{M}^n$. Here we take $y_0 \in T_{x_0}S \subseteq Y$, with $y(t)$ the solution of (2.4), again with $z_t$ replaced by $x_t$, and $\tilde{M}^n y_0 = y_{np}$. The bounds (3.1) and (3.2) still hold, and as $\|y_0\|_x \leq \|y_0\|_Y$, we again obtain uniform bounds on $\|y_{np}\|_x$ and $\|\dot{y}_{np}\|_x$ for $y_0$ in bounded subsets of $T_{x_0}S$. We must establish equicontinuity of the elements $\dot{y}_{np}$ for such $y_0$ in order to conclude compactness of $\tilde{M}^n$. To this end, take $s_1, s_2 \in [np - R, np] \subseteq [0, np]$, and using (2.4) along with (2.8), we have the estimate

$$\left|\dot{y}(s_1) - \dot{y}(s_2)\right| \leq |Df(x_{s_1})y_{s_1} - Df(x_{s_2})y_{s_2}| + |Df(x_{s_1})y_{s_2} - Df(x_{s_2})y_{s_2}|$$

$$\leq B_0\|y_{s_1} - y_{s_2}\|_x + \|Df(x_{s_1}) - Df(x_{s_2})\|_{\mathcal{L}(Y, \mathbb{R}^m)}\|y_{s_2}\|_Y$$

$$\leq B_0e^{B_{0np}}(B_0|s_1 - s_2| + \|Df(x_{s_1}) - Df(x_{s_2})\|_{\mathcal{L}(Y, \mathbb{R}^m)})\|y_0\|_Y.$$

The continuity of $Df(x_s)$ in $s$, in the operator norm, with the above estimate, immediately gives the desired equicontinuity property. □

The following result clarifies the relation between the spectra of the operators $M$ and $\tilde{M}$.

**Proposition 3.2.** Assume all the conditions in the statement of Theorem 1.1 hold, and let $M$ and $\tilde{M}$ denote the monodromy operators of $x(t)$ with period $p$ as above. Then $M \in \mathcal{L}(X, X)$ and $\tilde{M} \in \mathcal{L}(T_{x_0}S, T_{x_0}S)$ have the same spectrum

$$\text{spec}(M) = \text{spec}(\tilde{M}). \quad (3.3)$$
Further, if \( \lambda \neq 0 \) then for every \( k \geq 1 \) we have that

\[
\ker(\lambda I - M)^k = \ker(\lambda I - \tilde{M})^k, \tag{3.4}
\]

so in particular, the algebraic multiplicities of \( \lambda \) as an eigenvalue of \( M \) and of \( \tilde{M} \) are equal.

**Proof.** Fix \( n \geq 1 \) so that \( np \geq R \). Then both \( M^n \) and \( \tilde{M}^n \) are compact operators by Proposition 3.1, and so for both these operators the essential spectrum is just the point \( \{0\} \), and every nonzero point of the spectrum is an isolated point of finite algebraic multiplicity. From this we see that (3.3) follows directly from (3.4).

To prove (3.4), fix any \( \lambda \neq 0 \) and \( k \geq 1 \). Clearly \( \ker(\lambda I - \tilde{M})^k \subseteq \ker(\lambda I - M)^k \) as \( M \) is an extension of \( \tilde{M} \). To prove the opposite inclusion, let us first observe that

\[
\text{range}(M^n) \subseteq T_{x_0}S. \tag{3.5}
\]

The proof of (3.5) is simply the observation that if \( y(t) \) satisfies (2.5) (with \( x_t \) in place of \( z_t \)), then \( y_{np} \in Y \) as \( np \geq R \), and also \( y(np) = D_y f(x_{np})y_{np} = D_y f(x_0)y_{np} = Df(x_0)y_{np} \). This implies that \( y_{np} \in T_{x_0}S \), as desired.

Next we write \( (\lambda I - M)^k = \lambda^k I - M \Psi(M) \) where \( \Psi \) is a polynomial depending on \( \lambda \) and \( k \). Similarly \( (\lambda I - \tilde{M})^k = \lambda^k I - \tilde{M} \Psi(\tilde{M}) \). Take any \( \psi \in \ker(\lambda I - M)^k \). Then \( M \Psi(M)\psi = \lambda^k \psi \) and thus \( M^n \Psi(M)^n \psi = \lambda^{kn} \psi \). This implies, with (3.5), that \( \psi \in T_{x_0}S \), which is the domain of \( \tilde{M} \). It follows that \( \tilde{M} \Psi(M)\psi = \lambda^k \psi \), equivalently, \( \psi \in \ker(\lambda I - \tilde{M})^k \). This completes the proof. \( \square \)

4. The Poincaré map

Here we construct a Poincaré map in the solution manifold \( S \) for our periodic solution \( x(t) \). We continue to assume that the conditions in the statement of Theorem 1.1 hold. In addition, we assume the (not necessarily minimal) period \( p \) satisfies

\[
p > R. \tag{4.1}
\]

Let us note that from the point of view of Theorem 1.1 and Corollary 1.2, condition (4.1) is in fact no restriction at all. Indeed, replacing the period \( p \) with any multiple \( np \) does not affect the inequality (1.14) in the statement of Theorem 1.1, as this would replace \( \lambda_0 \) with \( \lambda_0^n \) leaving the right-hand side of that inequality unchanged. Thus, in proving Theorem 1.1 and Corollary 1.2, we may assume without loss of generality that \( p \) is any multiple of the minimal period for which (4.1) holds.

Let \( U \subseteq S \) be an open set (in the relative topology of \( S \)) which contains the initial point \( x_0 \) of the periodic orbit, and let \( H : U \to \mathbb{R} \) be a \( C^1 \)-smooth function with

\[
H(x_0) = 0, \quad DH(x_0)x_0 \neq 0. \tag{4.2}
\]

Define the set

\[
P = \{ \varphi \in U \mid H(\varphi) = 0 \}. \tag{4.3}
\]

Then in a sufficiently small neighborhood of \( x_0 \), the set \( P \) is a \( C^1 \) manifold containing \( x_0 \) and which is transverse to the periodic orbit in \( S \) at \( x_0 \). The tangent space of \( P \) at \( x_0 \) is the subspace

\[
T_{x_0}P = \ker(DH(x_0)) = \{ \psi \in T_{x_0}S \mid DH(x_0)\psi = 0 \}. \tag{4.4}
\]
which is a subspace of $T_{x_0}S$ of codimension one. (We shall only be interested in points of $\mathcal{P}$ in a sufficiently small neighborhood of $x_0$; outside a neighborhood of this point the set $\mathcal{P}$ need not be a manifold.) The set $\mathcal{P}$, in a sufficiently small neighborhood of $x_0$ in $S$, is a Poincaré section to the periodic orbit. As an example of such a function $H$, one could take $H(\varphi) = \Lambda(\varphi - x_0)$ where $\Lambda$ is any bounded linear functional on $Y$ for which $\Lambda x_0 \neq 0$.

The following result defines a return map to the Poincaré section $\mathcal{P}$ in a sufficiently small neighborhood of $x_0$.

**Proposition 4.1.** Assume all the conditions in the statement of Theorem 1.1 hold. Assume further that the period $p$ of $x(t)$ satisfies (4.1), and fix a function $H : U \to \mathbb{R}$ with Poincaré section $\mathcal{P}$ as above in (4.2), (4.3). Then there exists a neighborhood $V \subseteq U$ containing $x_0$, and a unique $C^1$ function $\delta : V \to \mathbb{R}$ with $\delta(x_0) = 0$, such that

$$Z(p + \delta(\varphi), \varphi) \in \mathcal{P}$$

for every $\varphi \in V$.

**Proof.** Consider the function $\overline{H}(\delta, \varphi) = H(Z(p + \delta, \varphi))$ defined in a sufficiently small neighborhood of $(\delta, \varphi) = (0, x_0)$ in $\mathbb{R} \times S$. Then due to condition (4.1), the function $\overline{H}$ is $C^1$ with $\overline{H}(0, x_0) = H(Z(p, x_0)) = H(x_0) = 0$. Moreover,

$$\frac{d}{d\delta} \overline{H}(\delta, x_0) = \frac{d}{d\delta} H(Z(p + \delta, x_0)) = \frac{d}{d\delta} H(x_{p+\delta}) = \frac{d}{d\delta} H(x_\delta) = D H(x_\delta) \dot{x}_\delta$$

(4.5)

for $|\delta|$ sufficiently small, and so the quantity (4.5) is nonzero at $\delta = 0$ by (4.2). The result now follows by applying the implicit function theorem to the function $\overline{H}$. \qed

Let us note that the neighborhood $V$ of $x_0$ in the above result is a neighborhood in the solution manifold $S$, and not just a neighborhood in the Poincaré section $\mathcal{P}$ which is locally a submanifold of $S$. Thus the above result allows for initial conditions $\varphi$ which do not necessarily lie on the Poincaré section. In this context we define the return map $\Gamma$ by

$$\Gamma(\varphi) = Z(p + \delta(\varphi), \varphi) \in \mathcal{P}$$

(4.6)

for $\varphi \in V$, and so $\Gamma : V \to \mathcal{P}$. Certainly $\Gamma(x_0) = x_0$, and in fact $\Gamma(x_t) = x_0$ if $|t|$ is sufficiently small. Because the range of $\Gamma$ lies in $\mathcal{P}$, we have that

$$\text{range}(D \Gamma(x_0)) \subseteq T_{x_0}\mathcal{P}.$$  (4.7)

Thus $D \Gamma(x_0) \in \mathcal{L}(T_{x_0}S, T_{x_0}\mathcal{P})$, although we may also regard $D \Gamma(x_0) \in \mathcal{L}(T_{x_0}S, T_{x_0}S)$ due to the inclusion $T_{x_0}\mathcal{P} \subseteq T_{x_0}S$ as a subspace.

The next result relates the spectrum of $D \Gamma(x_0)$ to the spectrum of the monodromy operators. In preparation for this, observe the direct sum decomposition

$$T_{x_0}S = \langle \dot{x}_0 \rangle \oplus T_{x_0}\mathcal{P},$$

(4.8)

which holds by (4.4) because $DH(x_0)\dot{x}_0 \neq 0$. Here $\langle \dot{x}_0 \rangle$ denotes the one-dimensional span of $\dot{x}_0$.

**Proposition 4.2.** Assume all the conditions in the statement of Theorem 1.1 hold. Assume further that the period $p$ of $x(t)$ satisfies (4.1), and fix a function $H$ with Poincaré section $\mathcal{P}$ as above in (4.2), (4.3). Let the neighborhood $V$ and map $\delta$ be as in the statement of Proposition 4.1, with return map $\Gamma$ as in (4.6). Then relative to the decomposition (4.8), the operators $\tilde{\mathcal{M}}$, $D \Gamma(x_0) \in \mathcal{L}(T_{x_0}S, T_{x_0}S)$ have the matrix representations...
\[ \hat{M} = \begin{pmatrix} 1 & A \\ 0 & \hat{M} \end{pmatrix}, \quad D \Gamma(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{M} \end{pmatrix}. \]

Next let \( \tilde{\lambda} \in \mathcal{L}(T_{x_0} \mathcal{P}, M) \) and \( \tilde{\lambda} \in \mathcal{L}(T_{x_0} \mathcal{P}, T_{x_0} \mathcal{P}) \), and further, \( \hat{M} \) and \( D \Gamma(x_0) \) are compact operators, as are \( \hat{M} \) and \( M \). If \( k \geq 1 \) denotes the algebraic multiplicity of \( \lambda = 1 \) as an eigenvalue of the operator \( \hat{M} \), or equivalently of \( M \), then

\[ \text{spec}(\hat{M}) = \text{spec}(D \Gamma(x_0)) \begin{cases} \text{spec}(\hat{M}) \setminus \{1\} = \text{spec}(M) \setminus \{1\} & \text{if } k = 1, \\ \text{spec}(\hat{M}) = \text{spec}(M) & \text{if } k > 1, \end{cases} \]

and if \( k > 1 \) then \( \lambda = 1 \) has algebraic multiplicity \( k - 1 \) as an eigenvalue of \( \hat{M} \) and of \( D \Gamma(x_0) \). Further, the multiplicity of any given \( \lambda \neq 0, 1 \) in the spectrum of \( \hat{M} \), \( D \Gamma(x_0) \), \( \hat{M} \), or \( M \) is the same for these four operators.

**Proof.** First, we have the trivial eigenvector \( \hat{M} \hat{x}_0 = \hat{x}_0 \), to give the first column of the matrix for \( \hat{M} \). Next let \( A \in \mathcal{L}(T_{x_0} \mathcal{P}, \mathcal{R}) \) and \( \tilde{\lambda} \in \mathcal{L}(T_{x_0} \mathcal{P}, T_{x_0} \mathcal{P}) \) be defined to be the matrix entries in the formula for \( \tilde{\lambda} \) in (4.9), that is, the first equation in (4.9). We must verify the remaining formulas in (4.9). Differentiating the formula (4.6) for \( \Gamma \) at \( x_0 \) gives

\[ D \Gamma(x_0) = D_1 Z(p, x_0) D \delta(x_0) + D_2 Z(p, x_0) = \dot{x}_p D \delta(x_0) + \hat{M} = \hat{x}_0 D \delta(x_0) + \hat{M} \]

which we may write as

\[ \hat{M} = -\hat{x}_0 D \delta(x_0) + D \Gamma(x_0). \]

The two terms in the right-hand side of (4.11) correspond to the decomposition (4.8), in particular because of (4.7). Thus letting \( P \in \mathcal{L}(T_{x_0} \mathcal{S}, T_{x_0} \mathcal{S}) \) denote the projection from \( T_{x_0} \mathcal{S} \) onto \( T_{x_0} \mathcal{P} \) with kernel \( \langle \delta_0 \rangle \), we have that \( P = \text{diag}(0,1) \) in matrix form and it follows that

\[ -\hat{x}_0 D \delta(x_0) = (I - P) \hat{M} = \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix}, \quad D \Gamma(x_0) = P \hat{M} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{M} \end{pmatrix}. \]

This gives the matrix representation of \( D \Gamma(x_0) \) in (4.9). One also immediately reads off from (4.12) the formulas for \( A \) and \( \hat{M} \) as claimed in (4.9).

The operators \( M \) and \( \hat{M} \) are compact by Proposition 3.1 and the inequality (4.1), and from the matrix representations (4.9) it follows that \( \hat{M} \) and \( D \Gamma(x_0) \) are also compact. The remaining claims about the spectra of the various operators follows simply and directly from the matrix representations (4.9), using in particular the upper triangular structure, and from Proposition 3.2. \( \Box \)

5. **Proofs of the main results**

In this section we prove both Theorem 1.1 and Corollary 1.2. Before doing this, we need the following result.

**Proposition 5.1.** Assume all the conditions in the statement of Theorem 1.1 hold. Assume further that the period \( p \) of \( x(t) \) satisfies (4.1), and fix a function \( H \) with Poincaré section \( \mathcal{P} \) as above in (4.2), (4.3). Let the neighborhood \( V \) and map \( \delta \) be as in the statement of Proposition 4.1, with return map \( \Gamma \) as in (4.6). Then given any \( \lambda \) satisfying \( \lambda_0 < \lambda < 1 \), where \( \lambda_0 \) is as in (1.13), there exists a neighborhood \( V^\lambda \subseteq V \) containing \( x_0 \), a \( C^1 \) function \( \delta^\lambda : V^\lambda \rightarrow \mathcal{R} \) with \( \delta^\lambda(x_0) = 0 \), and an integer \( n \geq 1 \), such that

\[ \Gamma^n(\varphi) = Z(n \varphi + \delta^\lambda(\varphi), \varphi) \in \mathcal{P}, \quad \| \Gamma^n(\varphi) - x_0 \|_Y \leq \lambda^n \| \varphi - x_0 \|_Y \]

for every \( \varphi \in V^\lambda \).
**Proof.** We have \( \text{spec}(D \Gamma(x_0)) = \text{spec}(M) \setminus \{1\} \) by (4.10) of Proposition 4.2, as the eigenvalue \( \lambda = 1 \) of \( M \) is assumed to have simple algebraic multiplicity. Thus the spectral radius of \( D \Gamma(x_0) \) equals \( \lambda_0 \), and so there exists \( n \geq 1 \) such that

\[
\| D \Gamma(x_0)^n \|_{\mathcal{L}(T_{x_0}, T_{x_0})} \leq \frac{\lambda^n}{2}. \quad (5.2)
\]

Fix any such \( n \). Then with \( V \) as in Proposition 4.1, there exists a neighborhood \( \tilde{V} \subseteq V \) containing \( x_0 \) such that if \( \varphi \in \tilde{V} \) then we have the iterates \( \Gamma^k(\varphi) \in V \) for every \( 1 \leq k \leq n - 1 \) and so \( \Gamma^n(\varphi) \) is well-defined. We claim that for such \( \varphi \) we have

\[
\Gamma^k(\varphi) = Z(kp + \delta(\varphi, k), \varphi), \quad \delta(\varphi, k) = \sum_{j=0}^{k-1} \delta(\Gamma^j(\varphi)), \quad (5.3)
\]

holding for \( 1 \leq k \leq n \). Indeed, one easily proves (5.3) by finite induction on \( k \). The claim holds by definition if \( k = 1 \). If (5.3) holds for some \( k \) with \( 1 \leq k \leq n - 1 \) then

\[
\Gamma^{k+1}(\varphi) = \Gamma^*(\Gamma^k(\varphi)) = Z(p + \delta(\Gamma^k(\varphi)), \Gamma^k(\varphi)) = Z(p + \delta(\Gamma^k(\varphi)), Z(kp + \delta(\varphi, k), \varphi))
\]

\[
= Z((k+1)p + \delta(\varphi, k) + \delta(\Gamma^k(\varphi)), \varphi) = Z((k+1)p + \delta(\varphi, k+1), \varphi),
\]

to give it for \( k + 1 \). Note that the semiflow property of \( Z \) is used in the penultimate equality above.

As \( x_0 \) is a fixed point of \( \Gamma \), we have that \( D \Gamma^n(x_0) = D \Gamma(x_0)^n \) for the derivative of the \( n \)th iterate \( \Gamma^n \). Thus by (5.2) there exists a neighborhood \( V^\lambda \subseteq \tilde{V} \) containing \( x_0 \) such that

\[
\| \Gamma^n(\varphi) - x_0 \|_Y \leq 2\| D \Gamma(x_0)^n \|_{\mathcal{L}(T_{x_0}, T_{x_0})} \| \varphi - x_0 \|_Y \leq \lambda^n \| \varphi - x_0 \|_Y
\]

for every \( \varphi \in V^\lambda \). With (5.3) and upon setting \( \delta^\lambda(\varphi) = \delta(\varphi, n) \), we have (5.1). \( \square \)

**Proof of Theorem 1.1.** Throughout this proof \( z(t) \), for \( t \geq 0 \), is the solution of (1.5) with initial condition \( z_0 = \varphi \), where \( \varphi \in \mathcal{S} \) as in the statement of the theorem. In particular, \( z_t = Z(t, \varphi) \). Also, as noted earlier, we may assume without loss that condition (4.1) on the period \( p \) of the periodic solution \( x(t) \) holds. Additionally, all norms in this proof are either the norm of \( Y \) or else \( C^1 \)-type norms of associated spaces of operators, unless noted otherwise. Finally, we shall introduce various constants \( K_j \), for \( 1 \leq j \leq 18 \). Some of these constants will depend on the choice of \( \mu \), in which case we write \( K_j(\mu) \). We write simply \( K_j \) if the constant is independent of \( \mu \).

Fix \( K_4 > 0 \) such that we have the bounds

\[
|\dot{x}(t)|, |\ddot{x}(t)| \leq K_4 \quad \text{for all } t \in \mathbb{R}, \quad (5.4)
\]

for the periodic solution \( x(t) \), and note that for any \( t_1, t_2 \in \mathbb{R} \) we have

\[
\|x_{t_1} - x_{t_2}\| \leq K_4|t_1 - t_2|, \quad (5.5)
\]

here using the \( Y \)-norm. In particular, the bounds \( \|x_{t_1} - x_{t_2}\|_X \leq K_4|t_1 - t_2| \) and \( \|\dot{x}_{t_1} - \ddot{x}_{t_2}\|_X \leq K_4|t_1 - t_2| \) in the \( X \)-norm follow from (5.4), and together these give (5.5).

We shall first prove that for every \( \mu > 0 \) satisfying the inequality (1.14) in the statement of the theorem, there exist quantities \( K_5(\mu), K_6(\mu), K_7(\mu) > 0 \) such that if \( \|\varphi - x_0\| \leq K_5(\mu) \) and \( \varphi \in \mathcal{S} \), then there exists \( \theta \in \mathbb{R} \) such that
\[ \|z_t - x_{t+\theta}\| \leq K_7(\mu)e^{-\mu t}\|\varphi - x_0\|, \quad |\theta| \leq K_6(\mu)\|\varphi - x_0\|, \quad (5.6) \]

for all \( t \geq 0 \). Note that the conclusion (5.6) is weaker than the desired conclusion (1.15), as the constants \( K_5(\mu) \) and \( K_6(\mu) \) depend on \( \mu \), unlike \( K_1 \) and \( K_2 \) in the statement of the theorem. Also, the initial condition \( \varphi \) is compared only to \( x_0 \), and not to \( x_\theta \) as in the statement of the theorem.

To prove this claim, first take any \( \mu \) as above and let \( \lambda = e^{-\mu p} \). Then \( \lambda_0 < \lambda < 1 \) holds and so Proposition 5.1 applies. Fix \( n \) as in Proposition 5.1, let \( V^\lambda \) and \( \delta^\lambda \) be as in that result, and let \( K_5(\mu), K_6(\mu) > 0 \) and \( W(\mu) \subseteq S \) be such that

\[ W(\mu) = \{ \varphi \in S \mid \|\varphi - x_0\| \leq K_5(\mu) \} \subseteq V^\lambda, \]

\[ |\delta^\lambda(\varphi)| \leq K_6(\mu)\|\varphi - x_0\| \quad \text{for every } \varphi \in W(\mu). \quad (5.7) \]

In particular, we take \( K_5(\mu) \) sufficiently small and we use the smoothness of \( \delta^\lambda(\varphi) \) in \( \varphi \) to obtain \( K_6(\mu) \). By possibly decreasing \( K_5(\mu) \) further, we may also assume that there exists \( K_9(\mu) > 0 \) such that

\[ \|Z(t, \varphi) - x_t\| \leq K_9(\mu)\|\varphi - x_0\| \quad \text{if } \varphi \in W(\mu) \text{ and } 0 \leq t \leq np + K_5(\mu)K_9(\mu). \quad (5.8) \]

Indeed, the fact that \( Z(t, \varphi) \) is \( C^1 \) on the set \( D_\nu \) of Proposition 2.3 ensures that the first inequality in (5.8) holds at least for \( \varphi \in W(\mu) \) and \( 2R \leq t \leq \max\{2R, np + K_5(\mu)K_9(\mu)\} \), provided that the radius \( K_5(\mu) \) of \( W(\mu) \) is small enough. Further, this inequality holds also at \( t = R \) for small enough \( K_5(\mu) \), as \( Z(t, \varphi) \) is \( C^1 \) in \( \varphi \) for fixed \( t = R \). But now with the first inequality in (5.8) holding at both \( t = R \) and \( t = 2R \), and assuming without loss that \( K_9(\mu) \geq 1 \) so that it also holds at \( t = 0 \), it is elementary to see that (5.8) holds throughout the interval \( 0 \leq t \leq 2R \) since (roughly speaking) the segment \( z_t = Z(t, \varphi) \in S \subseteq C^1[-R, 0] \) is composed of pieces taken from the segments \( z_0 = \varphi, z_R, \) and \( z_{2R} \). Thus (5.8) holds for all \( \varphi \) and \( t \) as stated.

Now suppose that \( \varphi \in W(\mu) \) for the initial condition of \( z(t) \). Then \( \varphi \in V^\lambda \), and so by Proposition 5.1 and the fact that \( W(\mu) \) is a ball, we have for every \( k \geq 1 \) that \( \Gamma^{kn}(\varphi) \in W(\mu) \) with

\[ \|\Gamma^{kn}(\varphi) - x_0\| \leq \lambda^{kn}\|\varphi - x_0\| = e^{-\mu kn}\|\varphi - x_0\|. \quad (5.9) \]

In particular, the solution \( z(t) \) exists for all \( t \geq 0 \). Further, we have from Proposition 5.1 that

\[ \Gamma^{kn}(\varphi) = Z(t_k, \varphi) = z_{t_k}, \quad t_k = knp + \delta^\lambda(\varphi, k), \quad \delta^\lambda(\varphi, k) = \sum_{j=0}^{k-1} \delta^\lambda(\Gamma^{jn}(\varphi)), \quad (5.10) \]

for every \( k \geq 1 \), as is easily proved by induction on \( k \), and where the above formulas define \( t_k \) and \( \delta^\lambda(\varphi, k) \). Let us also define \( t_0 = 0 \), and note that \( t_k < t_{k+1} \) for every \( k \geq 0 \). Now set

\[ \theta = -\lim_{k \to \infty} \delta^\lambda(\varphi, k), \quad (5.11) \]

where the estimate

\[ |\delta^\lambda(\Gamma^{jn}(\varphi))| \leq K_8(\mu)\|\Gamma^{jn}(\varphi) - x_0\| \leq K_9(\mu)e^{-\mu jnp}\|\varphi - x_0\| \quad (5.12) \]

follows from (5.7) and (5.9) and ensures the convergence of the limit (5.11). In fact, upon summing the right-hand side of (5.12) we obtain the estimates
It is thus enough that we establish the first inequality of (5.6) in each
implies that
there exists
holds. Given
rem. However, the initial condition
as in the statement of the theorem.
we have from (5.16) that for
every
and combining this with (5.15) gives
Then (5.8) applied with the initial point
We next refine the bounds (5.6) by showing that there exist
such that
in the above formula vis-à-vis the formula (5.8).) Also, from (5.5), (5.10), and (5.13) we have that
and combining this with (5.15) gives
for every
with the above equation defining
Observing further from (5.13) that
we have from (5.16) that for

to give the desired result (5.6) with
We next refine the bounds (5.6) by showing that there exist
such that the following holds. Given
as in (1.14), there exists
such that if
and
then there exists
such that
for all
Here
and
are independent of
as are
and
in the statement of the theorem. However, the initial condition
is still compared only to
rather than to the more general
as in the statement of the theorem.
To prove this claim, let us first fix some $\mu > 0$ satisfying (1.14). The quantity $\mu$ will stay fixed for the remainder of the proof, while the quantity $\mu$ as before is allowed to take on any positive value satisfying (1.14). Let us also assume, without loss, that

$$K_5(\mu) = K_6(\mu) < p$$

holds for every such $\mu$. This is accomplished for each $\mu$ by decreasing $K_5(\mu)$ sufficiently. With $\mu$ now fixed, set

$$K_{11} = K_5(\mu), \quad K_{12} = K_6(\mu),$$

and assume that $\varphi \in S$ satisfies $\|\varphi - x_0\| \leq K_{11}$ as in the claim. Then from the earlier and now established claim (5.6) in this proof, we have that

$$\|z_t - x_{t+\theta}\| \leq K_7(\mu)e^{-\mu t}\|\varphi - x_0\|$$

for all $t \geq 0$, for some $\theta \in \mathbb{R}$ satisfying the second inequality in (5.17). Now given any $\mu > 0$ satisfying (1.14), set

$$K_{14}(\mu) = \max \left\{ 0, \frac{1}{\mu} \log \left( \frac{K_7(\mu)K_{11}}{K_5(\mu)} \right) \right\}.$$

We see from (5.19) that if $t \geq K_{14}(\mu)$ then

$$\|z_t - x_{t+\theta}\| \leq K_7(\mu)e^{-\mu K_{14}(\mu)}\|\varphi - x_0\| \leq \frac{K_5(\mu)\|\varphi - x_0\|}{K_{11}} \leq K_5(\mu).$$

Further, let

$$K_{15}(\mu) = 1 + \left[ \frac{K_{11}K_{12} + K_{14}(\mu)}{p} \right], \quad t_* = K_{15}(\mu)p - \theta.$$

Here $[\cdot]$, for any $c \in \mathbb{R}$, denotes the unique integer $m$ satisfying $m \leq c < m + 1$. Then from the second inequality in (5.17) we have that $t_* \geq K_{15}(\mu)p - K_{11}K_{12} \geq K_{14}(\mu)$, and so from (5.20) we have

$$\|z_{t_*} - x_0\| = \|z_{t_*} - x_{t_*+\theta}\| \leq K_5(\mu).$$

in particular because $t_* + \theta$ is an integer multiple of the period $p$. Upon letting $\widetilde{z}(t) = z(t + t_*)$, and so $\|\widetilde{z}_0 - x_0\| \leq K_5(\mu)$, we have from the earlier claim (5.6) but for $\widetilde{z}(t)$, and then from (5.19), that

$$\|z_{t+t_*} - x_{t_*+\tilde{\theta}}\| = \|\widetilde{z}_t - x_{t_*+\tilde{\theta}}\| \leq K_7(\mu)e^{-\mu t}\|z_{t_*} - x_0\|$$

$$= K_7(\mu)e^{-\mu t}\|z_{t_*} - x_{t_*+\theta}\| \leq K_7(\mu)K_7(\mu)e^{-\mu t-\tilde{\theta}t_*}\|\varphi - x_0\|$$

for all $t \geq 0$, for some $\tilde{\theta} \in \mathbb{R}$ satisfying

$$|\tilde{\theta}| \leq K_6(\mu)\|z_{t_*} - x_0\| \leq K_5(\mu)K_6(\mu) < p,$$

by (5.6) and (5.22), and by (5.18). We have next that
\[
\| x_t - x_{t+\hat{\theta}} \| = \| x_{t+t_\theta} - x_{t} \| \leq \| x_{t+t_\theta} + \theta - z_{t+t_\theta} \| + \| z_{t+t_\theta} - x_{t+\hat{\theta}} \| \to 0
\]  
(5.25)

as \( t \to \infty \), by (5.19) with \( t \) replaced with \( t + t_\theta \) and by (5.23), and by again noting that \( t_\theta + \hat{\theta} \) is an integer multiple of \( p \). It follows from (5.25) that \( \hat{\theta} \) must be an integer multiple of \( p \), and thus \( \hat{\theta} = 0 \) by (5.24). Let us also note the upper bound \( t_\theta \leq (K_{15}(\mu) + 1)p \), which holds by (5.21) and because \( |\theta| \leq K_{11}K_{12} = K_5(\bar{\mu})K_6(\bar{\mu}) < p \). We see from these facts and from (5.23), wherein we replace \( t \) with \( t - t_\theta \), that

\[
\| z_t - x_{t+\theta} \| = \| z_t - x_{t-t_\theta} \| \leq K_7(\mu)K_7(\bar{\mu})e^{-\mu(t-t_\theta)-\bar{\theta}t_\theta}\| \phi - x_0 \|
\leq K_7(\mu)K_7(\bar{\mu})e^{\mu(K_{15}(\mu)+1)p-\mu t}\| \phi - x_0 \| = K_{16}(\mu)e^{-\mu t}\| \phi - x_0 \|
\]  
(5.26)

for \( t \geq t_\theta \), where the final equality above serves as the definition of \( K_{16}(\mu) \). Next, for \( 0 \leq t \leq t_\theta \) we have from (5.19) and again from the upper bound on \( t_\theta \), that

\[
\| z_t - x_{t+\theta} \| \leq K_7(\bar{\mu})e^{\mu(K_{15}(\mu)+1)p-\mu t}\| \phi - x_0 \|
\leq K_7(\bar{\mu})e^{\mu(K_{15}(\mu)+1)p-\mu t}\| \phi - x_0 \| = K_{17}(\mu)e^{-\mu t}\| \phi - x_0 \|.
\]  
(5.27)

with the final equality above serving to define \( K_{17}(\mu) \). Combining (5.26) and (5.27) and letting

\[
K_{13}(\mu) = \max\{ K_{16}(\mu), K_{17}(\mu) \},
\]

we have that the first inequality in (5.17) holds for all \( t \geq 0 \), as claimed.

To complete the proof of the theorem, let \( K_1, K_{18} > 0 \) be such that

\[
\| Z(t, \phi) - x_{t+\sigma} \| \leq K_{18}\| \phi - x_\sigma \| \quad \text{if } 0 \leq t \leq p, \quad \text{and}
\]

\( \phi \in \mathcal{S} \) is such that \( \| \phi - x_\sigma \| \leq K_1 \) for some \( \sigma \in \mathbb{R} \),

(5.28)

and also

\[
K_1K_{18} \leq K_{11}.
\]  
(5.29)

The proof of the existence of such \( K_1 \) and \( K_{18} \) follows the same lines as the proof of (5.8). Suppose now that \( \phi \in \mathcal{S} \) satisfies \( \| \phi - x_\sigma \| \leq K_1 \) for some \( \sigma \in \mathbb{R} \), and again let \( z(t) \) be the solution with initial condition \( \phi \). Also, fix any \( \mu > 0 \) satisfying (1.14), as in the statement of the theorem. Assuming without loss that \( 0 \leq \sigma < p \), we have from (5.28) that

\[
\| z_t - x_{t+\sigma} \| \leq K_{18}\| \phi - x_\sigma \| \leq K_{18}e^{\mu(p-t)}\| \phi - x_\sigma \|
\]  
(5.30)

for \( 0 \leq t \leq p - \sigma \). In particular, at \( t = p - \sigma \) we have, using (5.29), that

\[
\| z_{p-\sigma} - x_0 \| \leq K_{18}\| \phi - x_\sigma \| \leq K_1K_{18} \leq K_{11}.
\]

Thus from (5.17), but with initial condition \( z_{p-\sigma} \) there, there exists \( \theta \in \mathbb{R} \) such that for \( t \geq 0 \) we have

\[
\| z_{t+p-\sigma} - x_{t+\theta} \| \leq K_{13}(\mu)e^{-\mu t}\| z_{p-\sigma} - x_0 \| \leq K_{13}(\mu)K_{18}e^{-\mu t}\| \phi - x_\sigma \|,
\]

\[ |\theta| \leq K_{12}\| z_{p-\sigma} - x_0 \| \leq K_{12}K_{18}\| \phi - x_\sigma \|.
\]  
(5.31)
Upon setting $K_2 = K_{12} K_{18}$, we see from this that the second formula in (1.15) holds. Also, for $t \geq p - \sigma$ we have that

$$
\|z_t - x_{t+\sigma + \vartheta}\| \leq K_{13}(\mu) K_{18} e^{\mu (p - \sigma - t)} \|\varphi - x_{\sigma}\| \leq K_{13}(\mu) K_{18} e^{\mu (p - t)} \|\varphi - x_{\sigma}\|. \tag{5.32}
$$

Finally, from (5.30) and (5.31), and using (5.5), we have for $0 \leq t \leq p - \sigma$ that

$$
\|z_t - x_{t+\sigma + \vartheta}\| \leq \|z_t - x_{t+\sigma}\| + \|x_{t+\sigma} - x_{t+\sigma + \vartheta}\|
\leq K_{18} e^{\mu (p - t)} \|\varphi - x_{\sigma}\| + K_4 |\vartheta| \leq (1 + K_4 K_{12}) K_{18} e^{\mu (p - t)} \|\varphi - x_{\sigma}\|. \tag{5.33}
$$

Upon setting $K_3(\mu) = K_{18} e^{\mu R} \max\{K_{13}(\mu), 1 + K_4 K_{12}\}$, the desired conclusion (1.15) of the theorem follows directly from (5.32) and (5.33).

**Proof of Corollary 1.2.** Let $Q = \{x_t \mid t \in \mathbb{R}\} \subseteq S$ denote the periodic orbit, which is a compact subset of $S \subseteq Y$, and let $B'$ and $B''$ be as in condition (H1'). Take any $\varphi \in X$ satisfying

$$
\|\varphi - x_{\sigma}\|_X \leq K_1' \quad \text{with} \quad K_1' = \min\{B'', K_1, K_1 (B')^{-1}\} e^{-B' R}
$$

for some $\sigma \in \mathbb{R}$, where $K_1$ is as in the statement of Theorem 1.1 and where the above formula defines $K_1'$. Then $\|\varphi - x_{\sigma}\|_X < B''$ hence $\varphi \in U_X$ by (H1'). Let $z(t)$ be any solution of (1.5) with $z_0 = \varphi$ on its maximal interval, which we denote by $[0, \omega)$, where $0 < \omega \leq \infty$. (We recall that there is no assurance that such a solution is unique.) Also let $\omega_R = \min(\omega, R)$. We claim that

$$
\|z_t - x_{t+\sigma}\|_X \leq e^{B' t} \|\varphi - x_{\sigma}\|_X < e^{B' R} K_1' \tag{5.34}
$$

whenever $0 \leq t < \omega_R$. To prove this, let

$$
t_0 = \sup\{t \in [0, \omega_R) \mid \|z_{t_1} - x_{t_1+\sigma}\|_X < e^{B' R} K_1' \quad \text{for every} \ t_1 \in [0, t)\},
$$

noting that $\|z_0 - x_{\sigma}\|_X = \|\varphi - x_{\sigma}\|_X < e^{B' R} K_1'$. Then if $0 \leq t < t_0$ we have by (H1') that

$$
\left|z(t_1) - x(t_1 + \sigma)\right| \leq \|\varphi - x_{\sigma}\|_X + \int_0^{t_1} |f(z_s) - f(x_{s+\sigma})| \, ds
\leq \|\varphi - x_{\sigma}\|_X + B' \int_0^{t_1} \|z_s - x_{s+\sigma}\|_X \, ds
\leq \|\varphi - x_{\sigma}\|_X + B' \int_0^t \|z_s - x_{s+\sigma}\|_X \, ds.
\tag{5.35}
$$

provided that $0 \leq t_1 \leq t$. Letting

$$
\eta(t) = \sup_{s \in [-R, t]} |z(s) - x(s + \sigma)| = \sup_{s \in [0, t]} \|z_s - x_{s+\sigma}\|_X
$$

for such $t$, we have upon taking the supremum of the left-hand side of (5.35) for $-R \leq t_1 \leq t$ that
\[ \eta(t) \leq \| \varphi - x_\sigma \|_X + B' \int_0^t \eta(s) \, ds, \quad \text{hence } \eta(t) \leq e^{B't} \| \varphi - x_\sigma \|_X < e^{B'R} K_1' \]

by Gronwall’s inequality. If \( t_0 < \omega_R \) the above inequality is valid for \( 0 \leq t \leq t_0 \), however, this contradicts the choice of \( t_0 \). Thus \( t_0 = \omega_R \) and our claim, that (5.34) holds whenever \( 0 \leq t < \omega_R \), follows directly. From this we see further that if \( 0 \leq t < \omega_R \) then

\[
|\dot{z}(t) - \dot{x}(t + \sigma)| = |f(z_t) - f(x_{t+\sigma})| \leq B' \| z_t - x_{t+\sigma} \|_X \leq B' e^{B't} \| \varphi - x_\sigma \|_X < B' e^{B'R} K_1'
\]

and thus the limit \( \lim_{t \to \omega_R} z(t) = z(\omega_R) \) exists. It follows that \( \omega_R < \omega \), that is, \( R < \omega \). Moreover, from (5.34) and (5.36) we conclude that

\[
\| z_R - x_{R+\sigma} \|_Y \leq K_4' \| \varphi - x_\sigma \|_X \leq K_1' K_4' \leq K_1, \quad K_4' = \max\{1, B'\} e^{B'R},
\]

where the above equality serves as the definition of \( K_4' \). Thus from Theorem 1.1, and with any \( \mu \) as in the statement of that result, we have

\[
\| z_t - x_{t+\sigma+\delta} \|_Y \leq K_3(\mu) e^{-\mu(t-R)} \| z_R - x_{R+\sigma} \|_Y \leq K_3(\mu) K_4' e^{-\mu(t-R)} \| \varphi - x_\sigma \|_X,
\]

\[
|\theta| \leq K_2 \| z_R - x_{R+\sigma} \|_Y \leq K_2 K_4' \| \varphi - x_\sigma \|_X.
\]

(5.37)

for \( t \geq R \). Also, for \( 0 \leq t \leq R \) we have that

\[
\| z_t - x_{t+\sigma+\delta} \|_X \leq \| z_t - x_{t+\sigma} \|_X + \| x_{t+\sigma} - x_{t+\sigma+\delta} \|_X \leq e^{B't} \| \varphi - x_\sigma \|_X + K_4 |\theta| \\
\leq (e^{B'R} + K_2 K_4 K_4') \| \varphi - x_\sigma \|_X \leq (e^{B'R} + K_2 K_4 K_4') e^{-\mu(t-R)} \| \varphi - x_\sigma \|_X
\]

(5.38)

from (5.34) and (5.37), and where \( K_4 \) is as in (5.4). Thus from (5.37) and (5.38), we see that the desired conclusions (1.16) of the corollary hold with \( K_2' = K_2 K_4' \) and \( K_3'(\mu) = \max\{K_3(\mu) K_4' e^{\mu R}, (e^{B'R} + K_2 K_4 K_4') e^{\mu R} \} \). □

References