On What We Don't Know (About List Coloring)

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Introduction

There's "normal" graph coloring:

Def. A graph is *k-colorable* if there is a function $c : V(G) \to \{1, \ldots, k\}$ such that

$$v \sim w \Rightarrow c(v) \neq c(w)$$

Then we can define the *chromatic number* as

$$\chi(G) = \min\{k \mid G \text{ is } k\text{-colorable.}\}$$

Introduction

Then there is *list coloring*:

Def. A *list assignment* for a graph G is an assignment of a list L_v (usually a subset of $\mathbb{N}$) to each vertex $v \in G$.

Let

$$\mathcal{L} = \{L_v \mid v \in V(G)\}$$

and we define the *palette* as

$$P_{\mathcal{L}} = \bigcup_{v \in V(G)} L_v$$

We then say that G is $\mathcal{L}$-choosable if there is a function $c : V(G) \to P_{\mathcal{L}}$ such that

$$v \sim w \Rightarrow c(v) \neq c(w) \text{ and } c(v) \in L_v, c(w) \in L_w$$

List Coloring Definition

Def. For a function $f : V(G) \longrightarrow \mathbb{N}$, we say that G is f-choosable if for *any* list assignment $\mathcal{L}$ satisfying $|L_v| = f(v)$ for all $v \in V(G)$, G is $\mathcal{L}$-choosable.

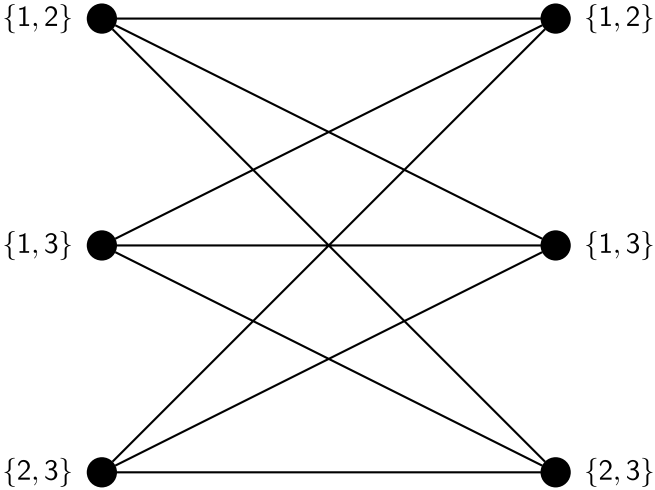
If $f \equiv k$ is a constant
function, then we say that G is k-choosable and say that $\chi_l(G) = k$.

Most of the interest so far in list coloring has dealt with k-choosability.

List Coloring *is* Different!

$\chi_l(G) \geq \chi(G)$ since "normal" coloring is equivalent to assigning the same list of colors to each vertex in the graph. However, notice:

The First Example, Always, With List Coloring



This list assignment shows that $\chi_l(K_{3,3}) = 3 \neq \chi(K_{3,3})$.

More Generally . . .

Fact.

$$\chi_l\left(K_{\binom{2n-1}{n},\binom{2n-1}{n}}\right) = n+1$$

Proof. We assign as lists on each side the n-subsets of $\{1, 2, \ldots, 2n-1\}$. Then we can color if and only if we use only $n-1$ colors on one side. However, for each choice of $n-1$ colors there is a
vertex that misses precisely those colors, and hence can't be colored.

Consequence: In general we cannot say anything about $\chi_l(G)$ given $\chi(G)$.

Conclusions for Planar Graphs

Theorem [Thomassen 1993]: Every planar graph is 5-choosable.

Theorem [Voigt 1993]: There are planar graphs that are not 4-choosable.

Voigt's example had

The smallest-known example of a non-4-choosable planar graph has 75 vertices [Gutner 1996].

What's Different About List Coloring?

There are some obvious statements about "normal" coloring whose list-coloring counterparts aren't so obvious. For example,

Obvious Fact. If $\chi(G) = t$ and $s < t$, then there is a subgraph $H \subseteq G$ such that

$$|V(H)| \geq \frac{s}{t}|V(G)|$$

and $\chi(H) = s$.

Proof. Color G with t colors and select the s largest color classes as H.

Conjecture 1: Albertson, Haas, Grossman [2000]

If $\chi_l(G) = t$ and $\mathcal{L}$ is a family of assignments where each vertex is assigned a list L_v of s colors ($s < t$), then there is a subgraph $H \subseteq G$ such that

$$|V(H)| \geq \frac{s}{t}|V(G)|$$

and H is $\mathcal{L}$-choosable.

Note: The more direct analogue is *not* true: there are graphs G with $\chi_l(G) = t$ and $s < t$ such that there are no subgraphs $H \subseteq G$ with $\chi_l(H) = s$ satisfying

$$|V(H)| \geq \frac{s}{t}|V(G)|$$

Progress on Conjecture 1

Theorem: If $s|t$, then the conjecture is true.

Proof: For sake of clarity, let $s = 2$ and $t = 4$. Each vertex $v \in G$ is given a list of two colors $L_v = \{a_v, b_v\}$. Append doppelgänger colors a'_v and b'_v to each list, so each new list is $L'_v = \{a_v, b_v, a'_v, b'_v\}$. If $\mathcal{L}'$ is the family of new lists, then G is $\mathcal{L}'$-choosable.

Progress of Conjecture 1 (Continued)

Color G using $\mathcal{L}'$.

Now, for each color c in the palette, some vertices may have been colored c and some may have been colored c'. Let V_c be the bigger of those two sets of vertices. Finally, let

$$H = \bigcup_{c \in P_{\mathcal{L}}} V_c$$

and notice that each vertex in H colored by a doppelgänger can be re-colored with its original color.

More Progress

Theorem [Chappell 1999]: If $\chi_I(G) = t$ and $s < t$ then there is a subgraph H with the required properties such that

$$|V(H)| \geq \frac{6}{7}\left(\frac{s}{t}|V(G)|\right)$$

Chappell's proof is based on simple probabilistic arguments.

The rest of the conjecture is still wide open. Even the case of $s = 2, t = 3$ remains a mystery.

Another Direction: Graphs where $\chi_l(G) = \chi(G)$.

The following graphs are known to satisfy $\chi_l(G) = \chi(G)$:

- (Galvin 1995) Line graphs of bipartite graphs.
- (Gravier, Maffray 1995) Complements of triangle-free graphs.
- (Ohba 2001) Graphs satisfying $|V(G)| \leq \chi(G) + \sqrt{2\chi(G)}$.
- (Reed, Sudakov 2005) Graphs satisfying $|V(G)| \leq \frac{5}{3}\chi(G) - \frac{4}{3}$.

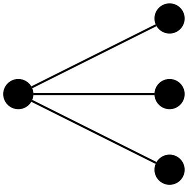
Hard Conjecture Number 1

Conjecture [Vizing 1976]: Every line graph satisfies $\chi_l(G) = \chi(G)$.

This conjecture is important enough to be called *The List Coloring Conjecture*.

Hard Conjecture Number 2

Conjecture [Gravier, Maffray 1997]: Every claw-free graph satisfies $\chi_l(G) = \chi(G)$.



Note that this conjecture is more general than hard conjecture number 1, and many people believe it is so general as to actually be false.

Ohba's Conjecture

Conjecture [Ohba 2001]: If $|V(G)| \leq 2\chi(G) + 1$ then $\chi_l(G) = \chi(G)$.

For Ohba's Conjecture it suffices to consider only complete partite graphs where equality holds.

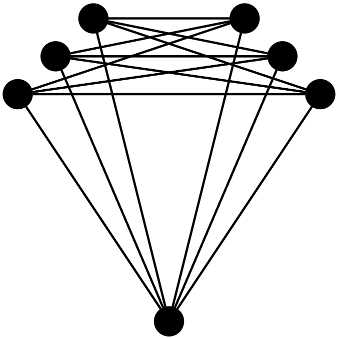
Complete Partite Graph Notation

Definition: $K(a_1, a_2, \ldots, a_k)$ is the complete k-partite graph with a_i vertices in part i. Usually we write it so $a_1 \geq a_2 \geq \cdots \geq a_k$. If there are repetitions, we also write as shorthand

$$K(a_1 * n_1, a_2 * n_2, \ldots, a_k * n_k)$$

Complete Partite Graph Example

So, for example, the following graph is $K(3,3,1) = K(3*2,1)$:



Motivation: The graph $G = K(4, 2*(k-1))$ satisfies $\chi(G) = k$, $|V(G)| = 2k+2$, and $\chi_I(G) = k+1$ iff k is even!

Progress Towards Ohba's Conjecture

Graphs for which Ohba's Conjecture is true:

- (Erdős, Rubin, Taylor 1979) $K(2 * k)$.
- (Gravier, Maffray 1998) $K(3, 3, 2 * (k - 2))$.
- (Enomoto, Ohba, Ota, Sakamoto 2002) $K(4, 2 * (k - 2), 1)$.
- (Cranston 2007) G such that $\alpha(G) = 3$, or G with one part of size 4.
- (Shen, He, Zheng, Wang, Zhang 2007) $K(5, 3, 2 * (k - 5), 1 * 3)$.
- (Enomoto, Ohba, Ota, Sakamoto 2002) $K(m, 2 * (k - s - 1), 1 * s)$ for $m \leq 2s - 1$.

Machinery (Old)

The following ideas are used heavily in the previous results:

1. (Hall 1935) If $G = (A, B)$ is a bipartite graph such that $|N(S)| \geq |S|$ for all $S \subseteq A$, then there is a matching that saturates A.

Machinery (New)

2. (Kierstead 2000) Let G be given with list assignment $\mathcal{L}$. Let X be a maximal set of vertices so that

$$|L(X)| := \left| \bigcup_{v \in X} L_v \right| < |X|$$

Then if X is $\mathcal{L}|_X$-choosable, then G is $\mathcal{L}$-choosable.

3. (Kierstead 2000, Reed, Sudakov 2001) If G is $\mathcal{L}$-choosable for all list assignments such that $|L_v| = k$ and $|P_{\mathcal{L}}| < |V(G)|$, then $\chi_l(G) \leq k$.

Where To Go From Here

Chappell's result suggests that the conjecture of Albertson, et. al. is true.

Ambiguous Philosophical Thought: Most results concerning Ohba's Conjecture rely on *heavy* case analysis. Can it be avoided?

Example Of What I'd Like To See More Of

Lemma. $K(4, 3, 1, 1)$ is 4-choosable.

Proof. From the machinery mentioned earlier, it suffices to consider when the palette has at most 8 colors. If that is the case, then there is a set C of at least 4 colors such that for each color $c \in C$, there are at least two vertices in the 4-set that has c in their list.

Case to Always Exclude: If there is a color that is shared by all the vertices of the 4-set or the 3-set, then use that color and you're in a much easier situation.

Example Of What I'd Like To See More Of (Continued)

Case to Exclude: If both singleton vertices have the same list of colors, and that list is also the same as some vertex in the 3-set, then we can color everything.

Now, take a color $c \in C$, and WLOG there are two vertices, v_1 and v_2, in the 3-set that have c in their list. Since we've excluded the singleton lists being equal and equal to a vertex in the 3-set, there is a choice of colors to color the singletons so that the remaining two vertices in the 4-set and the 3-set still have two valid colors remaining. So - what's left if $K(3, 2)$, which we know is 2-choosable.

Finally . . .

Thank you for listening!