Probabilistic proofs of hook length formulas involving trees

by

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Let T be a rooted tree with n distinguishable vertices. We use T to stand for the vertex set of T. An *increasing labeling* of T is a bijection $\ell : T \to \{1, 2, \ldots, n\}$ such that $\ell(v) \leq \ell(w)$ for all descendents w of v. Let f^T be the number of increasing labelings. The *hooklength*, h_v, of a vertex v is the number of descendents of v (including v itself). The hook length formula for trees states that

$$f^T = \frac{n!}{\prod_{v \in T} h_v}.$$

There is a similar formula for the number of standard Young tableaux of given shape where a hooklength is the cardinality of a set which resembles a physical hook. Greene, Nijenhuis, and Wilf gave a beautiful probabilistic proof of the tableau formula where the hooklenths enter in a very natural way.

Recently, Han discovered a formula which has the interesting property that hooklengths appear as exponents. Specifically, let $\mathcal{B}(n)$ be the set of all n-vertex binary trees (each vertex has no children, a left child, a right child, or both children). Han proved that

$$\sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \frac{1}{h_v 2^{h_v - 1}} = \frac{1}{n!}$$

using algebraic manipulations. We will show how to give a simple probabilistic proof of this equation as well as various generalizations. We will also pose some open questions raised by this work.