Automated Proof and Discovery in Three Combinatorial Problems

Ph.D. Thesis Defense

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August 14, 2009
For Partial Fulfillment of Ph.D. Requirements

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Outline

1. Avoiding Differences
   - Introduction
   - Definition and Recurrence
   - Asymptotic Triangle Conjecture

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Avoiding Differences

Starting and ending with The Triangle Conjecture

- We will investigate the quantity $f_\Delta(n)$, defined by

$$f_\Delta(n) = \max\{|X| \mid X \subseteq [n] \text{ and } X \text{ avoids differences in } \Delta\}.$$

- Motivated by the Triangle Conjecture of Schützenberger and Perrin.

Result Enumeration schemes for computing $\{f_\Delta(n)\}|_{n=1}^{\infty}$ and *proving* its behavior.

Result An asymptotic version of the Triangle Conjecture.

Spanning Trees in Grid Graphs

Graphs of the form $G \times P_n$ or $G \times C_n$

- We will extend the methods used by Desjarlais and Molina to compute the sequence $\{\tau_G(n)\}_{n=1}^{\infty}$, where

$$\tau_G(n) = \text{number of spanning trees of } G \times P_n.$$

- Enumeration schemes that calculate and *prove* full information:
  - Recurrence
  - Generating function
  - Closed-form formula.

Result Spanning tree sequences are *divisibility sequences*.

The Firefighter Problem

On $\mathbb{Z} \times \mathbb{Z}$

- We will introduce the problem.
- We will discuss results pertaining to an upper bound on the number of firefighters needed to contain the fire in the two-dimensional grid $\mathbb{Z} \times \mathbb{Z}$.
- We will discuss a partial result regarding the sharpness of this upper bound.

Outline

Introduction

The Triangle Conjecture

- The Triangle Conjecture deals with *codes*, which are subsets C of the set

$$\mathcal{A}_m = \{x^i y x^j \mid i + j < m\}$$

where $C^\star$ exhibits unique factorization.

Example

$\{y, xy, yx\} \subseteq \mathcal{A}_2$ is *not* a code, for

$$yxy = y \cdot xy$$
$$yxy = yx \cdot y$$

Conjecture (Schützenberger-Perrin 1980)

If $C \subseteq \mathcal{A}_m$ is a code, then $|C| \leq m$.

- Shortly afterward, Shor exhibited a code $C \subseteq \mathcal{A}_{15}$ with 16 elements.

Important Point

Shor's counterexample relied on finding large sets avoiding prescribed differences.

Definition

Generally, $f_\Delta(I;S)$ is the *size* of the largest subset of I that avoids differences in Δ and elements in S. Additionally, $f_\Delta(I) = f_\Delta(I;\emptyset)$ and $f_\Delta(n;S) = f_\Delta([n];S)$.

Lemma (R.)

If $1 \in S$ *then*

$$f_\Delta(n;S) = f_\Delta(n-1;S-1)$$

otherwise,

$$f_\Delta(n;S) = \max\{f_\Delta(n-1;S-1), 1 + f_\Delta(n-1;\Delta \cup (S-1))\}.$$

- Given Δ, S, the number of different parameters needed in the enumeration scheme to compute $f_\Delta(n; S)$ is finite.

Theorem (R.)

Given any finite Δ, S, the sequence $(f_\Delta(n; S))$ is eventually pseudoperiodic.

- A theorem-prover proving the structure of $(f_\Delta(n; S))$ has been implemented.

Definition

$$\mu(\Delta) = \lim_{n\to\infty} \frac{f_\Delta(n)}{n}.$$

$\mu(\Delta)$ is rational.

Theorem (R. - Asymptotic Version of Triangle Conjecture)

$$\mu(X - X) \leq \frac{1}{|X|}.$$

Proof.

Let $X = \{x_1, x_2, \ldots, x_k\}$ and consider

$$\begin{gathered}\{x_1 + 0, x_2 + 0, \ldots, x_k + 0\}\\ \{x_1 + 1, x_2 + 1, \ldots, x_k + 1\}\\ \{x_1 + 2, x_2 + 2, \ldots, x_k + 2\}\\ \vdots\end{gathered}$$

To avoid differences, we can only have one element from each set. Each $n \in \mathbb{N}$ is represented at most $k (= |X|)$ times in this family of sets. $\square$

Outline

History

- The Matrix Tree Theorem will compute the number of spanning trees of any graph G.
- The Matrix Tree Theorem does *not* provide any information about the number of spanning trees of an infinite family of graphs.

Definition

$\tau_G(n)$ is the number of spanning trees of $G \times P_n$.

History
(continued)

- Desjarlais and Molina created an enumeration scheme to compute $\tau_{P_2}(n)$.

- To compute $\tau_{P_2}(n)$, they also computed $\tau'_{P_2}(n)$, defined as the number of spanning *forests* of $P_2 \times P_n$ with the special property that the two vertices on the right end are in different components.

This yields the enumeration scheme

$$\tau_{P_2}(n) = 3\tau_{P_2}(n-1) + \tau'_{P_2}(n-1)$$
$$\tau'_{P_2}(n) = 2\tau_{P_2}(n-1) + \tau'_{P_2}(n-1)$$

History
(continued)

- From this, they deduced that $\tau_{P_2}(n)$ satisfies the recurrence

$$\tau_{P_2}(n) = 4\tau_{P_2}(n-1) - \tau_{P_2}(n-2)$$

with the initial conditions $\tau_{P_2}(1) = 1$, $\tau_{P_2}(2) = 4$. $\tau'_{P_2}(n)$ also satisfies the same recurrence but $\tau'_{P_2}(2) = 3$.

- We generalize and formalize this framework.

Contributions

- This can be extended to a formal framework if we consider the enumeration scheme that counts, for a graph G on n vertices, *all* values $\tau_G(n; P)$ for all partitions P of $[n]$.
- We compute, for all P and P', the number of different ways we can append edges to *transition* from a spanning tree represented in $\tau_G(n; P)$ to one represented in $\tau_G(n; P')$. This naturally yields a matrix, A_G.

Example

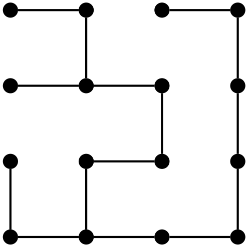
If $G = K_3$ our transition matrix is $\begin{bmatrix} 16 & 8 & 8 & 8 & 3 \\ 4 & 3 & 2 & 2 & 1 \\ 4 & 2 & 3 & 2 & 1 \\ 4 & 2 & 2 & 3 & 1 \\ 3 & 2 & 2 & 2 & 1 \end{bmatrix}$

Results

- Full sequence information for all graphs up to 5 vertices, with plans to find all information for all graphs on 6 vertices (about 25% complete).
- Interesting conjectures:
  - Coefficients of characteristic polynomial alternate in sign – suggests potential reformulation in terms of Inclusion-Exclusion.
  - Recurrence of minimum order for $P_k \times P_n$ has order 2^{k-1}.
  - Recurrence of minimum order for $K_k \times P_n$ has order k.

Spanning Tree Sequences are Divisibility Sequences

A spanning tree of $G \times P_{2n}$ can be split into three parts: a left tree, a right tree, and the middle edges.



COMP(P,MID) is the set of partitions that is compatible with the left-hand partition and the middle edges.

Lemma (Split-Merge Lemma)

$$\sum_{\text{MID} \in \binom{[\nu]}{k}} \sum_{P \in \mathcal{P}_\nu(p)} \tau_G(n;P) \sum_{P' \in \text{COMP}(P,\text{MID})} \tau_G(n;P')$$

$$=$$

$$\binom{k-1}{p-1} \tau_G(n) \sum_{P \in \mathcal{P}(e)} \left(\prod P\right) \tau_G(n;P).$$

$\tau_G(2n)$ is equal to the left-hand side of the above equation summed over all possible values of k and p.

Outline

Given a graph G, a fire is placed at a specified vertex and at discrete time intervals $t \geq 0$, $f(t)$ firefighters are placed on unoccupied vertices, and then the fire spreads to adjacent vertices that are not protected nor already on fire.

Question

If G is finite, can the fire be contained with vertices that are neither on fire nor protected? If G is infinite, can the fire be contained?

Theorem (Wang-Moeller)

If $f(t) = 1$, then no fire can be contained in the two-dimensional grid. If $f(t) = 2$, then any finite fire can be contained in the two-dimensional grid.

Theorem (Ng-R.)

If f is periodic and the average number of firefighters per turn is $1.5 + \varepsilon$, then any finite fire can be contained in the two-dimensional grid.

This theorem can be extended further to deal with non-periodic functions.

Theorem (R.)

The function f defined by

$$f(t) = \begin{cases} 3 & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even} \end{cases}$$

can not give a convex solution to a point fire in the two-dimensional grid.

Open Questions

- What is the behavior of the offset and period for the sequences $(f_\Delta(n))$?
- What's behind the structure of the characteristic polynomials of the transition matrices?
- Are non-convex solutions to the firefighter problem in $\mathbb{Z} \times \mathbb{Z}$ anything more than a pathology?

Thank you!

Questions?