Matrix Volume and its Applications

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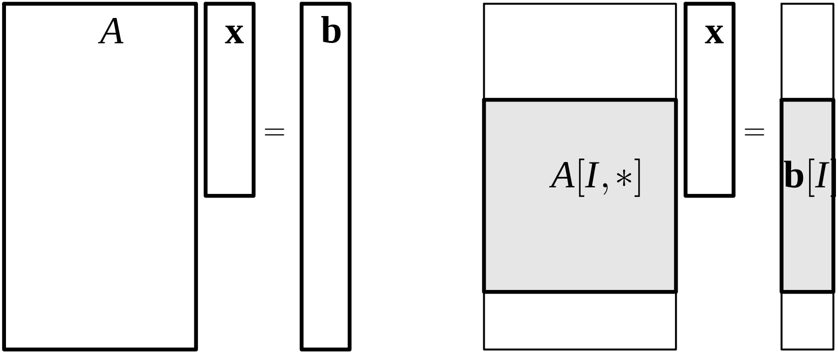
Outline

Least squares & basic solutions of $A\mathbf{x} = \mathbf{b}$, $A \in \mathbb{R}_r^{m\times r}$



Least squares solution $\mathbf{x}^* := \arg\min \|A\mathbf{x} - \mathbf{b}\|$

I–basic solution $\mathbf{x}_I := A[I,*]^{-1}\mathbf{b}[I]$.

A. Ben-Tal & M. Teboulle, [13]

Consider a system of linear equations

$$A\mathbf{x} = \mathbf{b}$$

$A \in \mathbb{R}_r^{m\times r}$, $\mathbf{b} \in \mathbb{R}^m$, and its least squares solution (LSS)

$$\mathbf{x}^* := \arg\min \|A\mathbf{x} - \mathbf{b}\|.$$

For any $I = \{i_1, i_2, \cdots, i_r\} \subset 1{:}m$, with $A[I,*]$ nonsingular, the I–basic solution is

$$\mathbf{x}_I := A[I,*]^{-1}\mathbf{b}[I].$$

Then LSS is a convex combination of the basic solutions

$$\mathbf{x}^* = \sum_I \lambda_I \mathbf{x}_I, \quad \lambda_I \propto \det{}^2 A[I,*]$$

$$\lambda_I = \frac{\det^2 A[I,*]}{\sum_J \det^2 A[J,*]}$$

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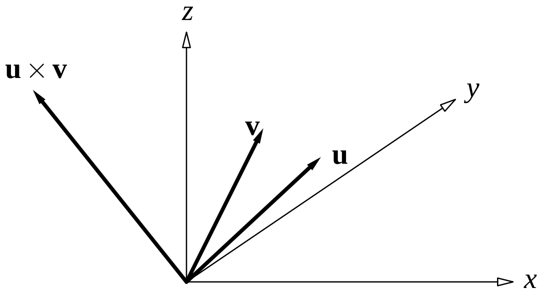
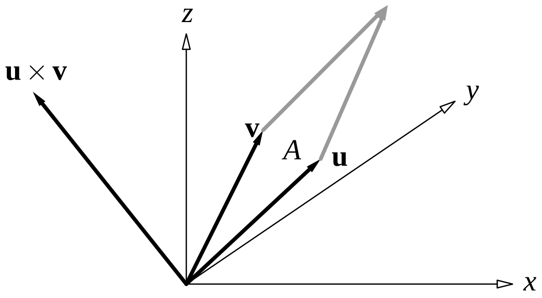
Why determinants? recall the cross product $\mathbf{u} \times \mathbf{v}$



Let $\mathbf{u} = (u_i), \mathbf{v} = (v_i) \in \mathbb{R}^3$, and the cross product

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\,\mathbf{i} + (u_1 v_3 - u_3 v_1)\,\mathbf{j} + (u_1 v_2 - u_2 v_1)\,\mathbf{k}.$$

$\mathbf{u} \times \mathbf{v}$ is the (signed) area of $\Diamond\{\mathbf{u}, \mathbf{v}\}$



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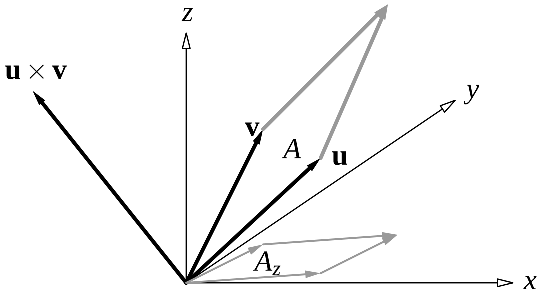
The area A of $\Diamond\{\mathbf{u}, \mathbf{v}\}$ is

$$A = \|\mathbf{u} \times \mathbf{v}\|$$

Why $\sum_J \det^2 A[J,*]$? A Pythagorean theorem for areas



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$$A = \|\mathbf{u}\times\mathbf{v}\| = \sqrt{A_x^2 + A_y^2 + A_z^2},$$ where $A_z = |\,u_1v_2 - u_2v_1\,|$

is the area of the projection of $\Diamond\{\mathbf{u},\mathbf{v}\}$ on the xy–plane, etc.

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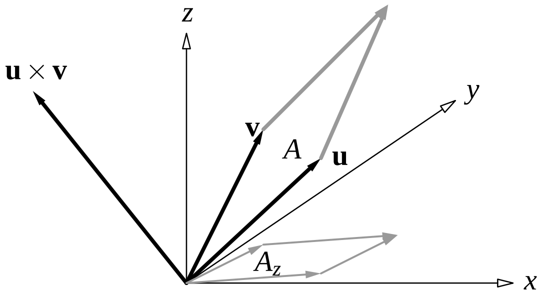

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MNLSS

Notation:

$$1{:}n := \{1, 2, \cdots, n-1, n\}$$

$$\mathbf{Q}(r,n) = \{I = \{i_1, \cdots, i_r\} \in 1{:}n \mid 1 \le i_1 < i_2 < \cdots < i_r \le n\}$$

$$\mathbf{I}(A) = \{I \in \mathbf{Q}(r,m) \mid \text{rank}\, A[I,*] = r\}$$

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$$\mathbf{M}(A) = \{(I,J) \in \mathbf{Q}(r,m) \times \mathbf{Q}(r,n) \mid \text{rank}\, A[I,J] = r\}$$

Let $A \in \mathbb{R}_r^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and consider the minimum norm least squares solution (MNLSS) $\mathbf{x}^*$ of

$$A\mathbf{x} = \mathbf{b}$$

For any $(I,J) \in \mathbf{M}(A)$, the solution $\mathbf{x}_{IJ}$ of

$$A[I,J]\mathbf{x} = \mathbf{b}[I]$$

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The MNLSS, $\mathbf{x}^* = A^\dagger \mathbf{b}$, is a convex combination of the $\mathbf{x}_{IJ}$'s

$$\mathbf{x}^* = \sum_{(I,J)\in\mathbf{M}(A)} \lambda_{IJ}\, \widehat{A[I,J]}^{-1}\, \mathbf{b}[I]$$

with convex weights proportional to $\det{}^2 A[I,J]$,

$$\lambda_{IJ} = \frac{\det{}^2 A[I,J]}{\sum_{(K,L)\in\mathbf{M}(A)} \det{}^2 A[K,L]}$$

Theorem 1

If $A \in \mathbb{R}_r^{m\times n}$ then

$$A^\dagger = \sum_{(I,J)\in\mathbf{M}(A)} \lambda_{IJ}\, \widehat{A[I,J]}^{-1}$$

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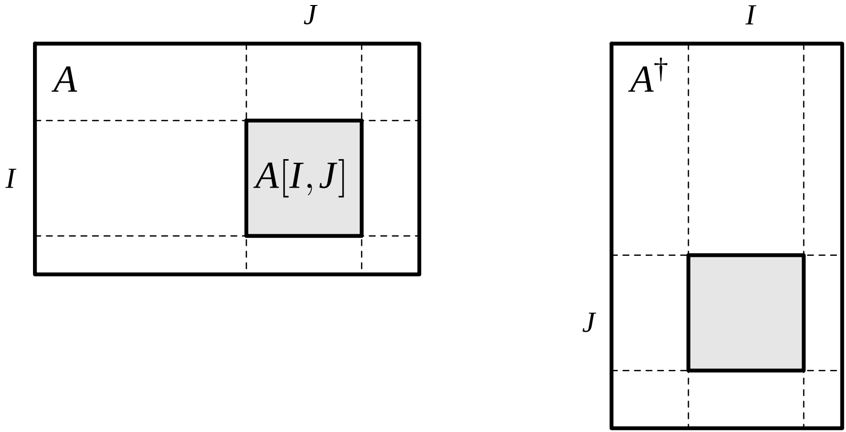
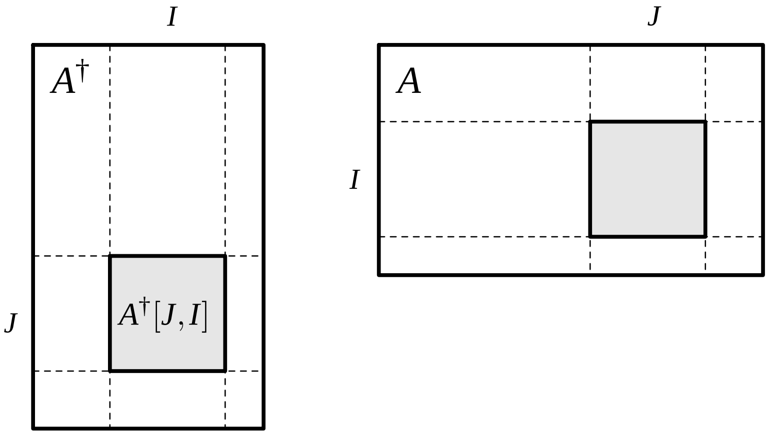


$A = A^{\dagger\dagger} = \sum_{(J,I)\in\mathbf{M}(A^\dagger)} \lambda_{JI}\,\widehat{A^\dagger[J,I]^{-1}}, \quad \lambda_{JI} \propto \det{}^2 A^\dagger[J,I]$



Outline

Definition 2

Let $A \in \mathbb{R}_r^{m\times n}$. The volume of A, $\mathrm{vol}\,(A)$, is defined as

$$\mathrm{vol}\,(A) := \begin{cases} 0, & r = 0; \\ \sqrt{\sum_{(I,J)\in \mathbf{M}(A)} \det{}^2 A[I,J]}, & r > 0. \end{cases}$$

Definition 3

Equivalently, if $r > 0$,

$$\mathrm{vol}\,(A) := \prod_{i=1}^{r} \sigma_i$$

the product of the singular values of A.

Definition 2 is applicable also to non–numerical matrices, e.g. Jacobians.

Definitions of $\text{vol}\,(A)$, $A \in \mathbb{R}_r^{m\times n}$

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Why volume?

Given $A \in \mathbb{R}_r^{m\times n}$, every unit cube in $\mathbf{R}(A^T)$ is mapped by A into a parallelepiped in $\mathbf{R}(A)$ of volume

$$\mathrm{vol}\,(A).$$

Given $A \in \mathbb{R}_r^{m\times n}$, every ball of volume 1 in $\mathbf{R}(A^T)$ is mapped by A into an ellipsoid in $\mathbf{R}(A)$ of volume

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Special cases

If $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\} \subset \mathbb{R}^n$ are l.i., the (signed) volume of the parallelepiped $\Diamond\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ is given by the determinant

$$\operatorname{vol}(\Diamond\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}) = \det W, \quad W = (\mathbf{w}_1, \cdots, \mathbf{w}_n),$$

$$\text{and} \quad \operatorname{vol}(W) = |\det W|.$$

If $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\} \subset \mathbb{R}^n$ are l.i., $k \leq n$, the volume of the parallelepiped $\Diamond\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ is given by the Gram determinant,

$$\operatorname{vol}^2(\Diamond\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}) = \det W^T W, \quad W = (\mathbf{w}_1, \cdots, \mathbf{w}_k),$$

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Outline

Full rank factorizations

A full rank factorization (FRF) of $A \in \mathbb{R}_r^{m\times n}$ is

$$A = CR,\ C \in \mathbb{R}_r^{m\times r},\ R \in \mathbb{R}_r^{r\times n}$$

If $A = CR$ is a FRF,

$$\mathbf{I}(A) = \mathbf{I}(C)$$
$$\mathbf{J}(A) = \mathbf{J}(R)$$
$$\mathbf{M}(A) = \mathbf{I}(A) \times \mathbf{J}(A)$$

$(I \in \mathbf{I}(A), J \in \mathbf{J}(A) \implies A[I,J] = C[I,*]R[*,J]$ nonsingular)

Examples: SVD, QR, CUR
The results below hold for any FRF.

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Examples: SVD, QR, CUR
The results below hold for any FRF.

The volume of $A \in \mathbb{R}_r^{m\times n}$ is $\quad \text{vol}\,(A) := \sqrt{\sum_{(I,J)\in\mathbf{M}(A)} \det{}^2 A[I,J]}$

Theorem 4

If $A \in \mathbb{R}_r^{m\times n}$, $r > 0$, and $A = CR$ is any FRF, then

$$\text{vol}\,(A) = \text{vol}\,(C)\,\text{vol}\,(R)$$

Proof.

$$\begin{aligned}
\text{vol}^2(A) &= \sum_{(I,J)\in\mathbf{M}(A)} \det{}^2 A[I,J] = \sum_{(I,J)\in\mathbf{M}(A)} \det{}^2 C[I,*]R[*,J] \\
&= \sum_{(I,J)\in\mathbf{M}(A)} \det{}^2 C[I,*] \det{}^2 R[*,J] \\
&= \left(\sum_{I\in\mathbf{I}(A)} \det{}^2 C[I,*]\right)\left(\sum_{J\in\mathbf{J}(A)} \det{}^2 R[*,J]\right) \\
&= \text{vol}^2(C)\,\text{vol}^2(R).
\end{aligned}$$

Volume of FRF

The volume of $A \in \mathbb{R}_r^{m\times n}$ is $\operatorname{vol}(A) := \sqrt{\sum_{(I,J)\in\mathbf{M}(A)} \det{}^2 A[I,J]}$

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If $A \in \mathbb{R}_r^{m\times n}$, $r > 0$, and $A = CR$ is any FRF, then

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Proof.

$$\begin{aligned}\operatorname{vol}^2(A) &= \sum_{(I,J)\in\mathbf{M}(A)} \det{}^2 A[I,J] = \sum_{(I,J)\in\mathbf{M}(A)} \det{}^2 C[I,*]R[*,J] \\ &= \sum_{(I,J)\in\mathbf{M}(A)} \det{}^2 C[I,*]\det{}^2 R[*,J] \\ &= \left(\sum_{I\in\mathbf{I}(A)} \det{}^2 C[I,*]\right)\left(\sum_{J\in\mathbf{J}(A)} \det{}^2 R[*,J]\right) \\ &= \operatorname{vol}^2(C)\operatorname{vol}^2(R).\end{aligned}$$

Singular values

Theorem 5

If $A \in \mathbb{R}_r^{m\times n}$ *with singular values* $\{\sigma_i \mid i \in 1{:}r\}$*, then*

$$\mathrm{vol}\,(A) = \prod_{i=1}^{r} \sigma_i$$

Proof.

Given a singular value decomposition (SVD) of A,

$$A = U\Sigma V^T$$

with $\Sigma = \mathrm{diag}(\sigma_i) \in \mathbb{R}_r^{r\times r}$, $U^TU = V^TV = I_r$

$$\begin{aligned} \therefore\ \mathrm{vol}\,(A) &= \mathrm{vol}\,(U)\,\mathrm{vol}\,(\Sigma V^T) \\ &= \mathrm{vol}\,(U)\,\mathrm{vol}\,(\Sigma)\,\mathrm{vol}\,(V^T) \\ &= \prod_{i=1}^{r} \sigma_i \end{aligned}$$

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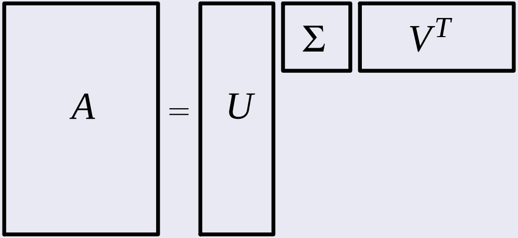
SVD of $A \in \mathbb{R}_r^{m\times n}$

$$A = U\Sigma V^T, \quad \Sigma = \text{diag}\{\sigma_i \mid i \in 1{:}r\},\ U^TU = V^TV = I_r$$



$$U = (\mathbf{u}_1, \cdots, \mathbf{u}_r),\ V = (\mathbf{v}_1, \cdots, \mathbf{v}_r)$$

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$$A : \square\{\mathbf{v}_1, \cdots, \mathbf{v}_r\} \to \square\{\sigma_1\mathbf{u}_1, \cdots, \sigma_r\mathbf{u}_r\},\ \text{of volume} \prod_{i=1}^{r} \sigma_i = \text{vol}\,(A)$$

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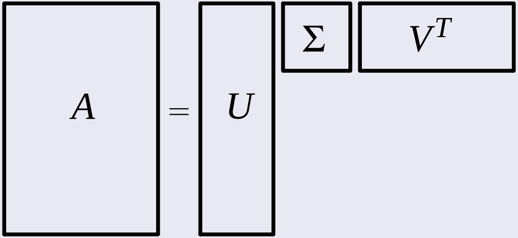

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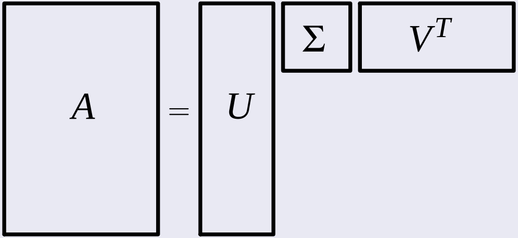

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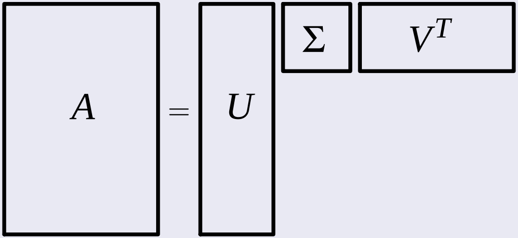

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Outline

Principal angles & vectors

Let L, M be subspaces in $\mathbb{R}^n$, $\dim L = \ell \le \dim M = m$.
The principal angles between L and M,

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_\ell \le \frac{\pi}{2}$$

are computed recursively as follows

$$\cos \theta_i = \frac{\langle \mathbf{x}_i,\ \mathbf{y}_i \rangle}{\|\mathbf{x}_i\|\,\|\mathbf{y}_i\|} = \max\left\{ \frac{\langle \mathbf{x},\ \mathbf{y} \rangle}{\|\mathbf{x}\|\,\|\mathbf{y}\|} \;\middle|\; \begin{array}{ll} \mathbf{x} \in L, & \mathbf{x} \perp \mathbf{x}_k, \\ \mathbf{y} \in M, & \mathbf{y} \perp \mathbf{y}_k, \end{array} \quad k \in 1{:}i-1 \right\}$$

where

$$(\mathbf{x}_i,\ \mathbf{y}_i) \ \in\ L \times M,\ i \in 1{:}\ell$$

are the corresponding pairs of principal vectors. We also define

$$\mathbf{sin}\{L,M\} := \prod_{i=1}^{\ell} \sin\,\theta_i, \qquad \mathbf{cos}\{L,M\} := \prod_{i=1}^{\ell} \cos\,\theta_i.$$

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The determinant of $A = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$ satisfies

$$|\det A| \leq \prod_{i=1}^{n} \|\mathbf{v}_i\|$$

equality $\iff$ the vectors are orthogonal, or include zero.

Theorem 6

Let $A = (A_1, A_2)$, $A_1 \in \mathbb{R}_{\ell}^{n \times n_1}$, $A_2 \in \mathbb{R}_{m}^{n \times n_2}$ *, rank* $A = \ell + m$ *. Then*

$$\mathrm{vol}_{\ell+m}(A) = \mathrm{vol}_{\ell}(A_1)\,\mathrm{vol}_m(A_2)\,\mathbf{sin}\{\mathbf{R}(A_1), \mathbf{R}(A_2)\}\ .$$

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Let $A = (A_1, A_2) \in \mathbb{R}^{n \times n}$*, with* $A_1 \in \mathbb{R}^{n \times \ell}$*,* $A_2 \in \mathbb{R}^{n \times m}$*. Then*

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Orthogonal projections

$V := \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ set of vectors in $\mathbb{R}^n$
$S := \text{span}\{V\}$, the subspace spanned by V
$\dim S = r$
Any $\mathbf{w} \in \mathbb{R}^n$ can be written as $\mathbf{w} = \mathbf{w}_S + \mathbf{w}_{S^\perp}$.

Theorem 8

Let V, S be as above. Then, for any $\mathbf{w} \in \mathbb{R}^n$,

$$\|\mathbf{w}_{S^\perp}\| = \frac{\text{vol}_{r+1}(\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{w})}{\text{vol}_r(\mathbf{v}_1, \cdots, \mathbf{v}_k)} ,$$

where $(\mathbf{v}_1, \cdots, \mathbf{v}_k)$ is the matrix with $\mathbf{v}_j$ as columns.

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If $\mathbf{w} \in S$, $0 = 0$. If $\mathbf{w} \notin S$ then

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For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|,$$

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Outline

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$V =$ finite–dimensional linear space over field F

An exterior product is an operation $\wedge : V \times V \to V$ that is
(a) anti–commutative, $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$
(b) $(\lambda \cdot \mathbf{u}) \wedge \mathbf{v} = \lambda \cdot (\mathbf{u} \wedge \mathbf{v})$
(c) distributive in both variables:

$$(\mathbf{u} + \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{w}$$
$$\mathbf{w} \wedge (\mathbf{u} + \mathbf{v}) = \mathbf{w} \wedge \mathbf{u} + \mathbf{w} \wedge \mathbf{v}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \lambda \in F$.

$\overset{k}{\wedge} V$ = the k_{th}–exterior space over V, spanned by all exterior products $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k$ of k elements in V

$$\dim \overset{k}{\wedge} \mathbb{R}^n = \binom{n}{k}$$

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Compound matrices

V, U = finite–dimensional linear spaces
$\mathbf{L}(V, U)$ = the linear transformations: $V \to U$
Linear transformations $\longleftrightarrow$ their matrix representations.

For $V = \mathbb{R}^n$, $U = \mathbb{R}^m$,
$A \in \mathbb{R}_r^{m \times n}$, $r > 0$, $k \in 1{:}r$,
the k–compound matrix of A is the matrix representing the linear transformation $C_k(A) \in \mathbf{L}(\overset{k}{\wedge} V, \overset{k}{\wedge} U)$, defined by

$$C_k(A)\,(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k) := A\mathbf{x}_1 \wedge \cdots \wedge A\mathbf{x}_k, \quad \forall\, \{\mathbf{x}_i\} \subset V,$$

The compound matrix $C_k(A)$ is $\binom{m}{k} \times \binom{n}{k}$ of rank $\binom{r}{k}$.

In particular, $C_r(A)$ is $\binom{m}{r} \times \binom{n}{r}$ of rank 1.

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If $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $X \in \mathbb{C}^{n\times n}$ then:

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Maple program for $C_k(A)$

```
with(LinearAlgebra):with(combinat):

Compound:=proc(A,k)
local m,n,i,j,rr,cc,P,Q:   global CO:
m:=RowDimension(A):n:=ColumnDimension(A)
P:=choose(m,k):Q:=choose(n,k):
rr:=numbcomb(m,k):cc:=numbcomb(n,k):
CO:=Matrix(rr,cc):
for i from 1 to rr do
convert(P[i],list):
for j from 1 to cc do
convert(Q[j],list):
CO(i,j):=evalf(Determinant(SubMatrix(A,P[i],Q[j])))
od:od:
print(CO):
end:
```

Let $A \in \mathbb{R}^{m\times n}$. The k–compound $C_k(A)$ is defined by

$$A\mathbf{x}_1 \wedge \cdots \wedge A\mathbf{x}_k = C_k(A)\,(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k),$$

for all $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\} \subset \mathbb{R}^n$.

If A is $n \times n$ then $C_n(A) = \det(A)$, i.e.,

$$A\mathbf{x}_1 \wedge \cdots \wedge A\mathbf{x}_n = \det(A)\,(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_n),$$

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Let $A \in \mathbb{R}_r^{m\times n}$. Then

$$A\mathbf{x}_1 \wedge \cdots \wedge A\mathbf{x}_r = \pm\mathrm{vol}(A)\,(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_r),$$

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If A is $n \times n$ then $C_n(A) = \det(A)$, i.e.,

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Let $A \in \mathbb{R}_r^{m\times n}$. Then

$$A\mathbf{x}_1 \wedge \cdots \wedge A\mathbf{x}_r = \pm \mathrm{vol}\,(A)\,(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_r),$$

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Let $A \in \mathbb{R}^{m\times n}$. The k–compound $C_k(A)$ is defined by

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Plücker coordinates, [77]

To any subspace $W \subset \mathbb{R}^n$, $\dim W = r$, there corresponds a 1–dimensional subspace $\overset{r}{\wedge} W \subset \overset{r}{\wedge} \mathbb{R}^n$, spanned by

$$\mathbf{w}^{\wedge} = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r$$

where $\{\mathbf{w}_1, \cdots, \mathbf{w}_r\}$ is any basis of W.

The $\binom{n}{r}$ components of $\mathbf{w}^{\wedge}$ (determined up to a multiplicative constant) are the Plücker coordinates of W.

Let $A \in \mathbb{R}_r^{m \times n}$ and $A = U \Sigma V^T$ its SVD,

$$U = (\mathbf{u}_1, \cdots, \mathbf{u}_r), \; V = (\mathbf{v}_1, \cdots, \mathbf{v}_r)$$

Plücker coordinates of $\mathbf{R}(A)$ and $\mathbf{R}(A^T)$ are

$$\mathbf{u}^{\wedge} = C_r(U) = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r, \quad \mathbf{v}^{\wedge} = C_r(V) = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$$

If A is square then

$$\cos\{\mathbf{R}(A), \mathbf{R}(A^T)\} = \cos \angle \{\mathbf{u}^{\wedge}, \mathbf{v}^{\wedge}\}$$

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Plücker coordinates preserve angles

Theorem 10

Let $\mathbf{U}, \mathbf{V}$ *be subspaces of* $\mathbb{R}^n$*, with the columns of*

$$U = (\mathbf{u}_1, \cdots, \mathbf{u}_r), \; V = (\mathbf{v}_1, \cdots, \mathbf{v}_r)$$

as bases, and Plücker coordinates

$$\mathbf{u}^{\wedge} = C_r(U) = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r, \quad \mathbf{v}^{\wedge} = C_r(V) = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r.$$

Then

$$\mathbf{cos}\{\mathbf{U}, \mathbf{V}\} = \cos \angle\{\mathbf{u}^{\wedge}, \mathbf{v}^{\wedge}\}$$

WLOG assume columns o.n., angles $\leq \frac{\pi}{2}$. Then

$$\det(U^T V) = \mathbf{cos}\{\mathbf{U}, \mathbf{V}\}, \text{ also,}$$

$$\det(U^T V) = C_r(U^T V) = C_r(U)^T C_r(V) = \langle \mathbf{u}^{\wedge}, \mathbf{v}^{\wedge} \rangle$$

□

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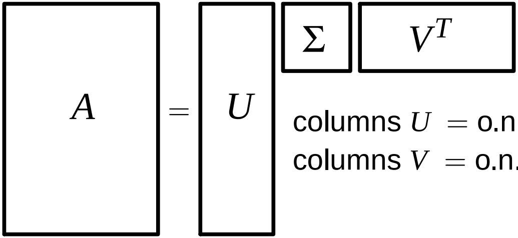
SVD of $A \in \mathbb{R}_r^{m\times n}$



columns U $=$ o.n. basis of $\mathbf{R}(A)$

columns V $=$ o.n. basis of $\mathbf{R}(A^T)$

$$A = U \; \Sigma \; V^T$$

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columns V = o.n. basis of $\mathbf{R}(A^T)$

$$C_r(A) = \mathrm{vol}\,(A)\; \mathbf{u}^\wedge \; (\mathbf{v}^\wedge)^T$$

$$\mathbf{u}^\wedge = C_r(U)$$

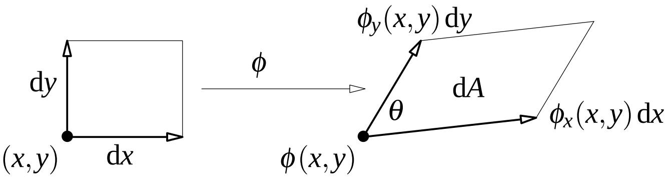
$$\mathbf{v}^\wedge = C_r(V)$$

$$C_r(A)\,\mathbf{v}^\wedge = \mathrm{vol}\,(A)\,\mathbf{u}^\wedge$$

$$C_r(AA^T)\,\mathbf{u}^\wedge = \mathrm{vol}^2(A)\,\mathbf{u}^\wedge$$

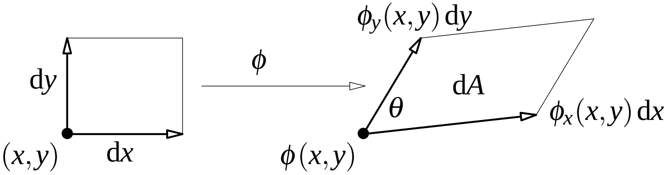
$$C_r(A^TA)\,\mathbf{v}^\wedge = \mathrm{vol}^2(A)\,\mathbf{v}^\wedge$$

Outline



$$E := \phi_x \cdot \phi_x = \|\phi_x\|^2$$
$$F := \phi_x \cdot \phi_y = \|\phi_x\|\|\phi_y\| \cos\theta$$
$$G := \phi_y \cdot \phi_y = \|\phi_y\|^2$$
$$\mathrm{d}s^2 = E\,\mathrm{d}x^2 + 2F\,\mathrm{d}x\,\mathrm{d}y + G\,\mathrm{d}y^2 \text{ (1st fundamental form)}$$
$$\mathrm{d}A = \|\phi_x \times \phi_y\|\,\mathrm{d}x\,\mathrm{d}y = \|\phi_x\|\|\phi_y\| \sin\theta\,\mathrm{d}x\,\mathrm{d}y$$
$$= \sqrt{EG - F^2}\,\mathrm{d}x\,\mathrm{d}y$$
$$\therefore \text{ Area}\,\phi(U) = \iint_U \sqrt{EG - F^2}\,\mathrm{d}x\,\mathrm{d}y$$

C.F. Gauss, [42]



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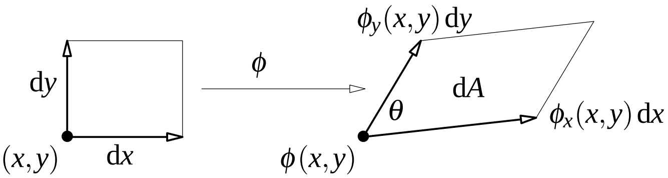

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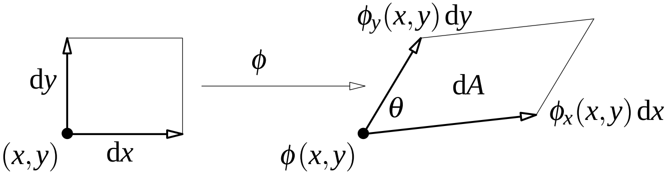

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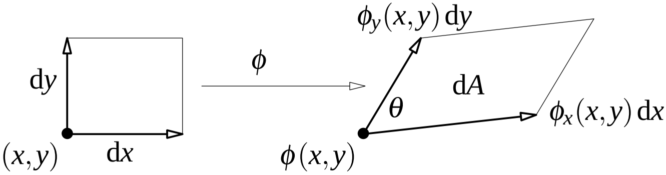

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E &:= \phi_x \cdot \phi_x = \|\phi_x\|^2 \\
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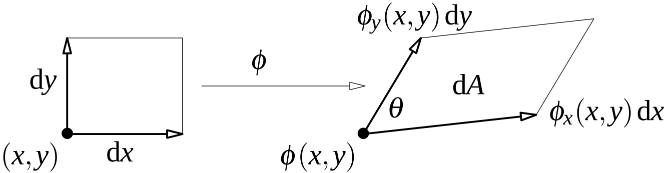
$$



$$\phi(x,y) = (x,y,f(x,y))$$

$$J_\phi(x,y) = \frac{\partial(x,y,z)}{\partial(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{pmatrix} = (\phi_x \ \phi_y)$$

$$EG - F^2 = \|\phi_x\|^2\|\phi_y\|^2 - (\phi_x \cdot \phi_y)^2 = 1 + f_x^2 + f_y^2 = \text{vol}^2(J_\phi(x,y))$$

$$\therefore \ dA = \text{vol}\,(J_\phi(x,y))\,dx\,dy$$

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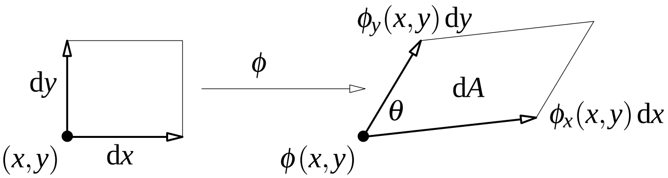

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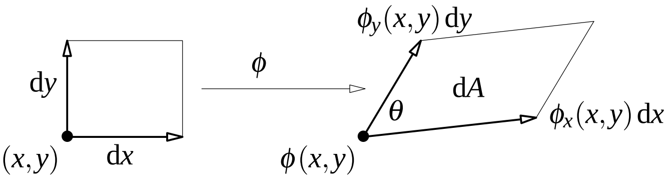
Monge patch: Surface in $\mathbb{R}^3$ given by $z = f(x,y)$



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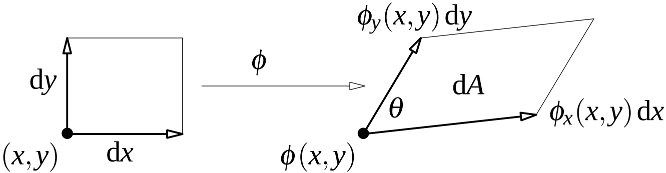

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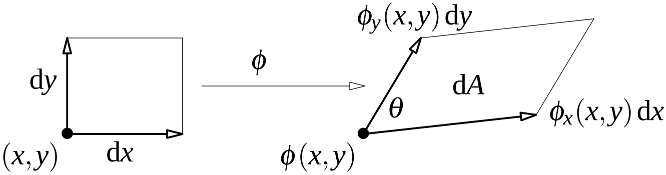

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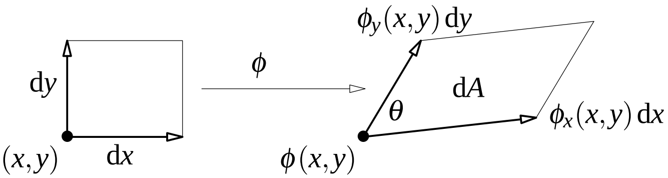

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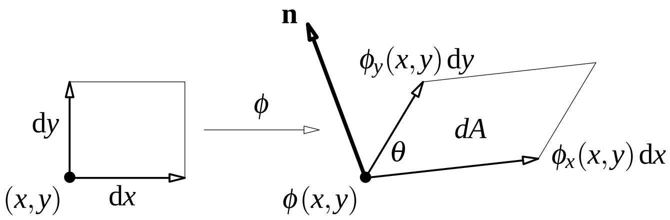
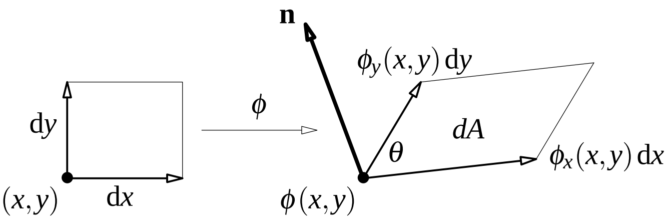


$$\mathbf{n} = \frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|}$$

$$J_\phi(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{pmatrix} \implies \mathbf{n} \propto \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}$$

$$\therefore\ \mathbf{n} = \frac{1}{\mathrm{vol}\,(J_\phi(x,y))} \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}.$$

The unit normal $\mathbf{n} = \mathbf{n}(x,y)$ at $\phi(x,y)$



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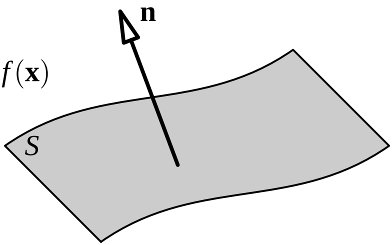
$$x_{n+1} = f(x_1, x_2, \cdots, x_n) = f(\mathbf{x})$$

$$S = \phi(U),\ U \subset \mathbb{R}^n$$

$$\phi(\mathbf{x}) = \begin{pmatrix} \mathbf{x} \\ f(\mathbf{x}) \end{pmatrix}$$



$$J_\phi(\mathbf{x}) = \begin{pmatrix} I \\ \nabla f(\mathbf{x})^T \end{pmatrix}, \quad \text{vol}\,(J_\phi(\mathbf{x})) = \sqrt{1 + \|\nabla f(\mathbf{x})\|^2}, \quad [17]$$

$$J_\phi(\mathbf{x}) = \begin{pmatrix} I \\ \nabla f(\mathbf{x})^T \end{pmatrix} \implies \mathbf{n} \propto \begin{pmatrix} -\nabla f(\mathbf{x}) \\ 1 \end{pmatrix} = \nabla(x_{n+1} - f(\mathbf{x}))$$

$$\therefore\ \mathbf{n} = \frac{1}{\sqrt{1 + \|\nabla f(\mathbf{x})\|^2}} \begin{pmatrix} -\nabla f(\mathbf{x}) \\ 1 \end{pmatrix}$$

n–dimensional surface, given by $\phi : \mathbb{R}^n \to \mathbb{R}^{n+1}$

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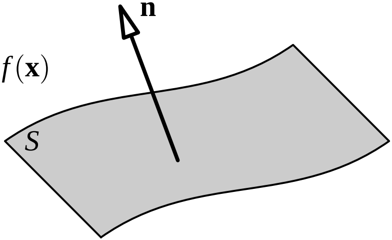

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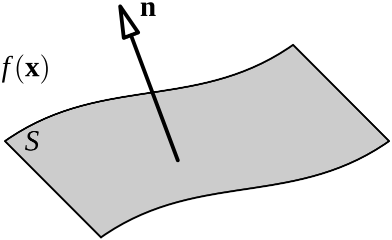

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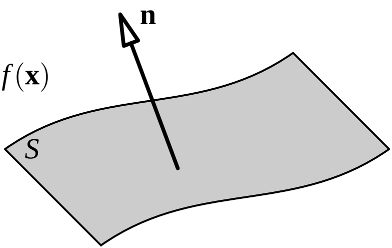

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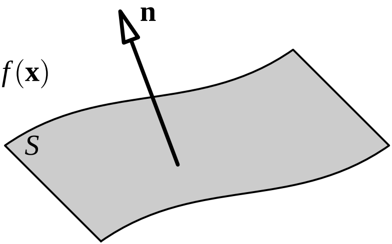

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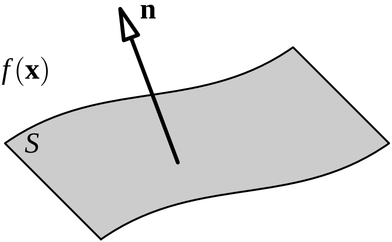

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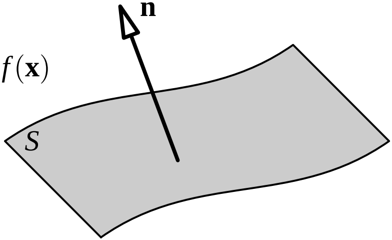

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Outline

C.G.J. Jacobi, [54]

- The change of variables formula is

$$\int_{\phi(U)} F(\mathbf{v})\,\mathrm{d}\mathbf{v} = \int_U (F \circ \phi)(\mathbf{u})\,\left|\det J_\phi(\mathbf{u})\right|\,\mathrm{d}\mathbf{u} \qquad \text{(A)}$$

$U, V \subset \mathbb{R}^n$,
$\phi : U \to V$ is a sufficiently well-behaved function,
F is integrable on $\phi(U)$,
$\mathrm{d}\mathbf{x}$ is the volume element $|\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n|$, and
J_ϕ is the Jacobi matrix (or Jacobian)

$$J_\phi := \left(\frac{\partial \phi_i}{\partial u_j}\right), \quad \text{also denoted} \quad \frac{\partial(v_1, v_2, \cdots, v_n)}{\partial(u_1, u_2, \cdots, u_n)},$$

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- The volume of an $m \times n$ matrix of rank r is

$$\text{vol}\,A := \sqrt{\sum_{(I,J)\in\mathbf{M}(A)} \det^2 A[I,J]}$$

$A[I,J]$ is the submatrix of A with rows I and columns J, $\mathbf{M}(A)$ is the index set of $r \times r$ nonsingular submatrices of A.

- If A is of full column rank, its volume is

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- If $m = n$ then $\text{vol}\,J_\phi = \left|\det J_\phi\right|$, and (B) reduces to (A).

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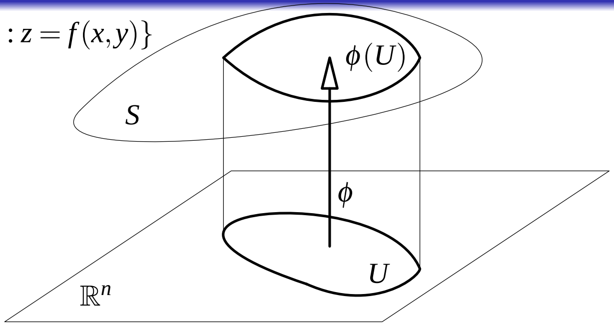
Surface integral of a scalar function $F(x,y,z)$ in $\mathbb{R}^3$

$$S = \{(x,y,z) : z = f(x,y)\}$$



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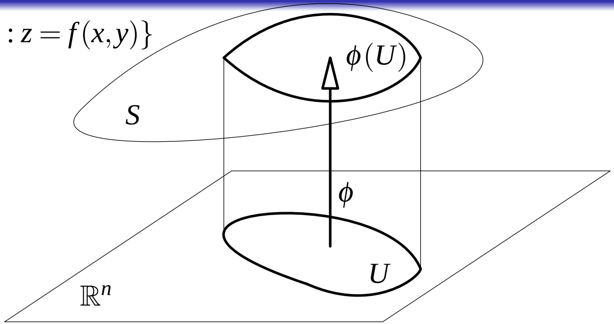

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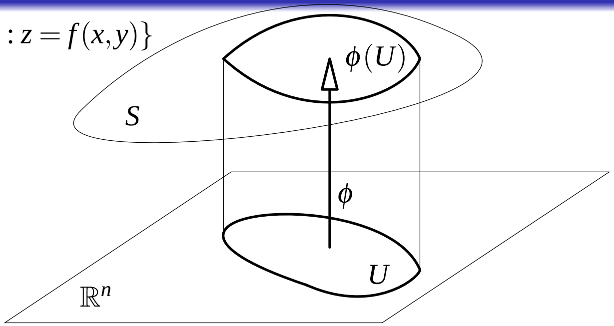

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Example: Cylindrical coordinates

Let S be a surface in $\mathbb{R}^3$ represented by $z = z(r, \theta)$ where $\{r, \theta\}$ are polar coordinates, or by the mapping

$$\phi(r, \theta) \longrightarrow (x, y, z) = (r\cos\theta, r\sin\theta, z(r, \theta))$$

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$$\therefore \; J_\phi(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{pmatrix}$$

$$\therefore \; \mathrm{vol}\,(J_\phi) = \sqrt{r^2 + r^2\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r\sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2}\,,$$

An integral over a domain $V \subset S$ is therefore

$$\iint_{\phi(U)} F(x,y,z)\,\mathrm{d}S =$$

$$\iint_U F(r\cos\theta, r\sin\theta, z(r,\theta))\, r\sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2}\,\mathrm{d}r\,\mathrm{d}\theta\,.$$

Example: Regular simplex in $\mathbb{R}^n$

Consider the simplex

$$\Delta_n = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \leq 1,\ \mathbf{x} \geq \mathbf{0}\},\ \text{with volume } V_n,$$

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Then $V_n = \dfrac{1}{n!}$, by induction $\quad \therefore\ A_j = V_{n-1} = \dfrac{1}{(n-1)!}$.

To calculate A_0, let $\phi : F_j \to A_0$ be given by $x_n = 1 - \sum_{j \neq i = 1}^{n-1} x_i$.

$$\therefore\ \operatorname{vol}(J_\phi) = \sqrt{n}. \qquad \therefore\ A_0 = \int_{F_j} \operatorname{vol}(J_\phi)\, d\mathbf{x} = \sqrt{n}\, A_j.$$

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Then by similarity, the volume of $\Delta_n(\theta)$,

$$\text{vol}\,(\Delta_n(\theta)) = \theta^n \text{vol}\,(\Delta_n) \longrightarrow 0,\ \text{as } n \longrightarrow \infty.$$

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Example: Radon transform

Let $\mathbf{H}(\mathbf{v},p)$ be a ("non–vertical") hyperplane in $\mathbb{R}^n$

$$\mathbf{H}(\mathbf{v},p) := \{\mathbf{x} : <\mathbf{v},\mathbf{x}> = p\} = \left\{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} v_i x_i \;=\; p\right\}, \quad (v_n \neq 0)$$

$$\mathbf{H}(\mathbf{v},p) = \phi(\mathbb{R}^{n-1}), \quad x_n \;:=\; \frac{p}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i$$

$$\text{vol}\, J_\phi = \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{v_i}{v_n}\right)^2} = \frac{\|\mathbf{v}\|}{|v_n|}$$

The Radon transform $(\mathbf{R}F)(\mathbf{v},p)$ of a function $F : \mathbb{R}^n \to \mathbb{R}$ is,

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Outline

The unit ball & sphere in $\mathbb{R}^n$

$\|\cdot\|$ the Euclidean norm,
$\mathbf{B}_n(r) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \le r\}$; $\mathbf{B}_n(1) = \mathbf{B}_n$, the unit ball in $\mathbb{R}^n$,
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For $n = 2, 3, \cdots$,

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$$A_n(r) = A_n\,r^{n-1}\ ,\ A_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}\ ,$$

where $\Gamma(\cdot)$ is the Gamma function

$$\Gamma(p) = \int_0^\infty x^{p-1}\,e^{-x}\,\mathrm{d}x\ .$$

$$\mathbf{S}_n^+ = \phi(\mathbf{B}_{n-1}), \quad \phi = (\phi_1, \phi_2, \cdots, \phi_n)$$
$$\phi_i(x_1, x_2, \cdots, x_{n-1}) = x_i \ , \ i \in 1{:}n-1 \ ,$$
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The Jacobi matrix and its volume,

$$J_\phi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\frac{x_1}{x_n} & -\frac{x_2}{x_n} & \cdots & -\frac{x_{n-1}}{x_n} \end{pmatrix}$$

$$\operatorname{vol} J_\phi = \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{x_i}{x_n}\right)^2} = \frac{1}{|x_n|} = \frac{1}{\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}} \ .$$

$\mathbf{S}_n^+$ the upper hemisphere

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The unit sphere in $\mathbb{R}^n$ (cont'd)

The area A_n is twice the area of the "upper hemisphere":

$$A_n = 2 \int\limits_{\mathbf{B}_{n-1}} \frac{\mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{n-1}}{\sqrt{1 - \sum_{i=1}^{n-1} x_i^2}} = 2 \int\limits_{r=0}^{1} \frac{\mathrm{d}V_{n-1}(r)}{\sqrt{1-r^2}}$$

$$= 2 \int\limits_{r=0}^{1} \frac{A_{n-1}(r)\,\mathrm{d}r}{\sqrt{1-r^2}} = 2 \int\limits_{r=0}^{1} \frac{A_{n-1}\, r^{n-2}\,\mathrm{d}r}{\sqrt{1-r^2}}$$

$$\therefore\ \frac{A_n}{A_{n-1}} = 2 \int\limits_{r=0}^{1} \frac{r^{n-2}\,\mathrm{d}r}{\sqrt{1-r^2}}$$

$$\therefore\ A_n = \frac{2\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

using well–known properties of the beta function,

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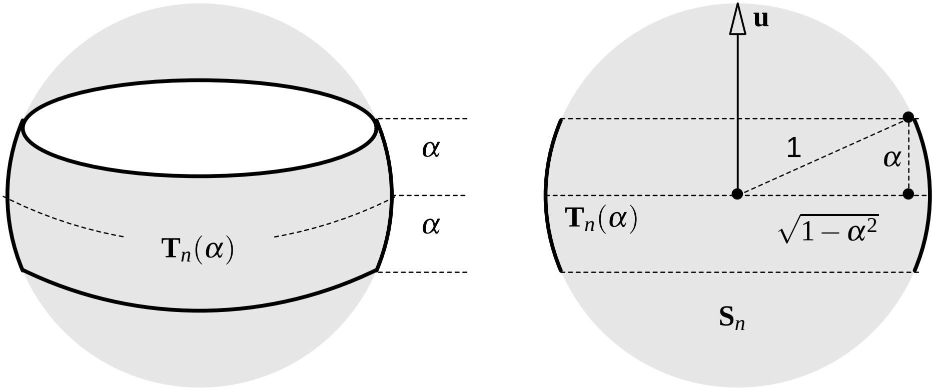
The unit sphere $\mathbf{S}_n$ and an equatorial belt $\mathbf{T}_n(\alpha)$, $\alpha > 0$

$$\mathbf{T}_n(\alpha) = \{\mathbf{x} \in \mathbf{S}_n : -\alpha \le x_j \le \alpha \quad \text{for some} \quad j \in 1{:}n\}$$



$$A(\mathbf{T}_n(\alpha)) := \text{area of } \mathbf{T}_n(\alpha)$$

$$\text{Prob}\{\mathbf{X} \in \mathbf{T}_n(\alpha)\} = \frac{A(\mathbf{T}_n(\alpha))}{A_n}$$

$$\begin{aligned} A(\mathbf{T}_n(\alpha)) &= 2 \int_{(1-\alpha^2)^{1/2}}^{1} \frac{\mathrm{d}v_{n-1}(r)}{\sqrt{1-r^2}} \\ &= 2 A_{n-1} \int_{(1-\alpha^2)^{1/2}}^{1} \frac{r^{n-2}}{\sqrt{1-r^2}} \, \mathrm{d}r \, , \\ &= A_{n-1} \int_{1-\alpha^2}^{1} x^{(n-3)/2} (1-x)^{-1/2} \, \mathrm{d}x \, , \ \text{for } x = r^2 \, . \end{aligned}$$

$$\begin{aligned} \therefore \ \text{Prob}\{\mathbf{X} \in \mathbf{T}_n(\alpha)\} &= \frac{A_{n-1}}{A_n} \int_{1-\alpha^2}^{1} x^{(n-3)/2} (1-x)^{-1/2} \, \mathrm{d}x \, , \\ &= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\,\Gamma(\frac{1}{2})} \int_{1-\alpha^2}^{1} x^{(n-3)/2} (1-x)^{-1/2} \, \mathrm{d}x \, . \end{aligned}$$

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Prob $\{\mathbf{X} \in \mathbf{T}_n(\frac{k}{\sqrt{n}})\}$ for $\mathbf{X} \sim \mathbf{U}(S_n)$

n	$k=1$	$k=2$	$k=3$
2	.5		
5	.6260990336	.9838699099	
10	.6565636037	.9632125020	.9999914613
100	.6802515257	.9550652747	.9976960345
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Values of Prob $\{\mathbf{X} \in \mathbf{T}_n(\frac{k}{\sqrt{n}})\}$ for some k, n

Theorem 11

Let $\mathbf{u} \in \mathrm{S}_n$ be fixed, $\mathbf{x} \sim \mathbf{U}(S_n)$. Then, as $n \to \infty$,

$$\text{Prob}\left\{|\langle \mathbf{u}, \mathbf{x}\rangle| \leq \frac{k}{\sqrt{n}}\right\} \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{-x^2/2}\, dx\,.$$

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Area of the intersection of two equatorial belts

Example 12 (Daniel Reem)

Consider two equatorial belts in $\mathbf{S}_{1000}$, say

$$\mathbf{T}_1 = \{\mathbf{x} \in \mathbf{S}_{1000} : -\frac{3}{\sqrt{1000}} \le x_1 \le \frac{3}{\sqrt{1000}}\}$$
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Then

$$\text{Area}\,(\mathbf{T}_1 \cap \mathbf{T}_2) \ge 0.994680132\ \text{Area}\,(\mathbf{S}_{1000})$$

Proof.

Let $\text{Area}\,(\mathbf{S}_{1000}) := A$. Then

$$\text{Area}\,\mathbf{T}_1 = \text{Area}\,\mathbf{T}_2 = .9973400661\,A$$

$$\therefore\ \text{Area}\,(\mathbf{T}_1 \cup \mathbf{T}_2) = 2\,(.9973400661\,A) - \text{Area}\,(\mathbf{T}_1 \cap \mathbf{T}_2) \le A$$

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Outline

Probability density of a function of RV's

$(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ RV's with a given joint density $f_{\mathbf{X}}(x_1, \cdots, x_n)$,
$h : \mathbb{R}^n \to \mathbb{R}$ well behaved, in articular $\dfrac{\partial h}{\partial x_n} \neq 0$,
It is required to find the density of the RV

$$\mathbf{Y} = h(\mathbf{X}_1, \cdots, \mathbf{X}_n).$$

Solve for x_n,

$$x_n = h^{-1}(y|x_1, \cdots, x_{n-1})$$

Change variables from $\{x_1, \cdots, x_n\}$ to $\{x_1, \cdots, x_{n-1}, y\}$, and use

$$\det\left(\frac{\partial(x_1, \cdots, x_n)}{\partial(x_1, \cdots, x_{n-1}, y)}\right) = \frac{\partial h^{-1}}{\partial y}$$

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A surface integral on $\mathbf{V}(y)$

Let $\mathbf{V}(y)$ be the surface in $\mathbb{R}^n$ given by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ h^{-1}(y|x_1, \cdots, x_{n-1}) \end{pmatrix} = \phi \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

Then the surface integral of $f_{\mathbf{X}}$ over $\mathbf{V}(y)$ is given by

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Theorem 13

If the ratio

$$A := \frac{\frac{\partial h^{-1}}{\partial y}}{\sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\partial h^{-1}}{\partial x_i}\right)^2}} \quad \text{does not depend on } x_1, \cdots, x_{n-1}, \quad \text{(C)}$$

then

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Proof. Compare

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$$\int_{\mathbf{V}(y)} f_{\mathbf{X}} =$$

$$\int_{\mathbb{R}^{n-1}} f_{\mathbf{X}}(x_1,\cdots,x_{n-1},h^{-1}(y|x_1,\cdots,x_{n-1}))\sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\partial h^{-1}}{\partial x_i}\right)^2}\,\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}$$

Condition (C)

the ratio

$$A := \frac{\frac{\partial h^{-1}}{\partial y}}{\sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\partial h^{-1}}{\partial x_i}\right)^2}} \quad \text{does not depend on } x_1,\cdots,x_{n-1} \qquad \text{(C)}$$

Functions $h : \mathbb{R}^n \to \mathbb{R}$ that satisfy (C):

$$h(x_1,\cdots,x_n) = \sum_{i=1}^{n} v_i\, x_i, \quad v_n \neq 0$$

$$h(x_1,\cdots,x_n) = \sum_{i=1}^{n} x_i^2$$

$$h(x_1,\cdots,x_n) = \exp\left\{\sum_{i=1}^{n} x_i\right\}$$

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Check condition (C) for $h(x_1,\cdots,x_n) := \sum_{i=1}^{n} v_i x_i,\; v_n \neq 0$

$$x_n = h^{-1}(y|x_1,\cdots,x_{n-1}) := \frac{y}{v_n} - \sum_{i=1}^{n-1} \frac{v_i}{v_n} x_i$$

with $\frac{\partial h^{-1}}{\partial y} = \frac{1}{v_n}$, $\sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\partial h^{-1}}{\partial x_i}\right)^2} = \sqrt{1+\sum_{i=1}^{n-1}\left(\frac{v_i}{v_n}\right)^2} = \frac{\|\mathbf{v}\|}{|v_n|}$

The density of $\mathbf{Y} = \sum v_i \mathbf{X}_i$ is the integral of $f_{\mathbf{X}}$ on the hyperplane

$$\mathbf{H}(\mathbf{v},y) := \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} v_i x_i = y\right\}$$

$$\therefore\; f_{\mathbf{Y}}(y) = \frac{1}{\|\mathbf{v}\|}(\mathbf{R}f_{\mathbf{X}})(\mathbf{v},y)$$

$$= \frac{1}{|v_n|}\int_{\mathbb{R}^{n-1}} f_{\mathbf{X}}\left(x_1,\cdots,x_{n-1},\frac{y}{v_n} - \sum_{i=1}^{n-1}\frac{v_i}{v_n}x_i\right) dx_1 dx_2 \cdots dx_{n-1}$$

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with $$\frac{\partial h^{-1}}{\partial y} = \frac{1}{v_n}\,, \quad \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\partial h^{-1}}{\partial x_i}\right)^2} = \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{v_i}{v_n}\right)^2} = \frac{\|\mathbf{v}\|}{|v_n|}$$

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Check condition (C) for $h(x_1, \cdots, x_n) := \sum_{i=1}^{n} x_i^2$

Two solutions of $y = h(x_1, \cdots, x_n) := \sum_{i=1}^{n} x_i^2$ for x_n,

$$x_n = h^{-1}(y|x_1, \cdots, x_{n-1}) := \pm\sqrt{y - \sum_{i=1}^{n-1} x_i^2}$$

$$\text{with} \quad \frac{\partial h^{-1}}{\partial y} = \pm\frac{1}{2\sqrt{y - \sum\limits_{i=1}^{n-1} x_i^2}}$$

$$\sqrt{1 + \sum_{i=1}^{n-1}\left(\frac{\partial h^{-1}}{\partial x_i}\right)^2} = \frac{\sqrt{y}}{\sqrt{y - \sum\limits_{i=1}^{n-1} x_i^2}}$$

Therefore the density of $\sum \mathbf{X}_i^2$ is expressed in terms of the integral of $f_{\mathbf{X}}$ on the sphere $\mathbf{S}_n(\sqrt{y})$ of radius $\sqrt{y}$.

Corollary 14

Let $(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ *have joint density* $f_{\mathbf{X}}(x_1, \cdots, x_n)$. *The density of*

$$\mathbf{Y} = \sum_{i=1}^{n} \mathbf{X}_i^2 \quad \textit{is} \quad f_{\mathbf{Y}}(y) = \frac{1}{2\sqrt{y}} \int_{\mathbf{S}_n(\sqrt{y})} f_{\mathbf{X}}$$

the integral is over the sphere $\mathbf{S}_n(\sqrt{y})$ *of radius* $\sqrt{y}$,

$$\int_{\mathbf{S}_n(\sqrt{y})} f_{\mathbf{X}} = \int_{\mathbf{B}_{n-1}(\sqrt{y})} \left[f_{\mathbf{X}}\left(x_1, \cdots, x_{n-1}, \sqrt{y - \sum_{i=1}^{n-1} x_i^2}\right) + \right.$$

$$\left. + f_{\mathbf{X}}\left(x_1, \cdots, x_{n-1}, -\sqrt{y - \sum_{i=1}^{n-1} x_i^2}\right) \right] \frac{\sqrt{y}\, \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}}{\sqrt{y - \sum_{i=1}^{n-1} x_i^2}}$$

The factor $1/2\sqrt{y}$ is the width of the spherical shell bounded by the two spheres $\mathbf{S}_n(\sqrt{y})$ and $\mathbf{S}_n(\sqrt{y + \mathrm{d}y})$, i.e. the difference of radii $\sqrt{y + \mathrm{d}y} - \sqrt{y} \approx \frac{\mathrm{d}y}{2\sqrt{y}}$

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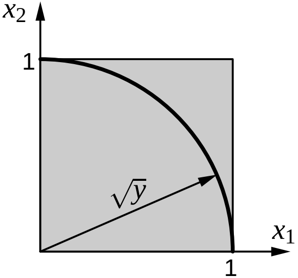
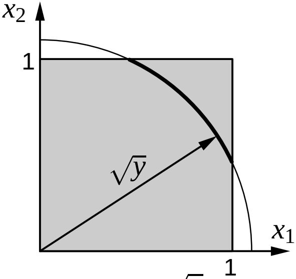
Example 15

Let $\mathbf{Y} = \mathbf{X}_1^2 + \mathbf{X}_2^2$, $\mathbf{X}_i$ i.i.d., $\mathbf{X}_i \sim \mathbf{U}[0,1]$. Then

$$f_{\mathbf{Y}}(y) = \begin{cases} \frac{\pi}{4}, & 0 \le y \le 1; \\ \frac{\pi}{4} - \arccos\left(\frac{1}{\sqrt{y}}\right), & 1 \le y \le 2. \end{cases}$$



$0 \le \sqrt{y} \le 1$



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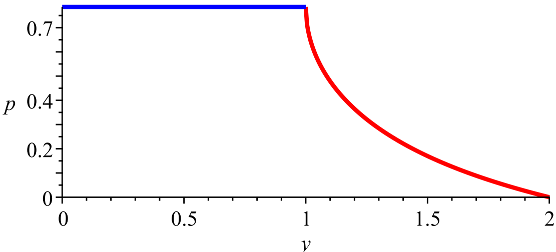
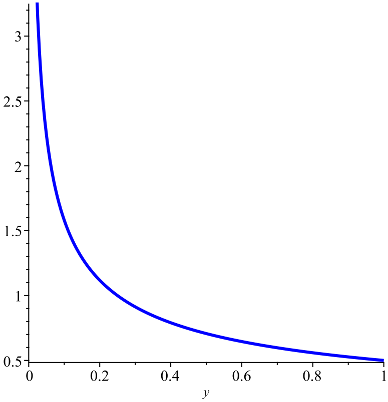
$$\mathbf{E}\{\mathbf{Y}\} = \frac{2}{3}, \quad \mathbf{Var}\{\mathbf{Y}\} = \frac{8}{45}$$



What makes this strange is that ···

the density of $\mathbf{Y} = \mathbf{X}^2$, $\mathbf{X} \sim \mathbf{U}[0,1]$ is

$$f_{\mathbf{Y}}(y) = \frac{1}{2\sqrt{y}}, \quad 0 < y \leq 1.$$



Random directions

Let $\mathbf{X} \sim \mathbf{U}(S_n)$. The probability of a surface element on S_n is,

$$\frac{\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{n-1}}{A_n\sqrt{1-\sum_{i=1}^{n-1}x_i^2}}$$

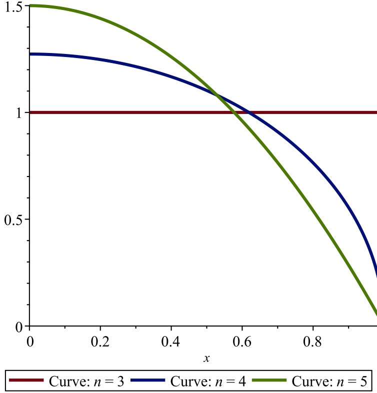
Let $\mathbf{L}_n$ be the length of the projection of $\mathbf{X}$ on a fixed line through the origin, say the x_n axis. Then $\mathbf{L}_n$ has the density

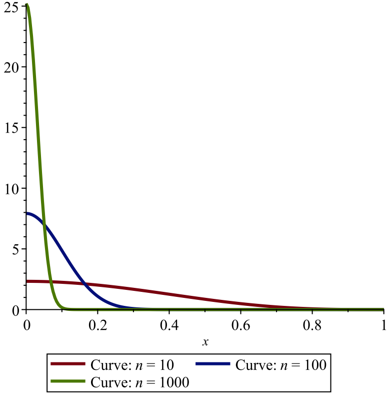
$$f_{\mathbf{L}_n}(x) = \frac{2}{B(\frac{1}{2},\frac{n-1}{2})}\left(1-x^2\right)^{\frac{n-3}{2}}$$

and expected value

$$\mathbf{E}\{\mathbf{L}_n\} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n+1}{2})}$$

Probability densities of $\mathbf{L}_n$ for $n = 3, 4, 5, 10, 100, 1000$

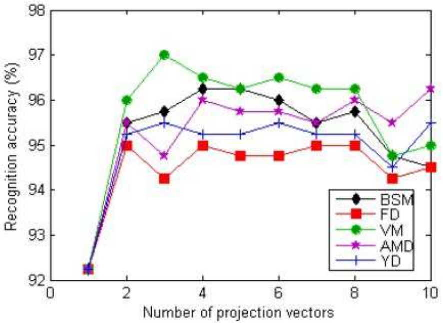




Outline

Face recognition [67], [114]

$\mathbf{Y} = \{Y_1, Y_2, \cdots, Y_N\}$ a set of known faces;
$Y =$ a face.
Question: $Y \in \mathbf{Y}$?
Answer: Yes, if $\min_{i \in 1:N} \text{vol}\,(Y - Y_i) < \varepsilon$
No, otherwise.



VM = volume measure
FD = Frpbenius distance
YD = Yang distance
AMD = assembled matrix distance
BSM = boosted similarity measure

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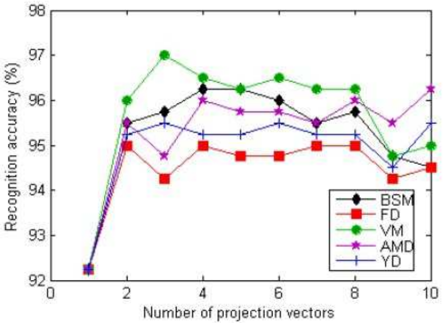

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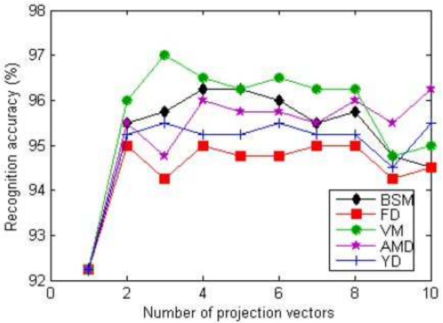

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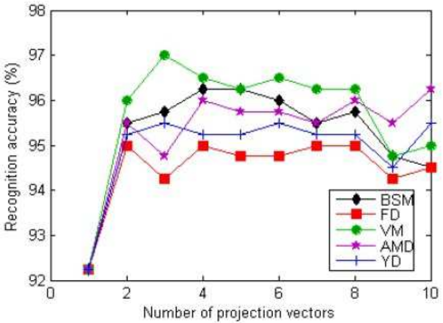

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Volume sampling [27]

Let $A \in \mathbb{R}^{m \times n}$,
S be a submatrix of k rows of A,
$\Delta(S)$ the k–simplex in $\mathbb{R}^n$ generated by S.

Theorem 16

If S is randomly chosen with probabilities

$$P_S = \frac{\mathrm{vol}^2(\Delta(S))}{\sum_T \mathrm{vol}^2(\Delta(T))}$$

then

$$\mathbf{E}_S\{\|A - \widehat{A}(S)\|_F^2\} \leq (k+1)\,\|A - A_k\|_F^2$$

where A_k is the best k–rank approximation of A,
$\widehat{A}(S)$ is the projection of A to the span of S.

$$\mathrm{vol}\,(\Delta(S)) = \frac{1}{k!}\mathrm{vol}\,(S)$$

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Applications to integration & integral operators

[2]: $\cdots = \int (1 + z_{x_1}^2 + \cdots + z_{x_n}^2)^{1/2}\, dx_1 \cdots dx_n$ (Theorem 2)

[10]: Rectangular Jacobian J_ϕ, replace $|\det(J_\phi)|$ by $\mathrm{vol}\,(J_\phi)$

[12]: Probability densities of $\sum v_i \mathbf{X}_i$ and $\sum \mathbf{X}_i^2$ for $\{\mathbf{X}_i\}$ i.i.d.

[17]: $\mathrm{vol}\,(J_\phi(\mathbf{x})) = \sqrt{1 + \|\nabla f(\mathbf{x})\|^2}$, etc. (P. 5105, bottom)

[18]: $L(S) \geq \iint \sqrt{J_1^2(u,v) + J_2^2(u,v) + J_3^2(u,v)}\, du\, dv$ (3)

[26]: Fourier integral operators of Schrödinger type

[37]: Bayesian learning of kernel embeddings

[38]: Kernel embeddings & associated probability measures

[66]: $L(S) = \iint (X^2 + Y^2 + Z^2) du\, dv = \iint (1 + p^2 + q^2)\, du\, dv$ (17.3)

[71]: Integrals on n–sphere.

[88]: $L(S) = \iint \left\{ \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 \right\}^{1/2} du\, dv$ (p. 76)

[92],[93]: Singular Jacobians in statistics

[95]: Rectangular Jacobian, $3N \times 6$, p. 401, section 2

[96]: Rectangular Jacobians, §A.1, Lemma 1

Outline

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