ON SYMMETRIC INTERSECTING FAMILIES OF VECTORS

SEAN EBERHARD, JEFF KAHN, BHARGAV NARAYANAN, AND SOPHIE SPIRKL

ABSTRACT. A family of vectors $\mathcal{A} \subset [k]^n$ is said to be intersecting if any two elements of $\mathcal{A}$ agree on at least one coordinate. We prove, for fixed $k \geq 3$, that the size of a symmetric intersecting subfamily of $[k]^n$ is $o(k^n)$, which is in stark contrast to the case of the Boolean hypercube (where $k = 2$). Our main contribution addresses limitations of existing technology: while there is now spectral machinery, developed by Ellis and the third author, to tackle extremal problems in set theory involving symmetry, this machinery relies crucially on the interplay between up-sets, biased product measures, and threshold behaviour in the Boolean hypercube, features that are notably absent in the problem at hand. To circumvent these barriers, we introduce the notion of a ‘measure flow’ on the product of ‘connected posets’, and prove a sharp threshold theorem for such flows, which may be of independent interest.

1. Introduction

We pursue a line of questioning initiated by Babai [2] and Frankl [9] about forty years ago concerning the role of symmetry in extremal set theory. Our starting point is the Erdős–Ko–Rado theorem [8], which asserts that for $n, k \in \mathbb{N}$ with $k < n/2$, the largest families of $k$-subsets of $[n]$ are precisely the trivial ones, namely those that consist of all $k$-sets containing some fixed element of $[n]$; many variants and generalisations of this theorem (involving different intersection conditions and different discrete structures such as permutations, vectors and graphs) have since been established. A common theme in this line of enquiry is that the extremal constructions are often highly asymmetric, depending strongly only on a small number of ‘coordinates’; see [15, 10, 1, 17], for example. It is therefore natural to ask what happens when one further imposes a symmetry requirement on the intersecting family under consideration, the most natural such requirement being that the family admit a transitive group of automorphisms. Indeed, this direction
was proposed by Babai [2] a few decades back, and has since been rather fruitful; see [9, 5] for some classical results, and [7, 6, 13] for more recent developments.

Here, we shall study intersecting families of vectors (or integer sequences). A family of vectors $A \subset [k]^n$ is said to be intersecting if any two elements of $A$ agree on at least one coordinate. By considering the orbits of the standard $\mathbb{Z}/k\mathbb{Z}$ action on $[k]^n$ (i.e., the orbits of the map shifting each coordinate cyclically by one), it is clear that any intersecting subfamily of $[k]^n$ has size at most $k^{n-1}$; furthermore, this bound is tight, as evidenced by considering the trivial family of vectors with one fixed coordinate, for example. These observations go back to Berge [3], Livingston [16] and Moon [17], and many generalisations of this fact are now known; see [11, 12, 19, 18] for a small sample of the literature.

In line with the preceding discussion, we again see that the extremal construction of intersecting families of vectors we have described is again quite asymmetric, in this case depending on a single coordinate, though there is one caveat: in the Boolean hypercube $[2]^n$ with $n$ odd, the family of vectors with more 2s than 1s is an intersecting family both of the maximum possible size of $2^{n-1}$, and with as large an automorphism group as possible; this turns out to be a special case, however, and when $k \geq 3$, it is known that the only maximum-sized constructions are the aforementioned trivial ones. In the light of this, we shall focus on the following question: for fixed $k \geq 3$, what is the maximum size of a symmetric intersecting subfamily of $[k]^n$? To put things on precise mathematical footing, let us define these concepts in full. For $n \in \mathbb{N}$, we write $[n] = \{1, 2, \ldots, n\}$ and $S_n$ for the symmetric group on $[n]$. For a permutation $\sigma \in S_n$ and a vector $x \in [k]^n$, we write $\sigma(x)$ for the image of $x$ under $\sigma$, and for $A \subset [k]^n$, we write $\sigma(A) = \{\sigma(x) : x \in A\}$; the automorphism group of a family $A \subset [k]^n$ is then $\text{Aut}(A) = \{\sigma \in S_n : \sigma(A) = A\}$, and we say that $A \subset [k]^n$ is symmetric if $\text{Aut}(A)$ is a transitive subgroup of $S_n$, i.e., if for all $i, j \in [n]$, there exists a permutation $\sigma \in \text{Aut}(A)$ such that $\sigma(i) = j$.

With this language in place, our main result asserts that for $k \geq 3$, symmetric intersecting families in $[k]^n$ are negligibly small compared to the trivial ones.

**Theorem 1.1.** For fixed $k \geq 3$, if $A \subset [k]^n$ is a symmetric intersecting family, then $|A| = o(k^n)$.

While the statement of Theorem 1.1 is inherently attractive (in our opinion), there are deeper considerations which make the result particularly interesting from the point of view of technique, as we now explain. Ellis and the third
author [7], in resolving a conjecture of Frankl [9] about symmetric 3-wise intersecting families, introduced some spectral machinery for tackling problems in extremal set theory involving symmetry; this framework has since been successfully adapted — see [6, 13], for example — to resolve a few other old extremal problems in the Boolean hypercube involving symmetry constraints. This approach depends crucially on exploiting the interplay between up-sets, biased product measures, and ‘sharp threshold’ behaviour in the Boolean hypercube; these features are absent in the problem under consideration here, and indeed, our main contribution is a method to circumvent these barriers.

Concretely, working in $[3]^n$ with the natural partial order induced by point-wise comparison, compressing an intersecting family ‘upwards’ preserves the intersection condition but not the symmetry group, while alternately, replacing a family by its ‘up-closure’ in the natural partial order preserves the symmetry group but not the intersection condition; furthermore, there appears to be no natural analogue in $[3]^n$ of the $p$-biased product measures on $[2]^n$ that are at the heart of the arguments in [7, 6, 13].

To get around these obstacles, we shall replace the space $[3]^n$ by a larger ‘covering space’, a suitable product of ‘connected posets’, where we can find a useful notion of being up-closed, as well as introduce a notion of ‘measure flow’ on this space that successfully plays the role of the biased product measures in the Boolean hypercube; finally, once we transfer the problem at hand to this larger space, we shall then deduce our main result using a suitable variant of the sharp threshold theorem of Friedgut and Kalai [14] tailored to this space (which will in turn follow from the results of Bourgain, Kahn, Kalai, Katznelson, and Linial [4]).

The paper is organised as follows. First, in Section 2, we prove the appropriate variant of the Friedgut–Kalai sharp threshold theorem in the generality of measure flow on a product of connected posets. The proof of Theorem 1.1 follows in Section 3. We conclude in Section 4 with a brief discussion of open problems.

2. Measure flow on a product of connected posets

We now present a general construction that is at the heart of our approach based on covering spaces and measure flows; in what follows, the reader might benefit from keeping the Boolean hypercube and the $p$-biased product measures on the Boolean hypercube in mind.
Let \((W, \preceq)\) be a finite poset; we say that \(A \subseteq W\) is an up-set if whenever \(x \in A\) and \(x \preceq y\), then \(y \in A\). Given probability measures \(\mu_0, \mu_1\) on \(W\), we say that \(\mu_0\) flows up to \(\mu_1\) if \(\mu_1(A) \geq \mu_0(A)\) for every up-set \(A \subseteq W\). We say that the flow from \(\mu_0\) to \(\mu_1\) has connectivity \(\kappa\) if in fact
\[
\mu_1(A) - \mu_0(A) \geq \kappa
\]
for every up-set \(A \subseteq W\) except \(\emptyset\) and \(W\). Now, define a measure flow from \(\mu_0\) to \(\mu_1\) by letting \(\mu_t = (1 - t) \mu_0 + t \mu_1\) be the measure at time \(t \in [0, 1]\) in the flow, and write \(\mu_t^n\) for the associated product probability measure on \(W^n\); when \(n\) is clear from the context, we shall abuse notation and abbreviate \(\mu_t^n\) by \(\mu_t\). With this language in place, we need the following variant of the Friedgut–Kalai [14] sharp threshold theorem.

**Theorem 2.1.** Let \(A \subseteq W^n\) be a symmetric up-set, and assume \(\mu_0\) flows up to \(\mu_1\) with connectivity \(\kappa > 0\). If \(0 \leq p < q \leq 1\) and \(\mu_p(A), \mu_q(A) \in [\varepsilon, 1 - \varepsilon]\), then
\[
q - p \leq C \kappa^{-1} \log(1/2\varepsilon)/\log n,
\]
where \(C > 0\) is a universal constant.

**Proof.** To prove this, we start with a variant of the Margulis–Russo formula, namely
\[
\frac{d}{dp} \mu_p^n(A) = \sum_{i=1}^n (\mu_p^{i-1} \times (\mu_1 - \mu_0) \times \mu_p^{n-i})(A).
\]

Next, recall the definition of influence: the influence \(I_{A,p}(i)\) of the coordinate \(i\) is the probability, given \(x \sim \mu_p\), that changing the value of \(x_i\) can affect whether \(x \in A\). In other words, \(I_{A,p}(i)\) is the probability that the slice
\[
A_i(x) = \{w \in W: (x_1, \ldots, x_{i-1}, w, x_{i+1}, \ldots, x_n) \in A\}
\]
is proper and nontrivial. By (†), we therefore have
\[
(\mu_p^{i-1} \times (\mu_1 - \mu_0) \times \mu_p^{n-i})(A) \geq \kappa I_{A,p}(i),
\]
implying that
\[
\frac{d}{dp} \mu_p^n(A) \geq \kappa \sum_{i=1}^n I_{A,p}(i).
\]
Now, as in [14], by symmetry and the Bourgain–Kahn–Kalai–Katznelson–Linial theorem [4], we have
\[ \sum_{i=1}^{n} I_{A,p}(i) = \Omega(\min(\mu_p^n(A), 1 - \mu_p^n(A)) \log n), \]
so it follows that
\[ \frac{d}{dp} \mu_p^n(A) = \Omega(\kappa \min(\mu_p^n(A), 1 - \mu_p^n(A)) \log n); \]
the claimed result now follows by elementary calculus. \[\square\]

3. Proof of the main result

We now prove our main result, starting with an outline of the argument. Let \( k \geq 3 \) be fixed and let \( A \subset [k]^n \) be a symmetric intersecting family; we wish to show that \( |A| = o(k^n) \). The proof will consist of the following steps.

1. Enlarge the space \([k]^n\) to a space \( W^n \), where \( W \) is a suitably chosen ‘covering poset’ equipped with an appropriate measure flow.
2. Use the fact that \( A \) is intersecting to conclude that the up-closure of \( A \) in \( W^n \) can have density at most \( 1/2 \) at a suitable time in the measure flow.
3. From the symmetry of \( A \), deduce by Theorem 2.1 that \( A \) must have been negligibly small in the original space \([k]^n\).

Proof of Theorem 1.1. Let \((W, \preceq)\) be the poset
\[ W = [k]^{(1)} \cup [k]^{(k-1)}, \]
and the set consisting of all subsets of \([k]\) of size 1 or \( k-1 \), with \( \preceq \) defined by inclusion. We embed \([k]\) in \( W \) by identifying \([k]\) with \([k]^{(1)}\) in the obvious way.

Let \( \mu_0 \) be the uniform probability measure on \([k]^{(1)}\) and let \( \mu_1 \) be the uniform probability measure on \([k]^{(k-1)}\). As before, we define \( \mu_t = (1-t)\mu_0 + t\mu_1 \), noting that \( \mu_{1/2} \) is the uniform probability measure on \( W \).

Claim 3.1. For \( k \geq 3 \), the flow from \( \mu_0 \) to \( \mu_1 \) has connectivity \( 1/k \).

Proof. Let \( A \subset W \) be a proper, nontrivial up-set. If either \( A \subset [k]^{(k-1)} \) or \( A \supset [k]^{(k-1)} \), then it is clear that
\[ (\mu_1 - \mu_0)(A) \geq 1/k. \]
The only other case is essentially the star
\[ \mathcal{A} = \{1\} \cup ([k]^{k-1} \setminus \{2, \ldots, k\}), \]
for which we have
\[ (\mu_1 - \mu_0)(\mathcal{A}) = 1 - 2/k \geq 1/k. \]

Next, we extend the notion of an intersecting family to \( W^n \) by saying that \( \mathcal{A} \subseteq W^n \) is intersecting if for any \( x, y \in \mathcal{A} \), there is a coordinate \( i \in [n] \) such that \( x_i \cap y_i \neq \emptyset \).

**Claim 3.2.** If \( \mathcal{A} \subseteq W^n \) is intersecting, then \( \mu_{1/2}(\mathcal{A}) \leq 1/2. \)

**Proof.** Note that if \( x \sim \mu_{1/2} \), then we have \( x^c \sim \mu_{1/2} \), where \( x^c = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \) is the point-wise complement of \( x \). Since at most one of \( x \) and \( x^c \) can belong to \( \mathcal{A} \), we have
\[ 2\mu_{1/2}(\mathcal{A}) = \mathbb{E}[|\mathcal{A} \cap \{x, x^c\}|] \leq 1. \]

With these constructions and claims in hand, we may now finish as follows. Let \( \mathcal{A} \subseteq [k]^n \) be a symmetric intersecting family, and let \( \mathcal{B} \) be its up-closure in \( W^n \). Observe that \( \mathcal{B} \) is also symmetric and intersecting, so we know that \( \mu_{1/2}(\mathcal{B}) \leq 1/2. \)

Therefore, by Theorem 2.1 with \( p = 0, q = 1/2, \) and \( \varepsilon = \mu_0(\mathcal{B}) \) we have
\[ 1/2 \leq C\kappa^{-1} \log(1/2\varepsilon)/\log n, \]
where \( \kappa = 1/k > 0 \) and \( C > 0 \) is the absolute constant as in the theorem. After rearranging, we get
\[ |\mathcal{A}|/k^n = \mu_0(\mathcal{B}) \leq \frac{1}{2} n^{-\kappa/2C}, \]
as claimed.

4. **Conclusion**

Our work raises a few different questions that we now discuss; since the case where \( k = 3 \) appears to capture all of the inherent difficulty, we restrict our attention to \([3]^n\) in what follows.

First, the most basic question that our main result raises is that of determining precisely how large a symmetric intersecting subfamily of \([3]^n\) can be. Theorem 1.1 asserts that the size of any symmetric intersecting subfamily of \([3]^n\) is \( o(3^n) \). Turning to lower bounds, attempting to generalise the natural construction in the Boolean hypercube leads us to the family of all the vectors in \([3]^n\) with at least \((n + 1)/2\)
coordinates equal to 3; this family has size \((3 - \varepsilon)^n\), where \(\varepsilon > 0\) is an absolute constant. We can do much better by considering projective-geometric constructions: we start with a finite projective plane on \(n\) points (assuming one exists, which we can ensure by taking \(n = q^2 + q + 1\) for some prime power \(q\), for example), and for each line \(L\) of the plane, we include in our family all the vectors in \([3]^n\) that have the value 3 in each of the coordinates corresponding to the points in \(L\); it is not hard to see that this produces a symmetric intersecting subfamily of \([3]^n\) of size at least \(3^n - \sqrt{n}\).

Our inability to find better constructions leads us to speculate that there exist universal constants \(c, \delta > 0\) such that for any symmetric intersecting family \(A \subset [3]^n\), we have

\[
\log_3 |A| \leq n - cn^\delta.
\]

Of course, we expect similar behaviour for each fixed \(k \geq 3\), but even the above estimate seems presently out of reach.

Next, the above construction has the curious property that if we form a family of sets \(B\) on \([n]\) from \(A \subset [3]^n\) as above by taking, for each \(x \in A\), the set of coordinates where \(x\) takes the value 3, then \(B\) is itself an intersecting family of sets; we say that such intersecting families of vectors, where all of the intersections are witnessed by a single element of the alphabet [3], are set-intersecting. It is natural to wonder if we can leverage the entire alphabet and build large symmetric intersecting families in \([3]^n\) which are not set-intersecting. One way to accomplish this is as follows: by choosing randomly, we may find three disjoint subsets \(x_1, x_2, x_3 \subset [n]\) of size \(100n^{1/2} \log n\) such that every one of these sets intersects each of its cyclic translates nontrivially; then take \(A \subset [3]^n\) to be the set of all vectors \(x\) that take the value \(i\) on some cyclic translate of \(x_i\) for at least two different values \(i \in [3]\). This family \(A\) is easily seen to be a symmetric intersecting family; however, we may construct a larger family than \(A\) just by taking all the vectors \(x\) that take the value 3 on some translate of \(x_3\), for example. This suggests the following question: could it be true that the largest symmetric intersecting families in \([3]^n\) are necessarily set-intersecting?

Finally, we anticipate the main technical contribution of this paper — studying intersection problems by situating them in the appropriate larger covering space equipped with a suitable measure flow — to be applicable to other questions in
extremal set theory as well; we expect to return to this circle of ideas in future work.

Acknowledgements

The second author wishes to acknowledge support from NSF grant DMS-1501962, the third author was partially supported by NSF grant DMS-1800521, and the fourth author was supported by NSF grant DMS-1802201.

References

2. L. Babai, Personal communication.

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK

*Email address: eberhard.math@gmail.com*

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA

*Email address: jkahn@math.rutgers.edu*

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA

*Email address: narayanan@math.rutgers.edu*

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA

*Email address: sophie.spirkl@rutgers.edu*