ON SYMMETRIC INTERSECTING FAMILIES

DAVID ELLIS, GIL KALAI, AND BHARGAV NARAYANAN

Abstract. We consider a question of Babai from the 1970s, which asks the following: for \( n, k \in \mathbb{N} \) with \( k \leq n/2 \), what is the largest possible cardinality \( s(n, k) \) of a symmetric intersecting family of \( k \)-element subsets of \( \{1, 2, \ldots, n\} \). (We say a family of subsets of \( \{1, 2, \ldots, n\} \) is symmetric if it admits a transitive group of automorphisms.) We establish both upper and lower bounds for \( s(n, k) \), and show in particular that \( s(n, k) = o(\binom{n-1}{k-1}) \) as \( n \to \infty \) if and only if \( k = n/2 - \omega(n)/(n/\log n) \) for some function \( \omega(\cdot) \) that increases without bound, thereby determining the threshold at which symmetric intersecting families are negligibly small compared to maximum-sized intersecting families. We also exhibit some connections to basic questions in group theory and additive number theory, and pose a number of related problems.

1. Introduction

A family of sets is said to be intersecting if any two sets in the family have nonempty intersection, and uniform if all the sets in the family have the same size. In this paper, we study uniform intersecting families. The most well-known result about such families is the Erdős–Ko–Rado theorem [11].

**Theorem 1.1.** Let \( n, k \in \mathbb{N} \) with \( k \leq n/2 \). If \( \mathcal{A} \) is an intersecting family of \( k \)-element subsets of \( \{1, 2, \ldots, n\} \), then \( |\mathcal{A}| \leq \binom{n-1}{k-1} \). Furthermore, if \( k < n/2 \), then equality holds if and only if \( \mathcal{A} \) consists of all the \( k \)-element subsets of \( \{1, 2, \ldots, n\} \) which contain some fixed element \( i \in \{1, 2, \ldots, n\} \).

Over the last fifty years, many variants of this theorem have been obtained, variants involving different intersection conditions (e.g. that any two sets in the family have intersection of size at least \( t \), for some fixed \( t \in \mathbb{N} \)) or different structures (e.g. families of permutations, or families of graphs). A common feature of many of these variants is that the maximum-sized families are ‘highly asymmetric’, in

*Date: 16th September 2019.*

2010 *Mathematics Subject Classification*. Primary 05D05; Secondary 05E18.
the sense that they depend strongly on a small number of coordinates. This is the case, for example, in the Erdős–Ko–Rado theorem itself, in the Hilton–Milner theorem \cite{18}, and in Frankl’s generalisation \cite{13} of these results. It is therefore natural to ask what happens to the maximum possible size of a uniform intersecting family when one imposes a symmetry requirement on the family. Perhaps the most natural symmetry requirement to impose is that the family have a transitive group of automorphisms. Indeed, about forty years ago, Babai \cite{1} asked for a determination of the maximum possible size of an intersecting family of \(k\)-element subsets of \(\{1, 2, \ldots, n\}\) which admits a transitive group of automorphisms, for each \(k, n \in \mathbb{N}\).

For the benefit of the reader, we proceed to define these concepts in full. For \(n \in \mathbb{N}\), we write \([n] = \{1, 2, \ldots, n\}\). We write \(S_n\) for the symmetric group on \([n]\) and \(P_n\) for the power-set of \([n]\). For a permutation \(\sigma \in S_n\) and a set \(x \subset [n]\), we write \(\sigma(x)\) for the image of \(x\) under \(\sigma\), and for \(A \subset P_n\), we write \(\sigma(A) = \{\sigma(x) : x \in A\}\). We define the automorphism group of a family \(A \subset P_n\) by

\[
\text{Aut}(A) = \{\sigma \in S_n : \sigma(A) = A\}.
\]

We say that \(A \subset P_n\) is symmetric if \(\text{Aut}(A)\) is a transitive subgroup of \(S_n\), i.e., if for all \(i, j \in [n]\), there exists a permutation \(\sigma \in \text{Aut}(A)\) such that \(\sigma(i) = j\).

For a pair of integers \(n, k \in \mathbb{N}\) with \(k \leq n/2\), let \([n]^{(k)}\) denote the family of all \(k\)-element subsets of \([n]\). In this paper, we will be concerned with investigating the quantity

\[
s(n, k) = \max\{|A| : A \subset [n]^{(k)}, A \text{ is symmetric and intersecting}\}.
\]

Of course, if \(k > n/2\), then \([n]^{(k)}\) itself is a symmetric intersecting family, so \(s(n, k) = \binom{n}{k}\); we may therefore restrict our attention to the case where \(k \leq n/2\).

With these definitions in place, we may restate the aforementioned question of Babai as follows.

**Problem 1.2.** For each \(k, n \in \mathbb{N}\) with \(k \leq n/2\), determine \(s(n, k)\).

Since the early 1980s, several results have been obtained on intersection problems for symmetric families in the non-uniform setting, that is, where one seeks to determine the maximum possible size of a symmetric subfamily of \(P_n\) satsifying some intersection condition; see for example the results of Frankl \cite{12} and of Cameron, Frankl and Kantor \cite{6}, and the more recent results of the first and third authors \cite{10}. Relatively little seems to be known in the uniform setting, however.
In this paper, we focus on determining when a symmetric uniform intersecting family must be \textit{significantly smaller} than the extremal families (of the same uniformity) in the Erdős–Ko–Rado theorem. A more precise formulation of this question is as follows.

\textbf{Question 1.3.} For which \( k = k(n) \leq n/2 \) is \( s(n, k) = o\left(\binom{n-1}{k-1}\right) \) as \( n \to \infty \)?

Utilising a well-known sharp threshold result of Friedgut and the second author \cite{friedgut2000threshold}, we obtain the following upper bound.

\textbf{Theorem 1.4.} There exists a universal constant \( c > 0 \) such that for any \( n, k \in \mathbb{N} \) with \( k \leq n/2 \), we have

\[ s(n, k) \leq \exp\left(-\frac{c(n - 2k) \log n}{k(\log n - \log k)}\right) \binom{n}{k}. \]

We also give a construction showing that Theorem 1.4 is sharp up to the value of \( c \), in the regime where \( k/n \) is bounded away from zero. This construction, in conjunction with Theorem 1.4, provides a complete answer to Question 1.3.

\textbf{Theorem 1.5.} If \( k = k(n) \leq n/2 \), then as \( n \to \infty \), \( s(n, k) = o\left(\binom{n-1}{k-1}\right) \) if and only if

\[ k = \frac{n}{2} - \omega(n) \left( \frac{n}{\log n} \right) \]

for some function \( \omega(\cdot) \) that increases without bound.

While Question 1.3 is the most basic question in the regime where the uniformity \( k \) is large compared to the size \( n \) of the ground set, the most basic question in the regime where \( k \) is small compared to \( n \) concerns the \textit{existence} of nonempty symmetric intersecting families. Note that if \( s(n, k) > 0 \), then \( s(n, l) > 0 \) for all \( l > k \); indeed, if \( \mathcal{A} \subset [n]^{(k)} \) is nonempty, symmetric and intersecting, then so is \( \{y \in [n]^{(l)} : x \subset y \text{ for some } x \in \mathcal{A}\} \). With this in mind, for each \( n \in \mathbb{N} \), we define

\[ g(n) = \min\{k \in \mathbb{N} : s(n, k) > 0\}. \]

It turns out that problem of determining the asymptotic behaviour of the function \( g(\cdot) \) is intimately connected to some long-standing problems in additive number theory. It is not hard to show, as we shall see, that \( g(n) \geq \sqrt{n} \) for all \( n \in \mathbb{N} \); we raise the question of whether this bound is asymptotically tight.

\textbf{Question 1.6.} Is it true that \( g(n) = (1 + o(1))\sqrt{n} \) for all \( n \in \mathbb{N} \)?
While we are able to show that the asymptotic in Question 1.6 holds along various (arithmetically special) infinite sequences of positive integers, we are currently unable to settle the question completely.

This paper is organised as follows. We give the proof of Theorem 1.4 in Section 2. In Section 3, we describe a combinatorial approach to constructing large symmetric intersecting families in the regime where \( k \) is not too small compared to \( n \), and deduce Theorem 1.5 as a consequence. In Section 4, we turn to the regime where \( k \) is small, and describe various algebraic constructions of symmetric intersecting families in this regime. We conclude in Section 5 with some open problems.

2. Upper bounds

We first describe briefly the notions and tools we will need for the proof of Theorem 1.4. In what follows, all logarithms are to the base \( e \).

We begin with the following simple observation which may be found in [6], for example; we include a proof for completeness.

**Proposition 2.1.** For all \( n, k \in \mathbb{N} \) with \( 1 < k \leq \sqrt{n} \), we have \( s(n, k) = 0 \).

**Proof.** The proposition follows from a simple averaging argument. Indeed, let \( k \leq \sqrt{n} \), suppose for a contradiction that \( A \subset [n]^{(k)} \) is a nonempty, symmetric intersecting family, and let \( x \in A \). If we choose \( \sigma \in \text{Aut}(A) \) uniformly at random, then since \( \text{Aut}(A) \) is transitive, we have

\[
\mathbb{E}[|x \cap \sigma(x)|] = \frac{k^2}{n} \leq 1.
\]

Since \( |x \cap \text{Id}(x)| = k > 1 \), there must exist a permutation \( \sigma \in \text{Aut}(A) \) such that \( x \cap \sigma(x) = \emptyset \), contradicting the fact that \( A \) is intersecting. \( \square \)

For \( 0 \leq p \leq 1 \), we write \( \mu_p \) for the \( p \)-biased measure on \( \mathcal{P}_n \), defined by

\[
\mu_p(\{x\}) = p^{|x|}(1 - p)^{n-|x|} \quad \forall x \subset [n].
\]

We say that a family \( F \subset \mathcal{P}_n \) is *increasing* if it is closed under taking supersets, i.e., if \( x \in F \) and \( x \subset y \), then \( y \in F \). It is easy to see that if \( F \subset \mathcal{P}_n \) is increasing, then \( p \mapsto \mu_p(F) \) is a monotone non-decreasing function on \([0, 1] \). For a family \( F \subset \mathcal{P}_n \), we write \( F^\uparrow \) for the smallest increasing family containing \( F \); in other words, \( F^\uparrow = \{ y \subset [n] : x \subset y \text{ for some } x \in F \} \).
We need the following fact, which allows one to bound from above the size of a family $F \subseteq [n]^{(k)}$ in terms of $\mu_p(F^\uparrow)$, where $p \approx k/n$; this was proved in a slightly different form by Friedgut [14]. We provide a proof for completeness.

**Lemma 2.2.** Let $n, k \in \mathbb{N}$ and suppose that $0 < p, \phi < 1$ satisfy

$$p \geq \frac{k}{n} + \frac{\sqrt{2n \log(1/\phi)}}{n}.$$ 

Then for any family $F \subseteq [n]^{(k)}$, we have

$$\mu_p(F^\uparrow) > (1 - \phi) \frac{|F|}{\binom{n}{k}}.$$ 

**Proof.** Let $\delta = |F|/\binom{n}{k}$ and let $X$ be a binomial random variable with distribution $\text{Bin}(n, p)$. For each $l \geq k$, the Local LYM inequality implies that

$$|F^\uparrow \cap [n]^{(l)}| \geq |F| \binom{n}{l} = \delta.$$ 

It follows from a standard Chernoff bound that

$$\mathbb{P}(X < (1 - \eta)np) < \exp(-\eta^2 np/2) \quad \forall \eta > 0. \tag{1}$$ 

Hence,

$$\mu_p(F^\uparrow) \geq \sum_{l=k}^n p^l (1-p)^{n-l} \binom{n}{l} \delta = \mathbb{P}(X \geq k) \delta > (1 - \phi) \delta,$$

using (1) with $k = (1 - \eta)np$. \qed

We also require the following sharp threshold result due to Friedgut and the second author [15].

**Theorem 2.3.** There exists a universal constant $c_0 > 0$ such that the following holds for all $n \in \mathbb{N}$. Let $0 < p, \varepsilon < 1$ and let $F \subseteq \mathcal{P}_n$ be a symmetric increasing family. If $\mu_p(F) > \varepsilon$, then $\mu_q(F) > 1 - \varepsilon$, where

$$q = \min \left\{ 1, p + c_0 \left( \frac{p \log(1/p) \log(1/\varepsilon)}{\log n} \right) \right\}. \tag{2}$$ 

The idea of the proof of Theorem 1.4 is as follows. Let $A \subseteq [n]^{(k)}$ be a symmetric intersecting family. We first use Lemma 2.2 to bound $|A|/\binom{n}{k}$ from above in terms of $\mu_p(A^\uparrow)$, where $p \approx k/n$; we then use Theorem 2.3, together with the simple fact that $\mu_{1/2}(A^\uparrow) \leq 1/2$, to bound $\mu_p(A^\uparrow)$, and hence $|A|$, from above.
Let us remark that this strategy of approximating the uniform measure on \([n]^{(k)}\) with the \(p\)-biased measure \(\mu_p\), where \(p \approx k/n\), and using the \(p\)-biased setting to obtain results about the \(k\)-uniform setting, has been employed in several previous works concerning uniform intersecting families; we refer the reader in particular to the work of Friedgut [14] and of Dinur and Friedgut [9].

Proof of Theorem 1.4. Let \(n, k \in \mathbb{N}\) with \(k \leq n/2\), let \(A \subset [n]^{(k)}\) be a symmetric intersecting family, and set \(\delta = |A|/(\binom{n}{k})\).

In the light of Proposition 2.1, we may assume that \(k > \sqrt{n}\). By the Erdős–Ko–Rado theorem, we have \(\delta \leq k/n \leq 1/2\), so we may also assume (by choosing \(c > 0\) to be sufficiently small, say \(c < (\log 2)^2\)) that \(k \leq n/2 - 10n/\log n\).

Applying Lemma 2.2 with \(p = k/n + \sqrt{(2\log 2)n}/n\) and \(\phi = 1/2\), we see that

\[
\mu_p(A^\uparrow) > \frac{\delta}{2}.
\]

Since \(A\) is symmetric, so is \(A^\uparrow\). We may therefore apply Theorem 2.3 with \(\varepsilon = \delta/2\) to deduce that \(\mu_q(A^\uparrow) > 1/2\), where

\[
q = \min\left\{1, p + c_0 \left(\frac{p \log(1/p) \log(2/\delta)}{\log n}\right)\right\}.
\]

Since \(A^\uparrow\) is increasing, the function \(r \mapsto \mu_r(A^\uparrow)\) is monotone non-decreasing on \([0, 1]\). Moreover, since \(A\) is intersecting, so is \(A^\uparrow\), and therefore \(\mu_{1/2}(A^\uparrow) \leq 1/2\). Since \(\mu_{1/2}(A^\uparrow) \leq 1/2\) and \(\mu_q(A^\uparrow) > 1/2\), the monotonicity of \(r \mapsto \mu_r(A^\uparrow)\) implies that

\[
p + c_0 \left(\frac{p \log(1/p) \log(2/\delta)}{\log n}\right) > \frac{1}{2}.
\]

Rearranging, we see that

\[
\delta < 2 \exp\left(-\frac{(1 - 2p) \log n}{2c_0p \log(1/p)}\right) \leq \exp\left(-c(n - 2k) \frac{\log n}{k(\log n - \log k)}\right),
\]

where the last inequality above holds for some universal constant \(c > 0\), provided \(\sqrt{n} < k \leq n/2 - 10n/\log n\) (which we may assume). This completes the proof. \(\square\)

3. Lower bounds for large \(k\)

In this section, we give a construction showing that Theorem 1.4 is sharp up to the value of the constant \(c\) in the exponent for many choices of \(k = k(n)\).
For a set $S \subset \mathbb{Z}_n$, we define its characteristic vector $\chi_S \in \{0, 1\}^{\mathbb{Z}_n}$ by $(\chi_S)_i = 1$ if $i \in S$ and $(\chi_S)_i = 0$ if $i \notin S$. Identifying $[n]$ with $\mathbb{Z}_n$, we define $\mathcal{F}(n, k)$ to be the family of all $k$-element subsets $S \subset \mathbb{Z}_n$ such that the longest run of (cyclically) consecutive ones in $\chi_S$ is longer than the longest run of (cyclically) consecutive zeros in $\chi_S$. In other words, slightly less formally, we take $\mathcal{F}(n, k)$ to consist of all the cyclic strings of $n$ zeros and ones which contain exactly $k$ ones and in which the longest run of consecutive ones is longer than the longest run of consecutive zeros.

It is clear that $\mathcal{F}(n, k)$ is symmetric, since any cyclic shift is an automorphism of $\mathcal{F}(n, k)$. It is also easy to check that $\mathcal{F}(n, k)$ is intersecting. Indeed, given $S, T \in \mathcal{F}(n, k)$, suppose without loss of generality that the longest run of consecutive ones in $S$ is at least as long as that in $T$. Choose a run of consecutive ones in $S$ of the maximum length; these cannot be all zeros in $T$ because otherwise $T$ would have a longer run of consecutive ones than $S$. Therefore, $S \cap T \neq \emptyset$.

We note that the non-uniform case of this construction, i.e., the family of all cyclic strings of $n$ zeros and ones in which the longest run of consecutive ones is longer than the longest run of consecutive zeros, shows that the Kahn–Kalai–Linial theorem [19] cannot be improved by more than a constant factor for intersecting families; see [20] for more details.

Theorem 1.4 implies that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $k/n \geq \epsilon$, then

$$s(n, k) \leq \exp\left(-\frac{\delta(n - 2k)\log n}{n}\right)\binom{n}{k}.$$  \hfill (2)

The following lower bound for $|\mathcal{F}(n, k)|$ shows that Theorem 1.4 is sharp up to the constant factor in the exponent when $k/n$ is bounded away from zero.

**Lemma 3.1.** For each $\epsilon > 0$, there exists $C > 0$ such that for any $n, k \in \mathbb{N}$ with $\epsilon \leq k/n \leq 1/2$, we have

$$|\mathcal{F}(n, k)| \geq C_0 \exp\left(-\frac{C(n - 2k)\log n}{n}\right)\binom{n}{k},$$

where $C_0 > 0$ is a universal constant.

In fact, we will prove the following stronger statement.

**Lemma 3.2.** There exists a universal constant $C_0 > 0$ such that the following holds. If $k = k(n) \leq n/2$ is such that $\sqrt{n\log n} = o(k)$ as $n \to \infty$, then

$$|\mathcal{F}(n, k)| \geq C_0 \exp\left(-\left(\frac{\log(n - k) - \log k}{\log n - \log(n - k)}\right)\log n\right)\binom{n}{k}.$$  \hfill (2)
To prove Lemma 3.2, we need the following.

**Lemma 3.3.** Let \( k, n \in \mathbb{N} \) with \( k < n \), and let
\[
l \geq \frac{\log n + 2 \log 2}{\log n - \log(n - k)}.
\]
Then the number of cyclic strings of \( n \) zeros and ones with exactly \( k \) ones and a run of consecutive zeros of length at least \( l \) is at most \( \frac{1}{4} \binom{n}{k} \).

**Proof.** The number of such strings is at most \( n \binom{n-k}{k} \), since (possibly overcounting) there are \( n \) choices for the position of the run of \( l \) consecutive zeros, and then \( \binom{n-k}{k} \) choices for the positions of the ones. We have
\[
\binom{n-k}{k} = \frac{n(n-k)(n-k-1)\ldots(n-k-l+1)}{(n-l+1)!} \leq n \left( \frac{n-k}{n} \right)^l \leq \frac{1}{4}
\]
provided \( l \geq (\log n + 2 \log 2)/\left(\log n - \log(n - k)\right) \), as required. \( \square \)

It is easy to see (e.g. from the Local LYM inequality) that if \( k \leq n/2 \), then the number of cyclic strings of length \( n \) with \( k \) ones and a run of consecutive ones of length at least \( l \) is at most the number of cyclic strings of length \( n \) with \( k \) ones and a run of consecutive zeros of length at least \( l \), so the following is now immediate.

**Corollary 3.4.** Let \( k \leq n/2 \). The number of cyclic strings of length \( n \) with \( k \) ones and no run of \( l \) consecutive zeros or ones is at least \( \frac{1}{2} \binom{n}{k} \), provided
\[
l \geq \frac{\log n + 2 \log 2}{\log n - \log(n - k)}.
\]
\( \square \)

We are now ready to prove Lemma 3.2.

**Proof of Lemma 3.2.** Choose \( l_0 \in \mathbb{N} \) such that
\[
l_0 - 1 \geq \frac{\log(n - l_0 - 2) + 2 \log 2}{\log(n - l_0 - 2) - \log(n - k - 2)}.
\]
(3)
Observe that \( \mathcal{F}(n, k) \) contains all cyclic strings of length \( n \) with \( k \) ones, precisely one run of \( l_0 \) consecutive ones, all other runs of consecutive ones having length at most \( l_0 - 2 \), and no run of \( l_0 \) consecutive zeros. We claim that if \( l_0 < n/2 \), then the number of such strings is at least
\[
\frac{n}{2} \binom{n-l_0-2}{k-l_0},
\]
Indeed, there are $n$ choices for the position of the run of $l_0$ consecutive ones, and there must be a zero on each side of this run of ones. Now, there are at least $\frac{1}{2}(n - l_0 - 2)$ choices for the remainder of the cyclic string (by Corollary 3.4), since if we take a cyclic string of length $n - l_0 - 2$ which contains no run of $l_0 - 1$ consecutive ones or zeros, and then insert (at some point) a run of $l_0$ consecutive ones with a zero on either side into this string, then the resulting string has the desired property provided $l_0 < n/2$.

It is easily checked that if $\sqrt{n \log n} = o(k)$, then we may choose $l_0 \in \mathbb{N}$ satisfying (3) such that

$$l_0 = (1 + O(1/\log n)) \frac{\log n}{\log n - \log(n - k)};$$

we then have $l_0 < n/2$ and

$$|\mathcal{F}(n, k)| \geq \frac{n}{2} \binom{n - l_0 - 2}{k - l_0} \geq \frac{n}{2} \binom{n - l_0 - 2}{k - l_0 - 2}$$

$$\geq \frac{n}{2} \binom{k - l_0 - 2}{n - l_0 - 2} l_0 + 2 \binom{n}{k} = \exp \left( - \left( \frac{\log(n - k) - \log n}{\log n - \log(n - k)} \right) \log n + O(1) \right) \binom{n}{k},$$

proving the lemma. □

We can now prove Lemma 3.1.

Proof of Lemma 3.1. Writing $\eta = (n - 2k)/2n$, we have $\eta \leq 1/2 - \epsilon$, and

$$\frac{\log(n - k) - \log k}{\log n - \log(n - k)} = \frac{\log(1 + 2\eta) - \log(1 - 2\eta)}{\log 2 - \log(1 + 2\eta)} = \frac{4\eta + O_{\epsilon}(\eta^3)}{\log 2 - \log(1 + 2\eta)} = \Theta_\epsilon(\eta).$$

Hence, it follows from Lemma 3.2 that

$$|\mathcal{F}(n, k)| \geq C_0 \exp(-2C\eta \log n) \binom{n}{k},$$

for some constant $C > 0$ depending on $\epsilon$ alone, as required. □

With the above estimates in hand, Theorem 1.5 is easily established.
Proof of Theorem 1.5. First, recall from Proposition 2.1 that \(s(n, k) = 0\) if \(k \leq \sqrt{n}\).
Next, suppose that \(\sqrt{n} < k \leq n / \log n\). Then, by Theorem 1.4, we have
\[
s(n, k) \leq \exp \left( -\frac{cn \log n}{2k(\log n - \log k)} \right) \binom{n}{k}
\]
\[
\leq \exp \left( -\frac{(\log n)^2}{\log \log n} \right) \binom{n}{k}
\]
\[
= o \left( \frac{1}{\sqrt{n}} \binom{n}{k} \right) = o \left( \frac{k}{n} \binom{n}{k} \right) = o \left( \binom{n - 1}{k - 1} \right).
\]

Now, suppose that \(n / \log n \leq k \leq n / 4\). Then, again by Theorem 1.4, we have
\[
s(n, k) \leq \exp \left( -\frac{4c \log n}{2 \log 4} \right) \binom{n}{k}
\]
\[
= o \left( \frac{1}{\log n} \binom{n}{k} \right) = o \left( \frac{k}{n} \binom{n}{k} \right) = o \left( \binom{n - 1}{k - 1} \right).
\]

Finally, suppose that \(n / 4 \leq k \leq n / 2\) and let \(\zeta > 0\). If \(n \geq \zeta(n - 2k) \log n\), then by applying Lemma 3.1 with \(\varepsilon = 1 / 4\), we see that there exists a universal constant \(C > 0\) such that
\[
s(n, k) \geq C_0 e^{-c/\zeta} \binom{n}{k} \geq 2C_0 e^{-c/\zeta} \binom{n - 1}{k - 1}. \tag{4}
\]
If, on the other hand, we have \(n \leq \zeta(n - 2k) \log n\), then it follows from (2) that there exists a universal constant \(\delta > 0\) such that
\[
s(n, k) \leq e^{-\delta/\zeta} \binom{n}{k} \leq 4 e^{-\delta/\zeta} \binom{n - 1}{k - 1}. \tag{5}
\]
The result is now immediate from (4) and (5).
\(\square\)

4. Nonempty symmetric intersecting families of small uniformity

We now turn our attention to the question of determining for which pairs of positive integers \((n, k)\) there exist nonempty symmetric intersecting subfamilies of \([n]^{(k)}\) — that is, we investigate the set
\[
\mathcal{S} = \{(n, k) \in \mathbb{N}^2 : s(n, k) > 0\}.
\]
Along the way, we will describe some (algebraic) constructions of symmetric intersecting families which are larger than \(\mathcal{F}(n, k)\) for certain values of \(n\) and \(k\).
As mentioned in the Introduction, if $s(n, k) > 0$, then $s(n, l) > 0$ for all $l > k$. Hence, defining

$$g(n) = \min\{k \in \mathbb{N} : s(n, k) > 0\}$$

for each $n \in \mathbb{N}$, we have $s(n, k) > 0$ iff $k \geq g(n)$.

It follows immediately from Proposition 2.1 that

$$g(n) \geq \lceil \sqrt{n} \rceil + 1 \quad \forall n \geq 2.$$  \hfill (6)

In the other direction, it is easy to check that $F(n, k) \neq \emptyset$ if and only if $n \leq \lfloor k/2 \rfloor^2 + k$, which implies that

$$g(n) \leq 2\sqrt{n} \quad \forall n \in \mathbb{N}.$$  \hfill (7)

We now seek to narrow the gap between (6) and (7). To improve (7), we note a strong connection between the problem of determining $g(n)$ and the problem of covering an Abelian group using a difference set. If $G$ is a finite Abelian group and $S \subset G$, we say that $S$ is a difference cover for $G$ if $S - S = G$. (Here, $S - S := \{i - j : i, j \in S\}$.)

We define

$$h(G) = \min\{|S| : S \text{ is a difference cover for } G\}.$$  \hfill (8)

Note that if $S \subset G$, then $S$ is a difference cover for $G$ if and only if the family of all the translates of $S$ is an intersecting family of subsets of $G$; this observation yields the following.

**Lemma 4.1.** For all $n \in \mathbb{N}$, we have $g(n) \leq h(\mathbb{Z}_n)$. If $n$ is prime, then equality holds.

**Proof.** Let $h = h(\mathbb{Z}_n)$ and write $\mathbb{Z}_n^{(h)}$ for the family of $h$-element subsets of $\mathbb{Z}_n$. By definition, there exists $S \in \mathbb{Z}_n^{(h)}$ such that $S - S = \mathbb{Z}_n$. Let $A = \{S + j : j \in \mathbb{Z}_n\} \subset \mathbb{Z}_n^{(h)}$ denote the family of all the translates of $S$. Then $A$ is clearly symmetric and intersecting. Hence, $g(n) \leq h$, proving the first part of the claim.

Now suppose that $n$ is prime, and let $g(n) = k$. Let $A \subset [n]^{(k)}$ be a nonempty, symmetric intersecting family. Since $\text{Aut}(A) \leq S_n$ is transitive, the orbit-stabilizer theorem implies that $n$ divides $|\text{Aut}(A)|$, and therefore by Sylow’s theorem, $\text{Aut}(A)$ has a cyclic subgroup $H$ of order $n$. Let $\sigma \in S_n$ be a generator of $H$; then $\sigma$ is an $n$-cycle, and by relabelling the ground set $[n]$ if necessary, we may assume that $\sigma = (1 \ 2 \ \ldots \ \ n)$ (in the standard cycle notation). Fix $x \in A$ and note that $B = \{x, \sigma(x), \ldots, \sigma^{n-1}(x)\}$ is also a nonempty, symmetric intersecting family as $H \leq \text{Aut}(B)$. Clearly, $B$ consists of all the cyclic translates, modulo $n$, of $x$. If we
regard \(x\) as a subset of \(\mathbb{Z}_n\), then since \(\mathcal{B}\) is intersecting, we have \(x - x = \mathbb{Z}_n\), i.e., \(x\) is a difference cover for \(\mathbb{Z}_n\). Hence, \(h(\mathbb{Z}_n) \leq k\) and it follows that \(h(\mathbb{Z}_n) = g(n)\) when \(n\) is prime, as required.

We now describe how existing constructions of difference covers lead to an improvement of (7). We say that \(S \subset \mathbb{Z}\) is a difference cover for \(n\) if \([n] \subset S \setminus S\). For each \(n \in \mathbb{N}\), let \(\pi_n : \mathbb{Z} \to \mathbb{Z}_n\) denote the natural projection modulo \(n\), defined by \(\pi_n(i) = i \mod n\) for all \(i \in \mathbb{Z}\). Note that if \(S \subset \mathbb{Z}\) is a difference cover for \([n/2]\), then \(\pi_n(S)\) is a difference cover for \(\mathbb{Z}_n\). Building on work of Rédei and Rényi [23] and of Leech [21], Golay [17] proved that for any \(n \in \mathbb{N}\), there exists a difference cover for \(n\) of size at most \(\sqrt{cn}\), where \(c < 2.6572\) is an absolute constant. It follows that for any \(n \in \mathbb{N}\), we have

\[
g(n) \leq h(\mathbb{Z}_n) \leq 1.1527\sqrt{n}.
\]

Unfortunately, one cannot hope to answer Question 1.6 in the affirmative purely by projecting difference covers for \([n/2]\) into \(\mathbb{Z}_n\) and using the fact that \(g(n) \leq h(\mathbb{Z}_n)\); this is a consequence of a result of Rédei and Rényi [23] which asserts that if \(S \subset \mathbb{Z}\) is a difference cover for \(n\), then

\[
|S| \geq \sqrt{\left(2 + \frac{4}{3\pi}\right)n}.
\]

In view of Lemma 4.1, we are led to the following natural question, which has occurred independently to others (see e.g. Banakh and Gavrylkiv [2]).

**Question 4.2.** Is it true that \(h(\mathbb{Z}_n) = (1 + o(1))\sqrt{n}\) for all \(n \in \mathbb{N}\)?

By Lemma 4.1, an affirmative answer to this question would imply an affirmative answer to Question 1.6. We remark that Question 4.2 is a ‘covering’ problem whose ‘packing’ counterpart has received a lot of attention. If \(G\) is an Abelian group and \(S \subset G\), we say that \(S\) is a Sidon set in \(G\) if for any non-identity element \(g \in G\), there exists at most one ordered pair \((s_1, s_2) \in S^2\) such that \(g = s_1 - s_2\). For \(n \in \mathbb{N}\), let

\[
\lambda(n) = \max\{|S| : S \subset \mathbb{Z}_n\text{ such that } S \text{ is a Sidon set}\}.
\]

The determination of \(\lambda(n)\) is a well-known open problem; see [7], for example. In particular, the following remains open.

**Question 4.3.** Is it true that \(\lambda(n) = (1 - o(1))\sqrt{n}\) for all \(n \in \mathbb{N}\)?
The constructions of Singer [26] and Bose [4] yield affirmative answers to Question 4.3 when \( n \) is of the form \( q^2 + q + 1 \) or \( q^2 - 1 \) respectively, where \( q \) is a prime power, and a construction due to Ruzsa [24] does so when \( n \) is of the form \( p^2 - p \), where \( p \) is prime. As observed by Banakh and Gavrylkiv [2], the constructions of Singer, Bose and Ruzsa yield efficient difference covers as well, providing affirmative answers to Questions 4.2 and 1.6 for all \( n \) of the above forms.

Returning to the question of determining \( g(n) \), we have shown that

\[
\lfloor \sqrt{n} \rfloor + 1 \leq g(n) \leq 1.1527 \sqrt{n}
\]

(8)

for all \( n \geq 2 \). It turns out that the precise value of \( g(n) \) has a nontrivial dependence on the arithmetic properties of \( n \); indeed, the lower bound in (8) is sharp for some positive integers, but strict for others. This is demonstrated by the series of constructions and observations below.

\( \textbf{G1} \) Observe that if \( d \geq 2 \) and there exists a transitive projective plane of order \( d \), then \( s(d^2 + d + 1, k) > 0 \) if and only if \( k \geq d + 1 \). Indeed, if \( k \leq d \), then by Proposition 2.1, we have \( s(d^2 + d + 1, k) = 0 \). On the other hand, if \( k \geq d + 1 \), then let \( \mathbb{P} \) be a transitive projective plane of order \( d \), let \( n = d^2 + d + 1 \), identify \( [n] \) with the set of points of \( \mathbb{P} \), and take \( \mathcal{A} \) to be the family of all \( k \)-element subsets of the points of \( \mathbb{P} \) that contain a line of \( \mathbb{P} \). It is clear that \( \mathcal{A} \) is nonempty, symmetric and intersecting. If \( k = d + 1 \), then \( |\mathcal{A}| = d^2 + d + 1 = n \), and if \( k > d + 1 \), then using the Bonferroni inequalities, we have

\[
|\mathcal{A}| \geq n\left(\frac{n-d-1}{k-d-1}\right) - \binom{n}{2}\left(\frac{n-2d-1}{k-2d-1}\right) \geq \frac{n}{2} \left(\frac{n-d-1}{k-d-1}\right).
\]

(9)

Of course, for any prime power \( q \), there exists a transitive projective plane of order \( q \), namely, the Desarguesian projective plane \( \mathbb{P}^2(\mathbb{F}_q) \) over the finite field \( \mathbb{F}_q \). Hence, for any prime power \( q \), we have \( s(q^2 + q + 1, k) > 0 \) if and only if \( k \geq q + 1 \). It follows that the lower bound in (8) is sharp whenever \( n = q^2 + q + 1 \) for some prime power \( q \), and consequently we get an affirmative answer to Question 1.6 for all positive integers of this form. Moreover, using (9), we have

\[
s(q^2 + q + 1, k) \geq |\mathcal{A}| \geq \frac{q^2 + q + 1}{2} \left(\frac{q^2}{k-q-1}\right)
\]

for all prime powers \( q \) and all \( k \geq q + 1 \). It can be checked from this that for any \( \delta > 0 \) and all prime powers \( q \) sufficiently large depending on \( \delta \), we
have \(|A| > |\mathcal{F}(q^2 + q + 1, k)|\) whenever \(q + 1 \leq k \leq (1 - \delta)q \log q\). This will inform our Conjecture 5.1 below.

\textbf{G2} On the other hand, the lower bound in (8) is not tight for \(n = 43\), for example. It was shown by Lovász \[22\] and Füredi \[16\] that if \(d \geq 2\) and \(d^2 + d + 1\) is prime, then \(s(d^2 + d + 1, d + 1) > 0\) if and only if there exists a transitive projective plane of order \(d\), and that in this case, we have \(s(d^2 + d + 1, d + 1) = d^2 + d + 1\). Consequently, \(s(43, 7) = 0\), since 43 is prime and there exists no projective plane of order 6, so the lower bound in (8) is not sharp in general. Moreover, we have \(s(d^2 + d + 1, d + 1) = d^2 + d + 1\) for all \(d \in \{2, 3, 5, 8, 17, 27, 41, 59, 71, 89\}\), since each such \(d\) is a prime power with \(d^2 + d + 1\) prime. The well-known (and widely believed) Generalized Buniakovsky conjecture (due to Schinzel and Sierpiński \[25\], generalizing simultaneously the conjecture of Buniakovsky \[5\] and the conjecture of Dickson \[8\]) would imply that \(d^2 + d + 1\) is prime for infinitely many primes \(d\), which would imply that \(s(d^2 + d + 1, d + 1) = d^2 + d + 1\) for infinitely many \(d \in \mathbb{N}\). The even stronger Bateman–Horn conjecture \[3\] would imply that \(d^2 + d + 1\) is prime for infinitely many positive integers \(d\) which are not themselves prime powers; taken together with the aforementioned observation of Lovász and Füredi along with the non-existence conjecture for projective planes whose order is not a prime power, this would imply that \(s(d^2 + d + 1, d + 1) = 0\) for infinitely many \(d \in \mathbb{N}\), and consequently that the lower bound in (8) is not sharp for infinitely many positive integers.

We can use other finite geometries in the place of projective planes to bound \(g(\cdot)\); this allows us to answer Question 1.6 in the affirmative for various other sequences of positive integers with suitable ‘arithmetic structure’.

\textbf{G3} Let \(q\) be a prime power. Recall that the affine plane \(\mathbb{A}^2(\mathbb{F}_q)\) over \(\mathbb{F}_q\) is the incidence geometry with point-set \(\mathbb{F}_q^2\) whose lines are the 1-dimensional affine subspaces of \(\mathbb{F}_q^2\); in other words, the lines of \(\mathbb{A}^2(\mathbb{F}_q)\) are the \(q^2 + q\) sets of the form \(\{x + \lambda v : \lambda \in \mathbb{F}_q\}\), where \(x \in \mathbb{F}_q^2\) and \(v \in \mathbb{F}_q^2 \setminus \{0\}\), so each point lies on \(q + 1\) lines. The dual affine plane \(\mathbb{D}A^2(\mathbb{F}_q)\) over \(\mathbb{F}_q\) is obtained by interchanging the point-set and the line-set of \(\mathbb{A}^2(\mathbb{F}_q)\) and preserving the incidence relation, so \(\mathbb{D}A^2(\mathbb{F}_q)\) has \(q^2 + q\) points and \(q^2\) lines, and each line contains \(q + 1\) points. If \(n = q^2 + q\) and \(k = q + 1\), then we identify \([n]\) with the point-set of \(\mathbb{D}A^2(\mathbb{F}_q)\), and take \(A \subset [n]^{(k)}\) to be the family of lines of
\( \mathbb{D}A^2(F_q) \). We claim that \( \mathcal{A} \) is a symmetric intersecting family. Indeed, any two points in \( A^2(F_q) \) lie on a common line in \( A^2(F_q) \), so \( \mathcal{A} \) is intersecting. For any two lines \( \ell_1, \ell_2 \) in \( A^2(F_q) \), there is an affine transformation \( \sigma \in \text{Aff}(F_q^2) \) (i.e., a map of the form \( v \mapsto Mv + c \) for some \( M \in \text{GL}(F_q^2) \) and \( c \in F_q^2 \)), such that \( \sigma(\ell_1) = \ell_2 \); clearly, \( \sigma \) defines an automorphism of \( \mathcal{A} \), so \( \mathcal{A} \) is symmetric. It follows that

\[
 s(q^2 + q, q + 1) \geq |\mathcal{A}| = q^2 > 0.
\]

In conjunction with Proposition 2.1, this implies that for any odd prime power \( q \), we have \( s(q^2 + q, k) > 0 \) if and only if \( k \geq q + 1 \). Therefore, we have an affirmative answer to Question 1.6 for any \( n = q^2 + q \), where \( q \) is a prime power; indeed, the lower bound in (8) is sharp for all \( n \) of this form.

These constructions based on projective planes and dual affine planes have natural analogues based upon higher-dimensional projective spaces and higher-dimensional dual affine spaces, enabling us to answer Question 1.6 affirmatively for some other infinite sequences of integers.

**G4** Fix \( r \in \mathbb{N} \) and let \( q \) be a prime power. If \( n = (q^{2r+1} - 1)/(q - 1) \) and \( k = (q^{r+1} - 1)/(q - 1) \), then we identify \([n]\) with the set of points of the \((2r)\)-dimensional projective space \( \mathbb{P}^{2r}(F_q) \), and take \( \mathcal{A} \) to be the family of all \( r \)-dimensional projective subspaces. Clearly, \( \mathcal{A} \) is symmetric and intersecting, so

\[
 s\left(\frac{q^{2r+1} - 1}{q - 1}, \frac{q^{r+1} - 1}{q - 1}\right) \geq \left[\frac{2r + 1}{r + 1}\right]_q = \prod_{i=0}^{r} \frac{2^{r+1-i} - 1}{q^{i+1} - 1} > 0.
\]

Note that if \( r \in \mathbb{N} \) is fixed, then \( k = (1 + o(1))\sqrt{n} \) as \( q \to \infty \); this gives an affirmative answer to Question 1.6 for all \( n \in \{(q^{2r+1} - 1)/(q - 1) : q \text{ is a prime power}\} \).

**G5** Fix \( r \in \mathbb{N} \) and let \( q \) be a prime power. Let \( A^{2r}(F_q) \) denote the \((2r)\)-dimensional affine space over \( F_q \) so that for each \( i \in [2r - 1] \cup \{0\} \), the \( i \)-flats of \( A^{2r}(F_q) \) are the \( i \)-dimensional affine subspaces of \( F_q^{2r} \); in particular, the 0-flats are the points and the \((2r - 1)\)-flats are the affine hyperplanes, so there are \( q^{2r} \) points and \( q(q^{2r} - 1)/(q - 1) \) affine hyperplanes. Two flats are said to be *incident* if one is contained in the other. It is easy to see that a fixed \((r - 1)\)-flat of \( A^{2r}(F_q) \) is contained in \((q^{r+1} - 1)/(q - 1)\) hyperplanes,
and that there are
\[
q^{r+1}\left[\frac{2r}{r-1}\right] = q^{r+1}\prod_{i=0}^{r-2} \frac{q^{2r-i} - 1}{q^{i+1} - 1}
\]

\((r - 1)\)-flats. The \((2r)\)-dimensional dual affine space \(D\mathbb{A}^{2r}(\mathbb{F}_q)\) is the space whose \(i\)-flats are the \((2r - i - 1)\)-flats of \(\mathbb{A}^{2r}(\mathbb{F}_q)\) for each \(i \in [2r - 1] \cup \{0\}\). In particular, the points of \(D\mathbb{A}^{2r}(\mathbb{F}_q)\) are the affine hyperplanes of \(\mathbb{A}^{2r}(\mathbb{F}_q)\) and the \(r\)-flats of \(D\mathbb{A}^{2r}(\mathbb{F}_q)\) are the \((r - 1)\)-flats of \(\mathbb{A}^{2r}(\mathbb{F}_q)\). Again, two flats are incident if one is contained in the other. Now, if \(n = q(q^{2r} - 1)/(q - 1)\) and \(k = (q^{r+1} - 1)/(q - 1)\), then we identify \([n]\) with the point-set of \(D\mathbb{A}^{2r}(\mathbb{F}_q)\), and take \(A \subset [n]^{(k)}\) to be the family of all \(r\)-flats of \(D\mathbb{A}^{2r}(\mathbb{F}_q)\). We claim that \(A\) is a symmetric intersecting family. Indeed, any two \((r - 1)\)-flats of \(\mathbb{A}^{2r}(\mathbb{F}_q)\) are contained in a common affine hyperplane in \(\mathbb{A}^{2r}(\mathbb{F}_q)\), so \(A\) is intersecting. Also, for any two affine hyperplanes \(V_1\) and \(V_2\) in \(\mathbb{A}^{2r}(\mathbb{F}_q)\), there exists an affine transformation \(\sigma \in \text{Aff}(\mathbb{F}_q^{2r})\) (i.e., a map of the form \(v \mapsto Mv + c\), where \(M \in \text{GL}(\mathbb{F}_q^{2r})\) and \(c \in \mathbb{F}_q^{2r}\)), such that \(\sigma(V_1) = V_2\); clearly, \(\sigma\) defines an automorphism of \(A\), so \(A\) is symmetric. Hence,

\[
\sigma\left(\frac{q(q^{2r} - 1)}{q - 1}, \frac{q^{r+1} - 1}{q - 1}\right) \geq q^{r+1}\left[\frac{2r}{r-1}\right] = q^{r+1}\prod_{i=0}^{r-2} \frac{q^{2r-i} - 1}{q^{i+1} - 1} > 0.
\]

Note as before that if \(r \in \mathbb{N}\) is fixed, then \(k = (1 + o(1))\sqrt{n}\) as \(q \to \infty\); this gives an affirmative answer to Question 1.6 for all \(n \in \{q(q^{2r} - 1)/(q - 1) : q\) is a prime power\}.

We now demonstrate, using a tensor product construction, that \(S\) is closed under taking pointwise products. For a set \(x \subset [n]\), we define its characteristic vector \(\chi_x \in \{0, 1\}^n\) by \((\chi_x)_i = 1\) if \(i \in x\) and \((\chi_x)_i = 0\) otherwise. Given two sets \(x \subset [n]\) and \(y \subset [m]\), we define their tensor product \(x \otimes y\) to be the subset of \([nm]\) whose characteristic vector \(\chi_{x \otimes y}\) is given by

\[
(\chi_{x \otimes y})_{(i-1)m+j} = (\chi_x)_i(\chi_y)_j \quad (i \in [n], j \in [m]).
\]

For two families \(A \subset \mathcal{P}_n\) and \(B \subset \mathcal{P}_m\), we define their tensor product by

\[
A \otimes B = \{x \otimes y : x \in A, y \in B\};
\]

note that \(A \otimes B \subset \mathcal{P}_{nm}\) and that \(|A \otimes B| = |A||B|\). Now observe that if \(A \subset [n]^{(k)}\) and \(B \subset [m]^{(l)}\), then \(A \otimes B \subset [nm]^{(kl)}\), and furthermore, if \(A\) and \(B\) are symmetric
and intersecting, then so is $A \otimes B$. It follows that
\[
s(nm, kl) \geq s(n, k)s(m, l)
\]
for all $k, l, m, n \in \mathbb{N}$, and in particular, if $(n, k), (m, l) \in \mathcal{S}$, then $(nm, kl) \in \mathcal{S}$.

**G6** The above observation implies that $g(\cdot)$ is submultiplicative, i.e., we have
\[
g(nm) \leq g(n)g(m)
\]
for all $n, m \in \mathbb{N}$. This fact may be used to answer Question 1.6 affirmatively for some additional sequences of positive integers; for example, we deduce an affirmative answer to Question 1.6 for all $n = (q_1^2 + q_1 + 1)(q_2^2 + q_2 + 1)$ with $q_1$ and $q_2$ both prime powers, and so on.

Finally, for completeness, we remind the reader of the following, mentioned earlier.

**G7** The observation of Banakh and Gavrylkiv [2] mentioned above shows that $g(n) = (1 + o(1))\sqrt{n}$ whenever $n = q^2 - 1$ for some prime power $q$, or $n = p^2 - p$ for some prime $p$. Consequently, we have an affirmative answer to Question 1.6 for any $n \in \mathbb{N}$ of these forms.

5. Conclusion

A number of interesting open problems remain. Theorem 1.4 and Lemma 3.2 together determine the order of magnitude of $\log((\binom{n}{k})/s(n, k))$ when $k/n$ is bounded away from zero by a positive constant. The gap between our upper and lower bounds for $s(n, k)$ is somewhat worse for smaller $k$, and it would be of interest to improve Theorem 1.4 in the regime where $k = o(n)$.

Determining $s(n, k)$ precisely for all $k \leq n/2$ would appear to be a challenging problem. We conjecture the following.

**Conjecture 5.1.** For any $\delta > 0$, if $n$ is sufficiently large depending on $\delta$ and $(1 + \delta)\sqrt{n}\log n \leq k \leq n/2$, then
\[
s(n, k) = |\mathcal{F}(n, k)|.
\]

Note that if $n$ is sufficiently large depending on $\delta$, then the family $\mathcal{F}(n, k)$ is larger than any of the symmetric intersecting families constructed in Section 4, provided $(1 + \delta)\sqrt{n}\log n \leq k \leq n/2$. 17
Determining the asymptotic behaviour of \( g(n) \) is another problem that merits further investigation. We have established various bounds in Section 4, but even the question of whether \( g(n)/\sqrt{n} \) converges as \( n \to \infty \), remains open.

Acknowledgements

The second author wishes to acknowledge support from ERC Advanced Grant 320924, NSF grant DMS-1300120 and BSF grant 2014290, and the third author was partially supported by NSF Grant DMS-1800521.

References

1. L. Babai, Personal communication.
School of Mathematical Sciences, Queen Mary, University of London, London E14NS, UK

Email address: d.ellis@qmul.ac.uk

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond J. Safra Campus, Jerusalem 91904, Israel

Email address: kalai@math.huji.ac.il

Department of Mathematics, Rutgers University, Piscataway NJ 08854, USA

Email address: narayanan@math.rutgers.edu