An improved lower bound for Folkman's theorem

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Abstract. Folkman's theorem asserts that for each $k \in \mathbb{N}$, there exists a natural number $n = F(k)$ such that whenever the elements of $[n]$ are two-coloured, there exists a set $A \subset [n]$ of size k with the property that all the sums of the form $\sum_{x \in B} x$, where B is a nonempty subset of A, are contained in $[n]$ and have the same colour. In 1989, Erdős and Spencer showed that $F(k) \geq 2^{ck^2/\log k}$, where $c > 0$ is an absolute constant; here, we improve this bound significantly by showing that $F(k) \geq 2^{2^{k-1}/k}$ for all $k \in \mathbb{N}$.

Schur's theorem, proved in 1916, is one of the central results of Ramsey theory and asserts that whenever the elements of $\mathbb{N}$ are finitely coloured, there exists a monochromatic set of the form $\{x, y, x+y\}$ for some $x, y \in \mathbb{N}$. About fifty years ago, a wide generalisation of Schur's theorem was obtained independently by Folkman, Rado and Sanders, and this generalisation is now commonly referred to as Folkman's theorem (see [2], for example). To state Folkman's theorem, it will be convenient to have some notation. For $n \in \mathbb{N}$, we write $[n]$ for the set $\{1, 2, \ldots, n\}$, and for a finite set $A \subset \mathbb{N}$, let

$$S(A) = \left\{ \sum_{x \in B} x : B \subset A \text{ and } B \neq \varnothing \right\}$$

denote the set of all *finite sums* of A. In this language, Folkman's theorem states that for all $k, r \in \mathbb{N}$, there exists a natural number $n = F(k, r)$ such that whenever the elements of $[n]$ are r-coloured, there exists a set $A \subset [n]$ of size k such that $S(A)$ is a monochromatic subset of $[n]$; of course, it is easy to see that Folkman's theorem, in the case where $k = 2$, implies Schur's theorem.

In this note, we shall be concerned with lower bounds for the two-colour Folkman numbers, i.e., for the quantity $F(k) = F(k, 2)$. In 1989, Erdős and Spencer [1] proved that

$$F(k) \geq 2^{ck^2/\log k} \tag{1}$$

for all $k \in \mathbb{N}$, where $c > 0$ is an absolute constant; here, and in what follows, all logarithms are to the base 2. Our primary aim in this note is to improve (1).

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Before we state and prove our main result, let us say a few words about the proof of (1). Erdős and Spencer establish (1) by considering uniformly random two-colourings. In particular, they show that if $[n]$ is two-coloured uniformly at random and additionally $n \le 2^{ck^2/\log k}$ for some suitably small absolute constant $c > 0$, then with high probability, there is no k-set $A \subset [n]$ for which $S(A)$ is monochromatic. On the other hand, it is not hard to check that if $n \ge 2^{Ck^2}$ for some suitably large absolute constant $C > 0$, then a two-colouring of $[n]$ chosen uniformly at random is such that, with high probability, there exists a set $A \subset [n]$ of size k for which $S(A)$ is monochromatic; indeed, to see this, it is sufficient to restrict our attention to sets of the form $\{p, 2p, \dots, kp\}$, where p is a prime in the interval $[n/\log^2 n, 2n/\log^2 n]$, and notice that the sets of finite sums of such sets all have size $k(k+1)/2$ and are pairwise disjoint. With perhaps this fact in mind, in their paper, Erdős and Spencer also describe some of their attempts at removing the factor of $\log k$ in the exponent in (1); nevertheless, their bound has not been improved upon since.

Our main contribution is a new, doubly exponential, lower bound for $F(k)$, significantly strengthening the bound due to Erdős and Spencer.

Theorem 1. *For all $k \in \mathbb{N}$, we have*

$$F(k) \ge 2^{2^{k-1}/k}. \tag{2}$$

Proof. The result is easily verified when $k \le 3$, so suppose that $k \ge 4$ and let $n = \lfloor 2^{2^{k-1}/k} \rfloor$. In the light of our earlier remarks, a uniformly random colouring of $[n]$ is a poor candidate for establishing (2). Instead, we generate a (random) red-blue colouring of $[n]$ as follows: we first red-blue colour the odd elements of $[n]$ uniformly at random, and then extend this colouring uniquely to all of $[n]$ by insisting that the colour of $2x$ be different from the colour of x for each $x \in [n]$; hence, for example, if 5 is initially coloured blue, then 10 gets coloured red, 20 gets coloured blue, and so on.

Fix a set $A \subset [n]$ of size k with $S(A) \subset [n]$. We have the following estimate for the probability that $S(A)$ is monochromatic in our colouring.

Claim 2. $\mathbb{P}(S(A)$ *is monochromatic*$) \le 2^{1-2^{k-1}}$.

Proof. First, if $|S(A)| \le 2^k - 2$, then it is easy to see from the pigeonhole principle that there exist two subsets $B_1, B_2 \subset A$ such that $\sum_{x \in B_1} x = \sum_{x \in B_2} x$, and by removing $B_1 \cap B_2$ from both B_1 and B_2 if necessary, these sets may further be assumed to be disjoint; in particular, this implies that $S(A)$ contains two elements one of which is twice the other. It therefore follows from the definition of our colouring that $S(A)$ cannot be monochromatic unless $|S(A)| = 2^k - 1$. Next, suppose that $|S(A)| = 2^k - 1$. For each odd integer $m \in \mathbb{N}$, we define $G_m = \{m, 2m, 4m, \dots\} \cap [n]$, and note that these geometric progressions partition $[n]$. Observe that $S(A)$ intersects at least 2^{k-1} of these progressions. Indeed, if there is an odd integer $r \in A$ for example, then $S(A)$

contains exactly 2^{k-1} distinct odd elements and these elements must lie in different progressions. More generally, if each element of A is divisible by 2^s and s is maximal, then there exists an element r of A with $r = 2^s t$, where t is odd; it is then clear that precisely 2^{k-1} elements of $S(A)$ are divisible by 2^s but not by 2^{s+1} and these elements must necessarily lie in different progressions. With this in mind, we define B_A to be a maximal subset of $S(A)$ with the property $|B_A \cap G_m| \leq 1$ for each m; for example, we may take B_A to consist of the least elements (where they exist) of the sets $S(A) \cap G_m$. Clearly, our red-blue colouring restricted to B_A is a uniformly random colouring, so the probability that B_A is monochromatic is $2^{1-|B_A|}$; it follows that the probability that $S(A)$ is monochromatic is at most $2^{1-|B_A|} \leq 2^{1-2^{k-1}}$. □

It is now easy to see that if X is the number of sets $A \subset [n]$ of size k for which $S(A)$ is a monochromatic subset of $[n]$ in our colouring, then

$$\mathbb{E}[X] \leq \binom{n}{k} 2^{1-2^{k-1}} \leq \left(\frac{en}{k}\right)^k 2^{1-2^{k-1}} \leq \left(\frac{e2^{2^{k-1}/k}}{k}\right)^k \left(2^{1-2^{k-1}}\right) = 2\left(\frac{e}{k}\right)^k < 1,$$

where the last inequality holds for all $k \geq 4$. Hence, there exists a red-blue colouring of $[n]$ without any set A of size k for which $S(A)$ is a monochromatic subset of $[n]$, proving the result. □

We conclude this note with two remarks. First, using the original arguments of Erdős and Spencer [1] in conjunction with an inverse Littlewood–Offord theorem of Nguyen and Vu [3], it is possible to improve (1) (up to removing the factor of $\log k$ in the exponent) by just considering uniformly random two-colourings. Second, we note that while (2) improves significantly on (1), this lower bound is still considerably far from the best upper bound for $F(k)$, which is of tower type; see [4], for instance.

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