We prove a generalization of the original statement. That is, we claim the following is true:

The number of partitions of n having k parts divisible by m is the same as the number of partitions of n in which the largest repeating at least m times part is k. (Notice if we let m be 2, then we get the original statement).

One can prove this generalization using generating functions or purely bijective methods. I found the latter to be shorter and prettier. So I will provide a purely bijective proof of the generalization.

First a lemma (generalization of Euler's Odd Distinct, Glaisher proved it in "a theorem in partitions"): The number of partitions of n with parts not divisible by m is equal to the number of partitions of n where each part is repeated less than m times. And here is a bijective proof of it (this is a analog of the well known bijective proof of the Euler's Odd Distinct):

(1) First the forward direction. Starting with a partition of n with parts not divisible by m, denoted by $b_1^{\beta_1}b_2^{\beta_2}...b_s^{\beta_s}$ where $b_i^{\beta_i}$ means b_i is repeated β_i many times and $b_1 > b_2 > ... > b_s$. We then write β_i in base m. Let $\beta_i = c_{i1} * m^{\beta_{i1}} + c_{i2} * m^{\beta_{i2}} + ...$. And so we have $n = (c_{11} * m^{\beta_{11}} + c_{12} * m^{\beta_{12}} + ...)b_1 + ... + (c_{s1} * m^{\beta_{s1}} + c_{s2} * m^{\beta_{s2}} + ...)b_s$. In other words, we get c_{ij} copies of $m^{\beta_{ij}} * b_i$, where $c_{ij} < m$, and the "$m^{\beta_{ij}} * b_i$" are obviously distinct for different (i, j) pair. Thus we get a partition of n where each part is repeated less than m times. We call this function performing this algorithm GOddDist (since it is the generalized version of odd to distinct). For example, if the partition is $[7, 7, 7, 7, 7, 4, 4, 1]$(that is $7^5 4^2 1^1$ in our notation) and $m = 3$, then we write the power of 7, namely 5, as 12 in base 3. So $7 + 7 + 7 + 7 + 7 = (1 * 3^1 + 2 * 3^0)7 = 1 * (3 * 7) + 2 * (3^0 * 7) = 1 * 21 + 2 * 7 = 21 + 7 + 7$. So the partition becomes $[21, 7, 7, 4, 4, 1]$.

(2) Now the other direction (call the function GDistOdd). Starting with a partition of n where each part is repeated less than m times, say $b_1^{\beta_1}b_2^{\beta_2}...b_s^{\beta_s}$. Since we can write each b_i as $c_i * m^{\beta_i}$ where c_i is not divisible by m, we simply divide b_i into m^{β_i} many copies of c_i. For example, if the partition is $[21, 7, 7, 4, 4, 1]$ and $m = 3$, we divide the parts that are multiples of 3 into equal sized smaller parts not divisible by 3. In this case we get $[7, 7, 7, 7, 7, 4, 4, 1]$.

Just as in the proof of Euler's Odd Distinct Identity, it is easy to see that GDistOdd and GOddDist are inverses of each other. Thus our bijective proof of the lemma is completed.

Now using this lemma as a tool, we are ready to construct a bijective proof of the generalization. Again we construct two functions GBijDR and GBijRD that are inverses of each other.

(a)First we construct GBijDR, which maps a partition of n having k parts divisible by m to a partition of n in which the largest repeating at least m times part is k.

Starting with a partition of n having k parts divisible by m (and some other parts not divisible by m), we first separate the k parts that are divisible by m, denoted by $P_1 = a_1^{\alpha_1}a_2^{\alpha_2}...a_r^{\alpha_r}$ and the parts that are not divisible by m, denoted by $P_2 = b_1^{\beta_1}b_2^{\beta_2}...b_s^{\beta_s}$. $(a_1 > a_2 > ... > a_r; b_1 > b_2 > ... > b_s)$

Then, we conjugate P_1 to get $c_1^{\phi_1}c_2^{\phi_2}...c_t^{\phi_t}$ $(c_1 > c_2 > ... > c_t)$. Clearly, $c_1 = k$ and is repeated at least m times since there are k parts in P_1 and each a_i is a positive multiple of m. And we use GOddDist to map P_2 to $d_1^{\omega_1}d_2^{\omega_2}...d_u^{\omega_u}$ $(d_1 > d_2 > ... > d_u)$ where $\omega_i < m$ for all i. Finally, combine $c_1^{\phi_1}c_2^{\phi_2}...c_t^{\phi_t}$ and $d_1^{\omega_1}d_2^{\omega_2}...d_u^{\omega_u}$. After reordering the parts according to their sizes, we get a partition of n in which the largest repeating at least m times part is k, because each d_i is repeated less than m times, and $c_1 = k$ is the largest among the $c_i (1 \le i \le t)$.

Example: $[7, 7, 7, 7, 7, 6, 6, 3, 1] \to P_1 = [6, 6, 3]$ and $P_2 = [7, 7, 7, 7, 7, 1]$; $P_1 \to [3, 3, 3, 2, 2, 2]$ (conjugation) and $P_2 \to [21, 7, 7, 1]$ (GOddDist). Finally combining the two we get $[21, 7, 7, 3, 3, 3, 2, 2, 2, 1]$.

(b) Now we construct GBijRD–the inverse of GBijDR. Given a partition of n in which the largest repeating at least m times part is k, denoted by $x_1^{y_1}x_2^{y_2}...x_p^{y_p}$ $(x_1 > x_2 > ... > x_p)$. Write $y_i = g_i * m + z_i$ $(0 \le z_i < m)$ by the Euclidean algorithm. Seperate the original partition into two partitions $P_1 = x_1^{m*g_1}x_2^{m*g_2}...x_p^{m*g_p}$ and $P_2 = x_1^{z_1}x_2^{z_2}...x_p^{z_p}$. By conjugating P_1, we get a partition A that has exactly k parts divisible by m (since the largest part in P_1 has to be k and each part in P_1 is repeated a multiple of m many times). And then we use GDistOdd to map P_2 to a partition B where each part is not divisible by m. Finally, combining A and B and reordering the parts, we get a partition of n having k parts divisible by m.

Example: $[21, 7, 7, 3, 3, 3, 2, 2, 2, 1] \to P_1 = [3, 3, 3, 2, 2, 2]$ and $P_2 = [21, 7, 7, 1]$; $P_1 \to [6, 6, 3]$ (conjugation) and $P_2 \to [7, 7, 7, 7, 7, 1]$ (GDistOdd). Finally Combining the two we get $[7, 7, 7, 7, 7, 6, 6, 3, 1]$.

It is easy to see that GBijRD and GBijDR are inverses of each other, thus our proof is completed!

Reference

Glaisher, J. W. L. (1883). "A theorem in partitions," *Messenger of Math.* 12, 158-170.