Structure in Stack-Sorting

Colin Defant

Princeton University

Rutgers University Experimental Math Seminar

February 21, 2019

Permutations

Permutations

A *permutation* of length n is an ordering of the elements of $\{1, \ldots, n\}$. Let S_n be the set of permutations of length n.

Permutations

A *permutation* of length n is an ordering of the elements of $\{1, \ldots, n\}$. Let S_n be the set of permutations of length n.

Example

- 2613475 is a permutation of length 7.

Permutations

A *permutation* of length n is an ordering of the elements of $\{1, \ldots, n\}$. Let S_n be the set of permutations of length n.

Example

- 2613475 is a permutation of length 7.
- 1234567 is a different permutation of length 7. This is the *identity permutation* of length 7.

Permutations

A *permutation* of length n is an ordering of the elements of $\{1, \ldots, n\}$. Let S_n be the set of permutations of length n.

Example

- 2613475 is a permutation of length 7.
- 1234567 is a different permutation of length 7. This is the *identity permutation* of length 7.

A *descent* of a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ is an index i such that $\pi_i > \pi_{i+1}$.

Permutations

A *permutation* of length n is an ordering of the elements of $\{1, \ldots, n\}$. Let S_n be the set of permutations of length n.

Example

- 2613475 is a permutation of length 7.
- 1234567 is a different permutation of length 7. This is the *identity permutation* of length 7.

A *descent* of a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ is an index i such that $\pi_i > \pi_{i+1}$.

Example

- The descents of 2613475 are 2 and 6.

Permutations

A *permutation* of length n is an ordering of the elements of $\{1, \ldots, n\}$. Let S_n be the set of permutations of length n.

Example

- 2613475 is a permutation of length 7.
- 1234567 is a different permutation of length 7. This is the *identity permutation* of length 7.

A *descent* of a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ is an index i such that $\pi_i > \pi_{i+1}$.

Example

- The descents of 2613475 are 2 and 6.
- The identity permutation 1234567 has no descents.

Permutation Patterns

Permutation Patterns

Definition

Given permutations σ and τ, we say that σ *contains* the pattern τ if there are (not necessarily consecutive) entries in σ that have the same relative order as τ. Otherwise, we say σ *avoids* the pattern τ.

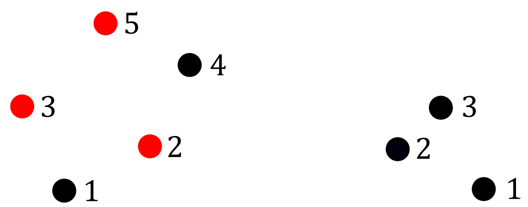
Permutation Patterns

Definition

Given permutations σ and τ, we say that σ *contains* the pattern τ if there are (not necessarily consecutive) entries in σ that have the same relative order as τ. Otherwise, we say σ *avoids* the pattern τ.

Example

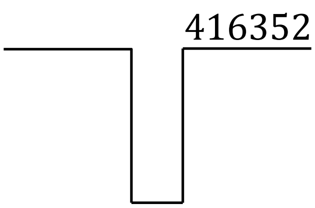
The permutation 31524 contains the pattern 231.
The permutation 31524 avoids the pattern 321.



West's Stack-Sorting Map

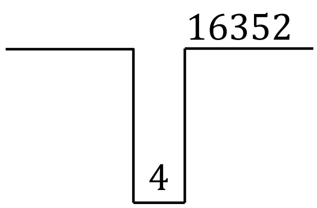
West's Stack-Sorting Map

$s : 416352 \mapsto$



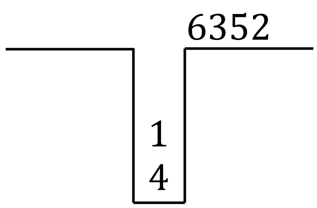
West's Stack-Sorting Map

$s : 416352 \mapsto$



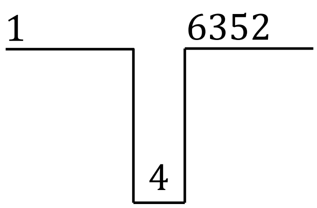
West's Stack-Sorting Map

$s : 416352 \mapsto$



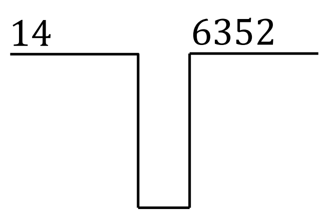
West's Stack-Sorting Map

$s : 416352 \mapsto$



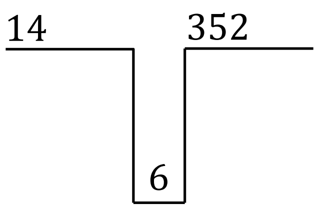
West's Stack-Sorting Map

$s : 416352 \mapsto$



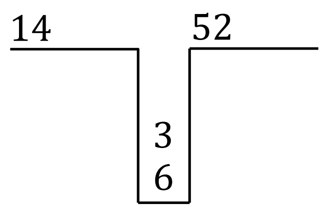
West's Stack-Sorting Map

$s : 416352 \mapsto$



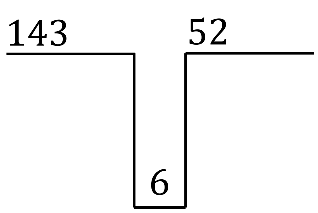
West's Stack-Sorting Map

$s : 416352 \mapsto$



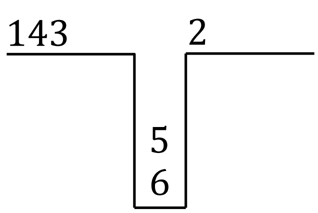
West's Stack-Sorting Map

$s : 416352 \mapsto$



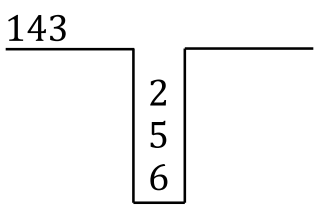
West's Stack-Sorting Map

$s : 416352 \mapsto$



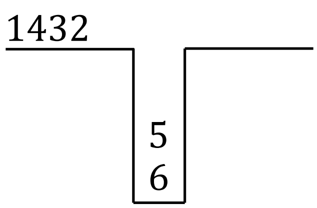
West's Stack-Sorting Map

$s : 416352 \mapsto$



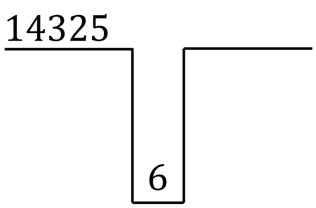
West's Stack-Sorting Map

$s : 416352 \mapsto$



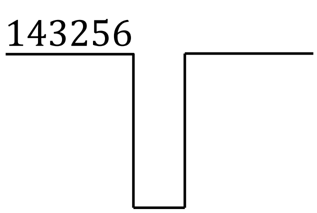
West's Stack-Sorting Map

$s : 416352 \mapsto$



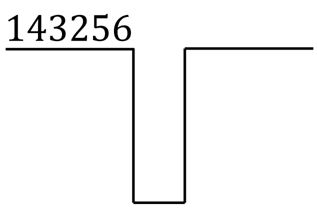
West's Stack-Sorting Map

$s : 416352 \mapsto$



West's Stack-Sorting Map

$s : 416352 \mapsto 143256$



t-Stack-Sortable Permutations

t-Stack-Sortable Permutations

Definition

Say a permutation $\pi \in S_n$ is *t-stack-sortable* if $s^t(\pi) = 123\cdots n$, where s^t denotes the t^{th} iterate of s. Let $W_t(n)$ denote the number of t-stack-sortable permutations in S_n.

t-Stack-Sortable Permutations

Definition

Say a permutation $\pi \in S_n$ is *t-stack-sortable* if $s^t(\pi) = 123 \cdots n$, where s^t denotes the t^{th} iterate of s. Let $W_t(n)$ denote the number of t-stack-sortable permutations in S_n.

Theorem (Knuth, 1968)

We have

$$W_1(n) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

t-Stack-Sortable Permutations

Definition

Say a permutation $\pi \in S_n$ is *t-stack-sortable* if $s^t(\pi) = 123\cdots n$, where s^t denotes the t^{th} iterate of s. Let $W_t(n)$ denote the number of t-stack-sortable permutations in S_n.

Theorem (Knuth, 1968)

We have

$$W_1(n) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem (Zeilberger, 1992)

We have

$$W_2(n) = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}.$$

Estimates for t-Stack-Sortable Permutations

Estimates for t-Stack-Sortable Permutations

Theorem (Stankova, West, 2002)

We have $W_t(n) \leq (t+1)^{2n}$.

Estimates for t-Stack-Sortable Permutations

Theorem (Stankova, West, 2002)

We have $W_t(n) \leq (t+1)^{2n}$.

Conjecture (Bóna)

We have $W_t(n) \leq \binom{(t+1)n}{n}$.

Estimates for t-Stack-Sortable Permutations

Theorem (Stankova, West, 2002)

We have $W_t(n) \leq (t+1)^{2n}$.

Conjecture (Bóna)

We have $W_t(n) \leq \binom{(t+1)n}{n}$.

Theorem (D., 2016)

We have $W_3(n) < 12.53296^n$ *and* $W_4(n) < 21.97225^n$.

Estimates for t-Stack-Sortable Permutations

Theorem (Stankova, West, 2002)

We have $W_t(n) \leq (t+1)^{2n}$.

Conjecture (Bóna)

We have $W_t(n) \leq \binom{(t+1)n}{n}$.

Theorem (D., 2016)

We have $W_3(n) < 12.53296^n$ *and* $W_4(n) < 21.97225^n$.

Theorem (D., 2019)

We have $\lim_{n\to\infty} \sqrt[n]{W_3(n)} \geq 7.96984$.

Tail Length Enumeration

Tail Length Enumeration

Suppose $\pi = \pi_1 \cdots \pi_n \in S_n$.

Tail Length Enumeration

Suppose $\pi = \pi_1 \cdots \pi_n \in S_n$.

Tail length: $\text{tl}(\pi)$ is the smallest nonnegative integer ℓ such that $\pi_{n-\ell} \neq n - \ell$. E.g., $\text{tl}(312456789) = 6$. Let $\text{tls}(\pi) = \text{tl}(s(\pi))$.

Tail Length Enumeration

Suppose $\pi = \pi_1 \cdots \pi_n \in S_n$.

Tail length: $\text{tl}(\pi)$ is the smallest nonnegative integer ℓ such that $\pi_{n-\ell} \neq n - \ell$. E.g., $\text{tl}(312456789) = 6$. Let $\text{tls}(\pi) = \text{tl}(s(\pi))$.

Zeilberger statistic: $\text{zeil}(\pi)$ is the largest integer m such that $n, n-1, \ldots, n-m+1$ appear in decreasing order in π.

Tail Length Enumeration

Suppose $\pi = \pi_1 \cdots \pi_n \in S_n$.

Tail length: $\text{tl}(\pi)$ is the smallest nonnegative integer ℓ such that $\pi_{n-\ell} \neq n - \ell$. E.g., $\text{tl}(312456789) = 6$. Let $\text{tls}(\pi) = \text{tl}(s(\pi))$.

Zeilberger statistic: $\text{zeil}(\pi)$ is the largest integer m such that $n, n-1, \ldots, n-m+1$ appear in decreasing order in π.

Zeilberger counted 2-stack-sortable permutations according to the additional statistic zeil and removed a "catalytic variable."

Tail Length Enumeration

Suppose $\pi = \pi_1 \cdots \pi_n \in S_n$.

Tail length: $\text{tl}(\pi)$ is the smallest nonnegative integer ℓ such that $\pi_{n-\ell} \neq n - \ell$. E.g., $\text{tl}(312456789) = 6$. Let $\text{tls}(\pi) = \text{tl}(s(\pi))$.

Zeilberger statistic: $\text{zeil}(\pi)$ is the largest integer m such that $n, n-1, \ldots, n-m+1$ appear in decreasing order in π.

Zeilberger counted 2-stack-sortable permutations according to the additional statistic zeil and removed a "catalytic variable."

I reproved the formula for $W_2(n)$ via a similar approach with zeil replaced by tls. This generalizes, allowing me to find a lower bound for $W_3(n)$ and to count several other things (mentioned later).

Tail Length Enumeration

Suppose $\pi = \pi_1 \cdots \pi_n \in S_n$.

Tail length: $\text{tl}(\pi)$ is the smallest nonnegative integer ℓ such that $\pi_{n-\ell} \neq n - \ell$. E.g., $\text{tl}(312456789) = 6$. Let $\text{tls}(\pi) = \text{tl}(s(\pi))$.

Zeilberger statistic: $\text{zeil}(\pi)$ is the largest integer m such that $n, n-1, \ldots, n-m+1$ appear in decreasing order in π.

Zeilberger counted 2-stack-sortable permutations according to the additional statistic zeil and removed a "catalytic variable."

I reproved the formula for $W_2(n)$ via a similar approach with zeil replaced by tls. This generalizes, allowing me to find a lower bound for $W_3(n)$ and to count several other things (mentioned later).

Interesting identity: $\text{zeil}(\pi) = \min\{\text{tls}(\pi), \text{rmax}(\pi)\}$.

The Fertility of a Permutation

The Fertility of a Permutation

Definition (West, 1990)

The *fertility* of a permutation π is $|s^{-1}(\pi)|$.

The Fertility of a Permutation

Definition (West, 1990)

The *fertility* of a permutation π is $|s^{-1}(\pi)|$.

Definition (Bousquet-Mélou, 2000)

A permutation π is called *sorted* if it has positive fertility (i.e., $s^{-1}(\pi) \neq \emptyset$).

The Fertility of a Permutation

Definition (West, 1990)

The *fertility* of a permutation π is $|s^{-1}(\pi)|$.

Definition (Bousquet-Mélou, 2000)

A permutation π is called *sorted* if it has positive fertility (i.e., $s^{-1}(\pi) \neq \emptyset$).

- West computed the fertilities of a few very specific types of permutations.

The Fertility of a Permutation

Definition (West, 1990)

The *fertility* of a permutation π is $|s^{-1}(\pi)|$.

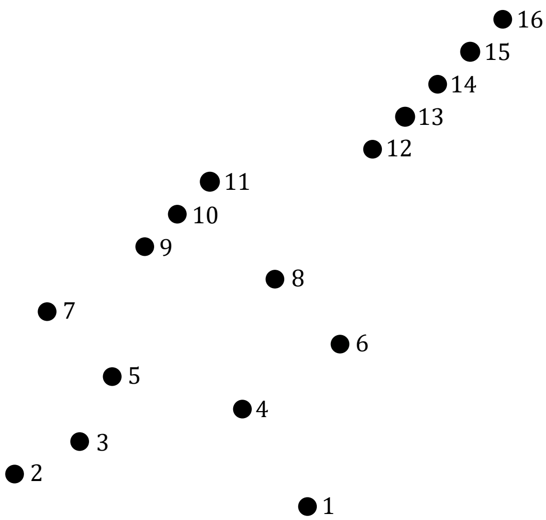
Definition (Bousquet-Mélou, 2000)

A permutation π is called *sorted* if it has positive fertility (i.e., $s^{-1}(\pi) \neq \emptyset$).

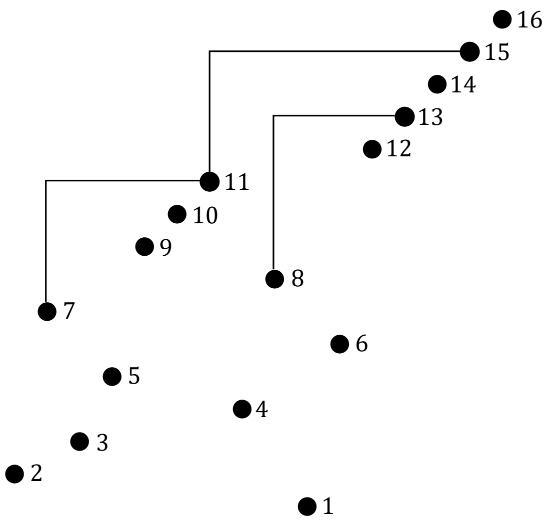
- West computed the fertilities of a few very specific types of permutations.
- Bousquet-Mélou gave an algorithm to decide whether or not a permutation is sorted. She asked for a general method that could be used to compute the fertility of any permutation.

Valid Hook Configurations

Valid Hook Configurations

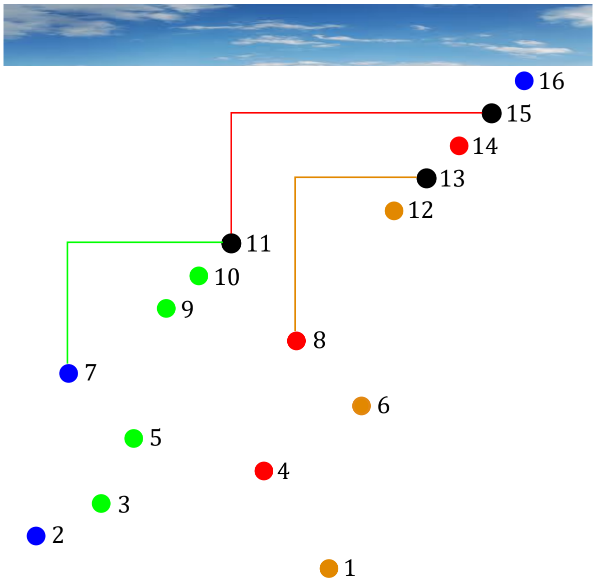


Valid Hook Configurations



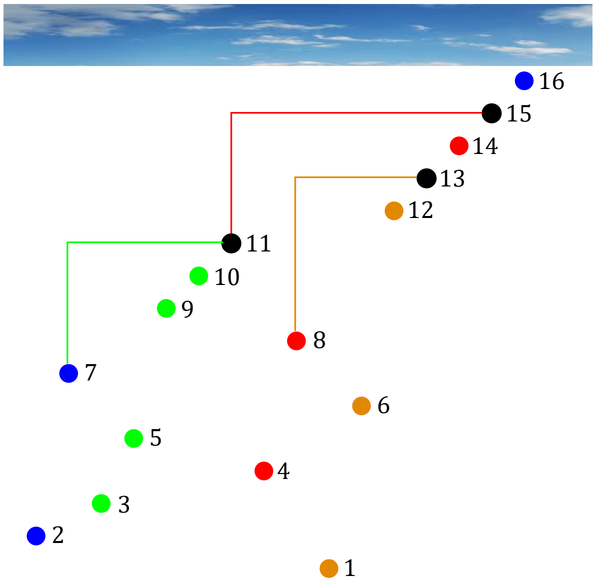
Configuration of Hooks
+ 3 Magical Properties
= Valid Hook Configuration

Valid Hook Configurations



Configuration of Hooks
$+$ 3 Magical Properties
$=$ Valid Hook Configuration

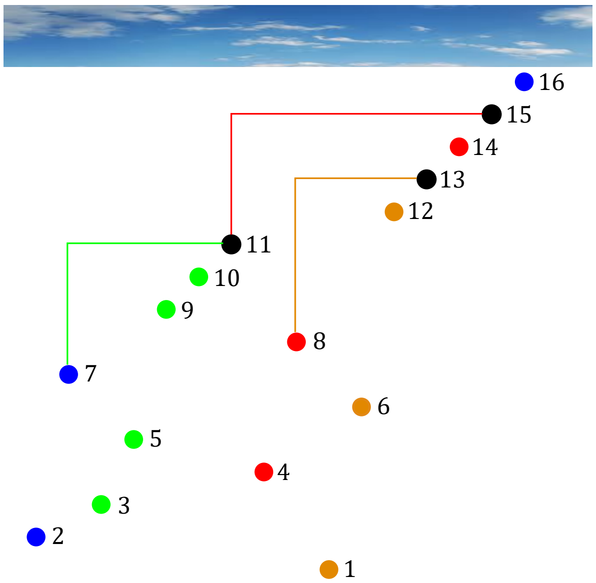
Valid Hook Configurations



Configuration of Hooks
$+$ 3 Magical Properties
$=$ Valid Hook Configuration

This valid hook configuration induces the valid composition $(3, 4, 3, 3)$. Let $\mathcal{V}(\pi)$ denote the set of valid compositions of π.

Valid Hook Configurations



Configuration of Hooks
+ 3 Magical Properties
= Valid Hook Configuration

This valid hook configuration induces the valid composition $(3, 4, 3, 3)$. Let $\mathcal{V}(\pi)$ denote the set of valid compositions of π.

Theorem (D., 2016)

The fertility of a permutation π is given by

$$|s^{-1}(\pi)| = \sum_{(q_0,\ldots,q_k)\in\mathcal{V}(\pi)} \prod_{t=0}^{k} C_{q_t}.$$

Uniquely Sorted Permutations

Uniquely Sorted Permutations

Definition

A permutation is *uniquely sorted* if its fertility is 1.

Uniquely Sorted Permutations

Definition

A permutation is *uniquely sorted* if its fertility is 1.

Theorem (D., Engen, Miller, 2018)

A permutation in S_n is uniquely sorted if and only if it is sorted and has exactly $\dfrac{n-1}{2}$ descents.

Lassalle's Sequence

Lassalle's Sequence

Definition

Lassalle's sequence $(A_m)_{m\geq 1}$ is defined by the recurrence

$$A_m = (-1)^{m-1}C_m + \sum_{j=1}^{m-1}(-1)^{j-1}\binom{2m-1}{2m-2j-1}A_{m-j}C_j$$

subject to the initial condition $A_1 = 1$.

Lassalle's Sequence

Definition

Lassalle's sequence $(A_m)_{m\geq 1}$ is defined by the recurrence

$$A_m = (-1)^{m-1}C_m + \sum_{j=1}^{m-1}(-1)^{j-1}\binom{2m-1}{2m-2j-1}A_{m-j}C_j$$

subject to the initial condition $A_1 = 1$.

Conjecture (Zeilberger)

The numbers A_m are positive and increasing.

Lassalle's Sequence

Definition

Lassalle's sequence $(A_m)_{m\geq 1}$ is defined by the recurrence

$$A_m = (-1)^{m-1} C_m + \sum_{j=1}^{m-1} (-1)^{j-1} \binom{2m-1}{2m-2j-1} A_{m-j} C_j$$

subject to the initial condition $A_1 = 1$.

Theorem (Lassalle, 2012)

The numbers A_m are positive and increasing.

Lassalle's Sequence

Definition

Lassalle's sequence $(A_m)_{m\geq 1}$ is defined by the recurrence

$$A_m = (-1)^{m-1}C_m + \sum_{j=1}^{m-1}(-1)^{j-1}\binom{2m-1}{2m-2j-1}A_{m-j}C_j$$

subject to the initial condition $A_1 = 1$.

Theorem (Lassalle, 2012)

The numbers A_m are positive and increasing.

The proof is algebraic and does not hint at any combinatorial interpretation for the numbers A_m.

Set Partitions, Acyclic Orientations, and Free Probability

Set Partitions, Acyclic Orientations, and Free Probability

By studying the cumulants of the “free semicircular law” and the “free Poisson law,” Josuat-Vergès gave a combinatorial interpretation of Lassalle’s sequence that involves set partitions and acyclic orientations of certain graphs.

The Main Bijection

The Main Bijection

Theorem (D., Engen, Miller, 2018)

There is a natural bijection Φ *from the set of all valid hook configurations to a set of objects that Josuat-Vergès considered. Restricting* Φ *gives a bijection from the set of uniquely sorted permutations to a special subset of Josuat-Vergès' objects.*

The Main Bijection

Theorem (D., Engen, Miller, 2018)

There is a natural bijection Φ from the set of all valid hook configurations to a set of objects that Josuat-Vergès considered. Restricting Φ gives a bijection from the set of uniquely sorted permutations to a special subset of Josuat-Vergès' objects.

Corollary (D., Engen, Miller, 2018)

There are $-k_{n+1}(-1)$ valid hook configurations of permutations in S_n. Here, $k_{n+1}(\lambda)$ is the $(n+1)^{st}$ cumulant of the free Poisson law with rate λ.

There are A_{k+1} uniquely sorted permutations in S_{2k+1}.

Symmetry in a Refinement of Lassalle's Sequence

Symmetry in a Refinement of Lassalle's Sequence

There are A_{k+1} uniquely sorted permutations in S_{2k+1}.

Definition

Let $A_{k+1}(\ell)$ be the number of uniquely sorted permutations in S_{2k+1} with first entry ℓ.

Symmetry in a Refinement of Lassalle's Sequence

There are A_{k+1} uniquely sorted permutations in S_{2k+1}.

Definition

Let $A_{k+1}(\ell)$ be the number of uniquely sorted permutations in S_{2k+1} with first entry ℓ.

Theorem (D., Engen, Miller, 2018)

For every $k \geq 1$, the sequence $A_{k+1}(1), A_{k+1}(2), \ldots, A_{k+1}(2k+1)$ is symmetric.

Symmetry in a Refinement of Lassalle's Sequence

There are A_{k+1} uniquely sorted permutations in S_{2k+1}.

Definition

Let $A_{k+1}(\ell)$ be the number of uniquely sorted permutations in S_{2k+1} with first entry ℓ.

Theorem (D., Engen, Miller, 2018)

For every $k \geq 1$, the sequence $A_{k+1}(1), A_{k+1}(2), \ldots, A_{k+1}(2k+1)$ is symmetric.

Conjecture

For every $k \geq 1$, the sequence $A_{k+1}(1), A_{k+1}(2), \ldots, A_{k+1}(2k+1)$ is log-concave.

Doubly and Triply Sorted Permutations

Doubly and Triply Sorted Permutations

What can we say about doubly sorted permutations (that is, permutations with fertility 2)?

Doubly and Triply Sorted Permutations

What can we say about doubly sorted permutations (that is, permutations with fertility 2)?

Using the fertility formula, it is easy to see that there are no doubly sorted permutations of odd length. Counting doubly sorted permutations of even length yields the sequence $1, 3, 31, 1186, \ldots$.

Doubly and Triply Sorted Permutations

What can we say about doubly sorted permutations (that is, permutations with fertility 2)?

Using the fertility formula, it is easy to see that there are no doubly sorted permutations of odd length. Counting doubly sorted permutations of even length yields the sequence $1, 3, 31, 1186, \ldots$.

What can we say about triply sorted permutations (permutations with fertility 3)?

Doubly and Triply Sorted Permutations

What can we say about doubly sorted permutations (that is, permutations with fertility 2)?

Using the fertility formula, it is easy to see that there are no doubly sorted permutations of odd length. Counting doubly sorted permutations of even length yields the sequence $1, 3, 31, 1186, \ldots$.

What can we say about triply sorted permutations (permutations with fertility 3)?

They don't exist!

Doubly and Triply Sorted Permutations

What can we say about doubly sorted permutations (that is, permutations with fertility 2)?

Using the fertility formula, it is easy to see that there are no doubly sorted permutations of odd length. Counting doubly sorted permutations of even length yields the sequence $1, 3, 31, 1186, \ldots$.

What can we say about triply sorted permutations (permutations with fertility 3)?

They don't exist!

Proof:

$$|s^{-1}(\pi)| = \sum_{(q_0,\ldots,q_k)\in\mathcal{V}(\pi)} \prod_{t=0}^{k} C_{q_t} \neq 3.$$

Fertility Numbers

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers:

Infertility Numbers:

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0,

Infertility Numbers:

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1,

Infertility Numbers:

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2,

Infertility Numbers:

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2,

Infertility Numbers: 3,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4,

Infertility Numbers: 3,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5,

Infertility Numbers: 3,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6,

Infertility Numbers: 3,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6,

Infertility Numbers: 3, 7,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8,

Infertility Numbers: 3, 7,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9,

Infertility Numbers: 3, 7,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10,

Infertility Numbers: 3, 7,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10,

Infertility Numbers: 3, 7, 11,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12,

Infertility Numbers: 3, 7, 11,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13,

Infertility Numbers: 3, 7, 11,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14,

Infertility Numbers: 3, 7, 11,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14,

Infertility Numbers: 3, 7, 11, 15,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16,

Infertility Numbers: 3, 7, 11, 15,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17,

Infertility Numbers: 3, 7, 11, 15,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18,

Infertility Numbers: 3, 7, 11, 15,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18,

Infertility Numbers: 3, 7, 11, 15, 19,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20,

Infertility Numbers: 3, 7, 11, 15, 19,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21,

Infertility Numbers: 3, 7, 11, 15, 19,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22,

Infertility Numbers: 3, 7, 11, 15, 19,

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22,

Infertility Numbers: 3, 7, 11, 15, 19, 23

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24,

Infertility Numbers: 3, 7, 11, 15, 19, 23

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25,

Infertility Numbers: 3, 7, 11, 15, 19, 23

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25, 26,

Infertility Numbers: 3, 7, 11, 15, 19, 23

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25, 26, 27,

Infertility Numbers: 3, 7, 11, 15, 19, 23

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25, 26, 27, 28,

Infertility Numbers: 3, 7, 11, 15, 19, 23

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25, 26, 27, 28, 29,

Infertility Numbers: 3, 7, 11, 15, 19, 23

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30

Infertility Numbers: 3, 7, 11, 15, 19, 23

Fertility Numbers

Definition

A nonnegative integer f is called a *fertility number* if there exists a permutations whose fertility is f.

Fertility Numbers: 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30

Infertility Numbers: 3, 7, 11, 15, 19, 23

Conjecture

There are "infinitely many" infertility numbers.

Fertility Numbers

Fertility Numbers

Theorem (D., 2018)

- *The set of fertility numbers is closed under multiplication.*

Fertility Numbers

Theorem (D., 2018)

- *The set of fertility numbers is closed under multiplication.*
- *If f is a fertility number, then there are "infinitely many" permutations with fertility f.*

Fertility Numbers

Theorem (D., 2018)

- *The set of fertility numbers is closed under multiplication.*
- *If f is a fertility number, then there are "infinitely many" permutations with fertility f.*
- *Every nonnegative integer that is not congruent to* 3 *modulo* 4 *is a fertility number. The lower asymptotic density of the set of fertility numbers is at least* $1954/2565 \approx 0.7618$.

Fertility Numbers

Theorem (D., 2018)

- *The set of fertility numbers is closed under multiplication.*
- *If f is a fertility number, then there are "infinitely many" permutations with fertility f.*
- *Every nonnegative integer that is not congruent to* 3 *modulo* 4 *is a fertility number. The lower asymptotic density of the set of fertility numbers is at least* $1954/2565 \approx 0.7618$.
- *The smallest fertility number that is congruent to* 3 *modulo* 4 *is* 27.

Fertility Numbers

Theorem (D., 2018)

- *The set of fertility numbers is closed under multiplication.*
- *If f is a fertility number, then there are "infinitely many" permutations with fertility f.*
- *Every nonnegative integer that is not congruent to* 3 *modulo* 4 *is a fertility number. The lower asymptotic density of the set of fertility numbers is at least* $1954/2565 \approx 0.7618$.
- *The smallest fertility number that is congruent to* 3 *modulo* 4 *is* 27.
- *If f is a fertility number, then there exists a permutation of length at most $f+1$ with fertility f.*

Fertility Numbers

Theorem (D., 2018)

- *The set of fertility numbers is closed under multiplication.*
- *If f is a fertility number, then there are "infinitely many" permutations with fertility f.*
- *Every nonnegative integer that is not congruent to* 3 *modulo* 4 *is a fertility number. The lower asymptotic density of the set of fertility numbers is at least* $1954/2565 \approx 0.7618$.
- *The smallest fertility number that is congruent to* 3 *modulo* 4 *is* 27.
- *If f is a fertility number, then there exists a permutation of length at most $f+1$ with fertility f.*

Conjecture

The second-smallest fertility number that is congruent to 3 modulo 4 is 95.

Permutation Class Preimages

Permutation Class Preimages

Let $\mathrm{Av}_n(\tau_1, \ldots, \tau_r)$ be the set of permutations of length n that avoid the patterns $\tau_1, \ldots, \tau_r$.

Permutation Class Preimages

Let $\text{Av}_n(\tau_1, \ldots, \tau_r)$ be the set of permutations of length n that avoid the patterns $\tau_1, \ldots, \tau_r$.

Theorem

We have

- $|s^{-1}(\text{Av}_n(123))| = 0$ *for* $n \geq 4$ *(Easy);*

Permutation Class Preimages

Let $\text{Av}_n(\tau_1, \ldots, \tau_r)$ be the set of permutations of length n that avoid the patterns $\tau_1, \ldots, \tau_r$.

Theorem

We have

- $|s^{-1}(\text{Av}_n(123))| = 0$ *for* $n \geq 4$ *(Easy);*
- $|s^{-1}(\text{Av}_n(213))| = C_n$ *(Knuth, 1968);*

Permutation Class Preimages

Let $\mathrm{Av}_n(\tau_1, \ldots, \tau_r)$ be the set of permutations of length n that avoid the patterns $\tau_1, \ldots, \tau_r$.

Theorem

We have

- $|s^{-1}(\mathrm{Av}_n(123))| = 0$ *for* $n \geq 4$ *(Easy);*
- $|s^{-1}(\mathrm{Av}_n(213))| = C_n$ *(Knuth, 1968);*
- $|s^{-1}(\mathrm{Av}_n(231))| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ *(Zeilberger, 1992);*

Permutation Class Preimages

Let $\text{Av}_n(\tau_1, \ldots, \tau_r)$ be the set of permutations of length n that avoid the patterns $\tau_1, \ldots, \tau_r$.

Theorem

We have

- $|s^{-1}(\text{Av}_n(123))| = 0$ *for* $n \geq 4$ *(Easy);*
- $|s^{-1}(\text{Av}_n(213))| = C_n$ *(Knuth, 1968);*
- $|s^{-1}(\text{Av}_n(231))| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ *(Zeilberger, 1992);*
- $|s^{-1}(\text{Av}_n(132))| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ *(Bouvel, Guibert, 2014);*

Permutation Class Preimages

Let $\text{Av}_n(\tau_1, \ldots, \tau_r)$ be the set of permutations of length n that avoid the patterns $\tau_1, \ldots, \tau_r$.

Theorem

We have

- $|s^{-1}(\text{Av}_n(123))| = 0$ *for* $n \geq 4$ *(Easy);*
- $|s^{-1}(\text{Av}_n(213))| = C_n$ *(Knuth, 1968);*
- $|s^{-1}(\text{Av}_n(231))| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ *(Zeilberger, 1992);*
- $|s^{-1}(\text{Av}_n(132))| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ *(Bouvel, Guibert, 2014);*
- $|s^{-1}(\text{Av}_n(312))| = \frac{2}{n(n+1)^2}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$ *(Bouvel, Guibert, 2014);*

Permutation Class Preimages

Let $\text{Av}_n(\tau_1, \ldots, \tau_r)$ be the set of permutations of length n that avoid the patterns $\tau_1, \ldots, \tau_r$.

Theorem

We have

- $|s^{-1}(\text{Av}_n(123))| = 0$ *for* $n \geq 4$ *(Easy);*
- $|s^{-1}(\text{Av}_n(213))| = C_n$ *(Knuth, 1968);*
- $|s^{-1}(\text{Av}_n(231))| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ *(Zeilberger, 1992);*
- $|s^{-1}(\text{Av}_n(132))| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ *(Bouvel, Guibert, 2014);*
- $|s^{-1}(\text{Av}_n(312))| = \frac{2}{n(n+1)^2}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$ *(Bouvel, Guibert, 2014);*
- $8.4199 \leq \lim_{n\to\infty} |s^{-1}(\text{Av}_n(321))|^{1/n} \leq 11.6569$ *(D., 2018).*

Permutation Class Preimages that are Permutation Classes

Permutation Class Preimages that are Permutation Classes

In some cases, the preimages of a permutation class under s form a permutation class. For example:

Permutation Class Preimages that are Permutation Classes

In some cases, the preimages of a permutation class under s form a permutation class. For example:

- $s^{-1}(\text{Av}(132, 231, 312, 321)) = \text{Av}(1342, 2341, 3142, 3241, 3412, 3421)$
- $s^{-1}(\text{Av}(132, 312, 321)) = \text{Av}(1342, 3142, 3412, 3421)$
- $s^{-1}(\text{Av}(231, 312, 321)) = \text{Av}(2341, 3241, 3412, 3421)$
- $s^{-1}(\text{Av}(312, 321)) = \text{Av}(3412, 3421)$
- $s^{-1}(\text{Av}(231, 321)) = \text{Av}(2341, 3241, 45231)$
- $s^{-1}(\text{Av}(321)) = \text{Av}(35241, 34251, 45231)$

Permutation Class Preimages

Permutation Class Preimages

Theorem (D., 2018)

We have

$$|s^{-1}(\mathrm{Av}_n(132,231,321))| = |s^{-1}(\mathrm{Av}_n(132,312,321))| = \binom{2n-2}{n-1}.$$

The number of elements of $s^{-1}(\mathrm{Av}_n(132,231,321))$ *(or* $s^{-1}(\mathrm{Av}_n(132,312,321))$*) with* m *descents is* $\binom{n-1}{m}^2$.

Permutation Class Preimages

Permutation Class Preimages

Theorem (D., 2018)

We have that $|s^{-1}(\mathrm{Av}_n(132, 231, 312))|$ *is the Fine number* F_{n+1}.

Permutation Class Preimages

Theorem (D., 2018)

We have that $|s^{-1}(\mathrm{Av}_n(132, 231, 312))|$ *is the Fine number* F_{n+1}.

We can also count the permutations in $s^{-1}(\mathrm{Av}_n(132, 231, 312))$ according to their numbers of descents or peaks, giving two reFinements of the Fine numbers.

Permutation Class Preimages

Permutation Class Preimages

Theorem (D., 2019)

We have

$$\sum_{n\geq 0}|s^{-1}(\text{Av}_n(231,321))|x^n = \frac{1}{1-xC(xC(x))},$$

where $C(x) = \dfrac{1-\sqrt{1-4x}}{2x}$. *One consequence is that*

$$|\text{Av}_n(2341,3241,45231)| = |\text{Av}_n(4321,4213)|.$$

Permutation Class Preimages

Permutation Class Preimages

Theorem (D., 2018)

We have

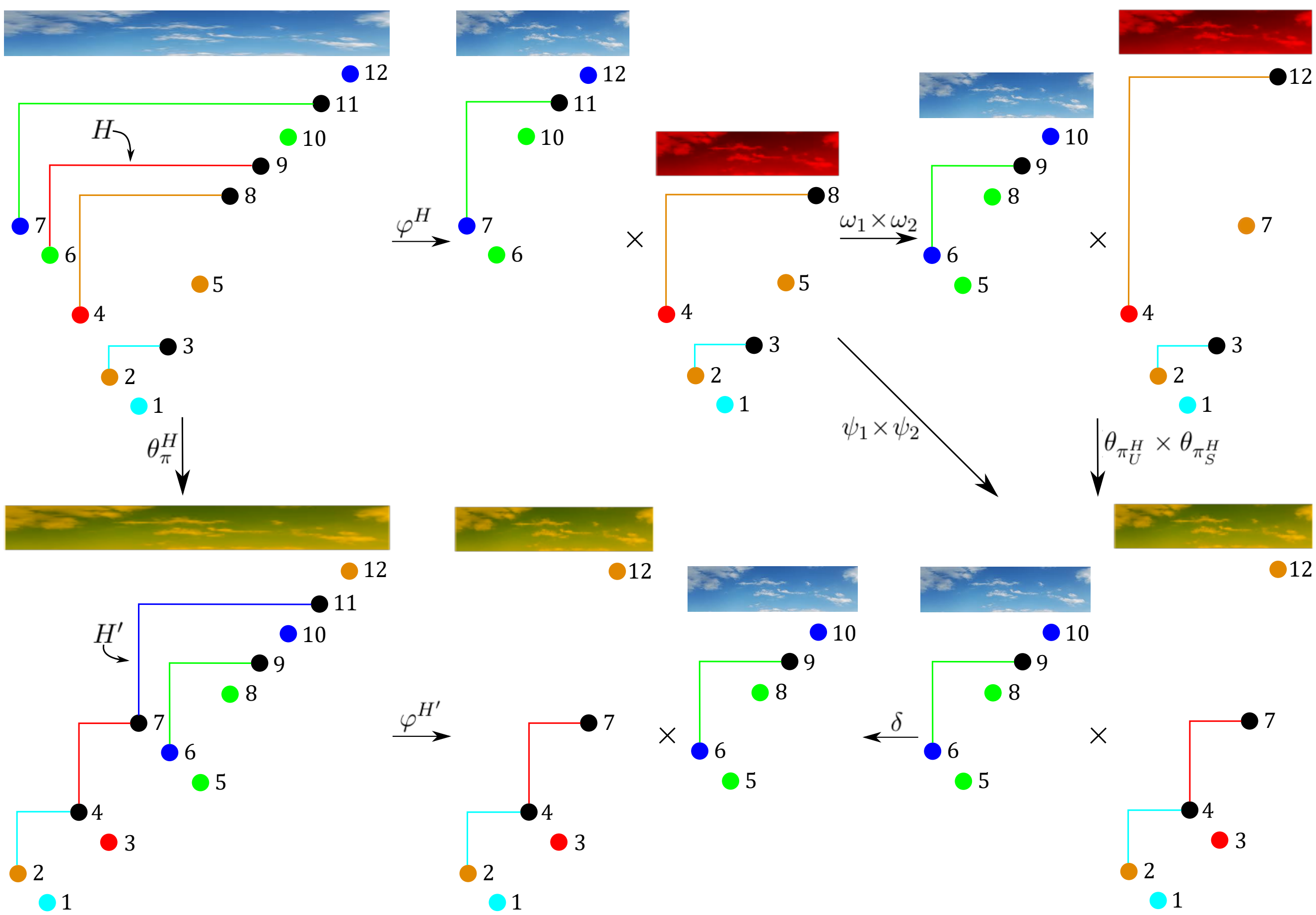
$$|s^{-1}(\mathrm{Av}_n(132,312))| = |s^{-1}(\mathrm{Av}_n(231,312))|$$

$$= |s^{-1}(\mathrm{Av}_n(132,231))|.$$

These numbers turn out to be what are called Boolean-Catalan numbers.

A Colorful Picture

A Colorful Picture



Stack-Sorting Words

Stack-Sorting Words

For us, the word “word” means a word with positive integer letters. We define stack-sorting for words as we did with permutations, except there is one point of ambiguity. Can a letter sit on top of another copy of itself in the stack?

Stack-Sorting Words

For us, the word "word" means a word with positive integer letters. We define stack-sorting for words as we did with permutations, except there is one point of ambiguity. Can a letter sit on top of another copy of itself in the stack?

Define fast : $\{\text{words}\} \to \{\text{words}\}$ by sending a word through the stack with the convention that a letter **can** sit on top of a copy of itself.

Stack-Sorting Words

For us, the word "word" means a word with positive integer letters. We define stack-sorting for words as we did with permutations, except there is one point of ambiguity. Can a letter sit on top of another copy of itself in the stack?

Define $\text{fast} : \{\text{words}\} \to \{\text{words}\}$ by sending a word through the stack with the convention that a letter **can** sit on top of a copy of itself.

Define $\text{slow} : \{\text{words}\} \to \{\text{words}\}$ by sending a word through the stack with the convention that a letter **cannot** sit on top of a copy of itself.

The Tortoise and the Hare

The Tortoise and the Hare

~~fast~~ hare   ~~slow~~ tortoise

$$3662451 \xrightarrow{\text{hare}} 3241566 \xrightarrow{\text{hare}} 2314566 \xrightarrow{\text{hare}} 2134566 \xrightarrow{\text{hare}} 1234566$$

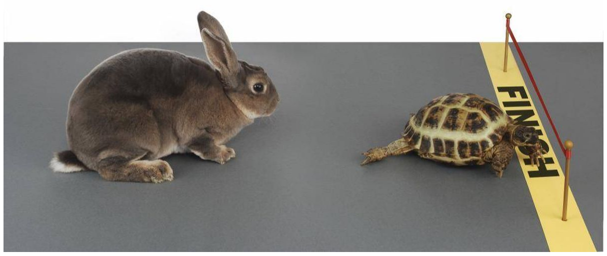
$$3662451 \xrightarrow{\text{tortoise}} 3624156 \xrightarrow{\text{tortoise}} 3214566 \xrightarrow{\text{tortoise}} 1234566$$

The Tortoise and the Hare

~~fast~~ hare ~~slow~~ tortoise

$$3662451 \xrightarrow{\text{hare}} 3241566 \xrightarrow{\text{hare}} 2314566 \xrightarrow{\text{hare}} 2134566 \xrightarrow{\text{hare}} 1234566$$

$$3662451 \xrightarrow{\text{tortoise}} 3624156 \xrightarrow{\text{tortoise}} 3214566 \xrightarrow{\text{tortoise}} 1234566$$



The Tortoise and the Hare

The Tortoise and the Hare

Let $\langle w \rangle_{\text{hare}}$ be the smallest nonnegative integer k such that $\text{hare}^k(w)$ is an identity word. Define $\langle w \rangle_{\text{tortoise}}$ similarly.

The Tortoise and the Hare

Let $\langle w \rangle_{\text{hare}}$ be the smallest nonnegative integer k such that $\text{hare}^k(w)$ is an identity word. Define $\langle w \rangle_{\text{tortoise}}$ similarly.

Theorem (D., Kravitz, 2018)

For any integer $n \geq 3$, there exists a word η_n of length $2n+1$ such that

$$\langle \eta_n \rangle_{\text{hare}} = 2n - 2 \quad \textit{and} \quad \langle \eta_n \rangle_{\text{tortoise}} = n.$$

The Tortoise and the Hare

Let $\langle w \rangle_{\text{hare}}$ be the smallest nonnegative integer k such that $\text{hare}^k(w)$ is an identity word. Define $\langle w \rangle_{\text{tortoise}}$ similarly.

Theorem (D., Kravitz, 2018)

For any integer $n \geq 3$, there exists a word η_n of length $2n+1$ such that

$$\langle \eta_n \rangle_{\text{hare}} = 2n-2 \quad \textit{and} \quad \langle \eta_n \rangle_{\text{tortoise}} = n.$$

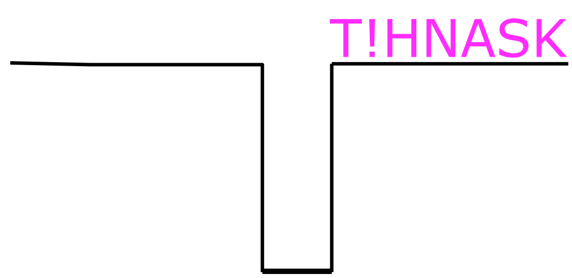
Conjecture

If w is a word of length m, then

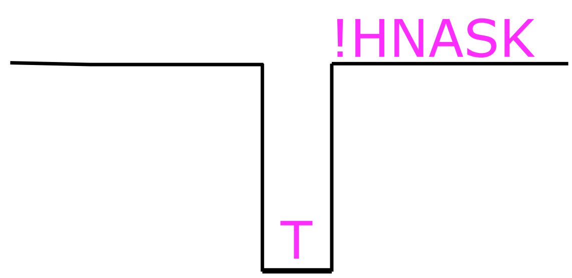
$$\langle w \rangle_{\text{hare}} - \langle w \rangle_{\text{tortoise}} \leq \frac{m-5}{2}$$

and

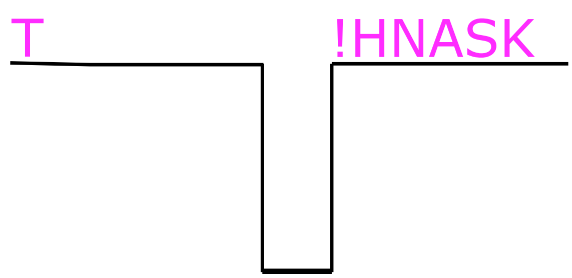
$$\langle w \rangle_{\text{hare}} \leq 2\langle w \rangle_{\text{tortoise}} - 2.$$



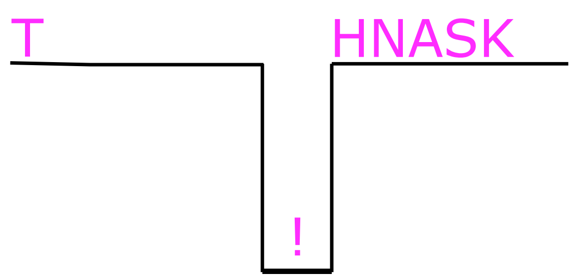
(where $\text{A} < \text{H} < \text{K} < \text{N} < \text{S} < \text{T} < \text{!}$)



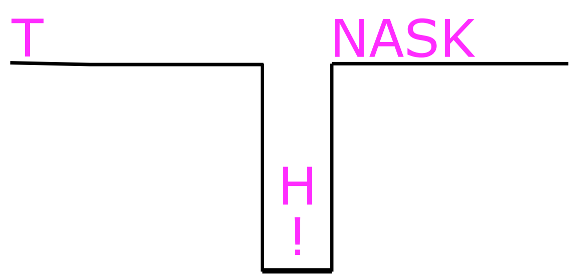
(where $\mathsf{A} < \mathsf{H} < \mathsf{K} < \mathsf{N} < \mathsf{S} < \mathsf{T} < !$)



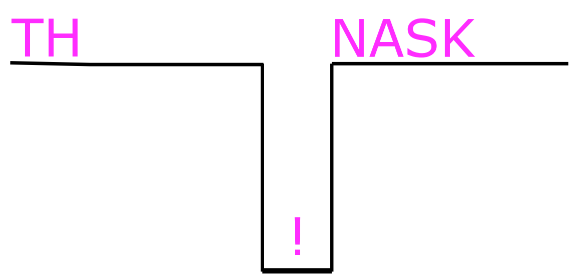
(where $\mathsf{A} < \mathsf{H} < \mathsf{K} < \mathsf{N} < \mathsf{S} < \mathsf{T} < !$)



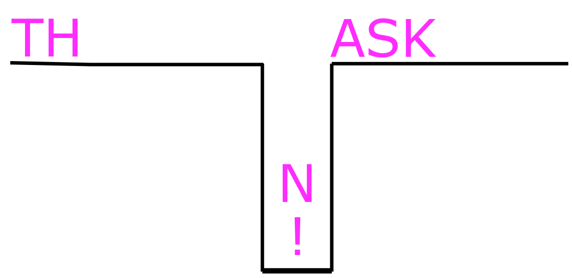
(where $\mathsf{A} < \mathsf{H} < \mathsf{K} < \mathsf{N} < \mathsf{S} < \mathsf{T} < !$)



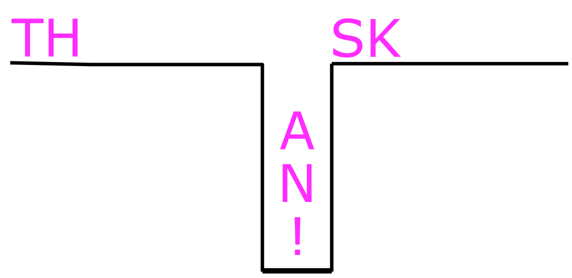
(where $\mathsf{A} < \mathsf{H} < \mathsf{K} < \mathsf{N} < \mathsf{S} < \mathsf{T} < \mathsf{!}$)



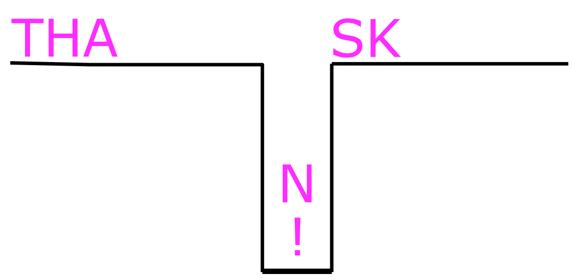
(where $\mathsf{A} < \mathsf{H} < \mathsf{K} < \mathsf{N} < \mathsf{S} < \mathsf{T} < \mathsf{!}$)



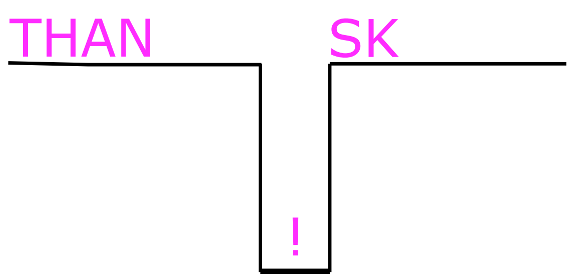
(where $\mathsf{A} < \mathsf{H} < \mathsf{K} < \mathsf{N} < \mathsf{S} < \mathsf{T} < \mathsf{!}$)



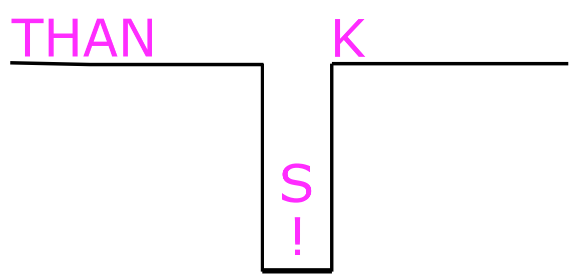
(where $A < H < K < N < S < T < !$)



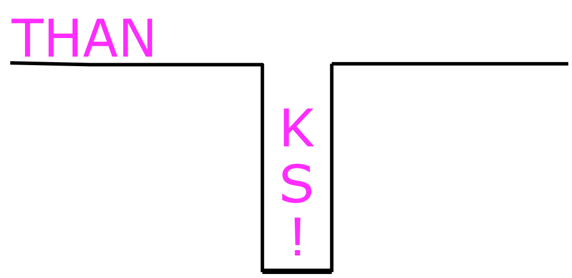
(where $A < H < K < N < S < T < !$)



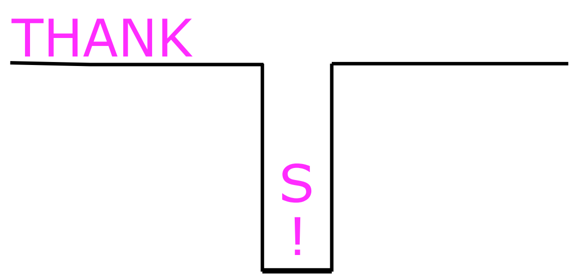
(where $A < H < K < N < S < T < !$)



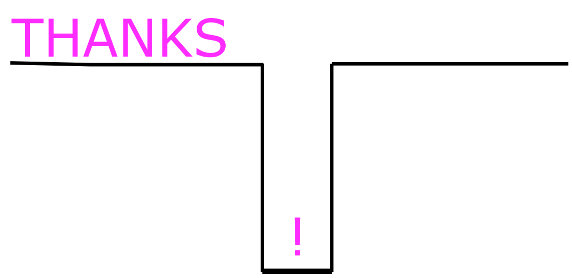
(where $A < H < K < N < S < T < !$)



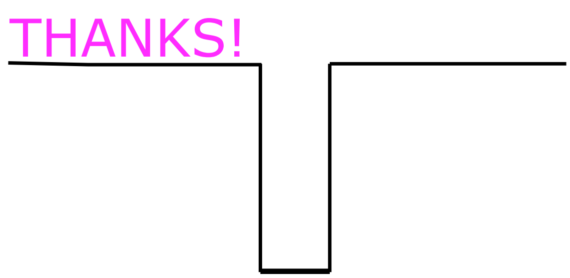
(where A $<$ H $<$ K $<$ N $<$ S $<$ T $<$!)



(where $A < H < K < N < S < T < !$)



(where $A < H < K < N < S < T < !$)



(where $A < H < K < N < S < T < !$)