When $1/\pi^2$ and Calabi-Yau meet

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To Gert Almkvist (1934-2018). In memoriam

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CONTENT

THIS TALK HAS THREE PARTS:

1. Ramanujan-like fomulas for $1/\pi^2$ (G.)
   I discovered this family of formulas in the years 2002-2003
2. Calabi-Yau operators (introduced by Almkvist and Zudilin in 2004).
3. Our joint work (Almkvist, G.)
   When $1/\pi^2$ and Calabi-Yau meet, an intriguing theory arises which is far of being understood...

PART 1

RAMANUJAN-LIKE FORMULAS FOR $1/\pi^2$.

In this part I explain a family of formulas that I discovered for $1/\pi^2$ *(apparently similar to the family of Ramanujan-type series for* $1/\pi$*).*

WZ method (proving formulas)

In the years 2002 and 2003, I proved with the Wilf–Zeilberger (WZ) method, the new formulas

$$\sum_{n=0}^{\infty} \binom{2n}{n}^5 (20n^2+8n+1)\frac{(-1)^n}{2^{12n}} = \frac{8}{\pi^2},$$

$$\sum_{n=0}^{\infty} \binom{2n}{n}^5 (820n^2+180n+13)\frac{(-1)^n}{2^{10n}} = \frac{128}{\pi^2},$$

$$\sum_{n=0}^{\infty} \binom{2n}{n}^4 \binom{4n}{2n} (120n^2+34n+3)\frac{1}{2^{16n}} = \frac{32}{\pi^2}.$$

WZ proof of the first formula:

$$\sum_{n=0}^{\infty} (-1)^n \frac{\binom{2n}{n}^5 \binom{2k}{k}^4}{\binom{n+k}{n}^4} \frac{20n^2+8n+1+4k(6n+2k+1)}{2^{12n}2^{8k}} = \frac{8}{\pi^2}.$$

PSLQ algorithm (discovering formulas)

I used the PSLQ algorithm to see if I could discover similar formulas, and in 2003 I discovered 4 new ones. One of them is

$$\sum_{n=0}^{\infty} \frac{(6n)!}{(n)!^6} (5418n^2 + 693n + 29) \frac{(-1)^n}{2880^{3n}} \stackrel{?}{=} \frac{128\sqrt{5}}{\pi^2}.$$

How I discovered the above formula without proving it?

$$Let \quad t_i = \sum_{n=0}^{\infty} \frac{(6n)!}{(n!)^6} (-1)^n \frac{n^i}{j^{3n}},$$

and use the PSLQ algorithm for finding integer relations among π^{-2}, $\sqrt{2}\,\pi^{-2}$, $\sqrt{3}\,\pi^{-2}$, $\sqrt{5}\,\pi^{-2}$, $\sqrt{6}\,\pi^{-2}$, t_0, t_1, t_2. Working with a precission of 100 digits, we find that for $j = 2880$:

$-128(\sqrt{5}\,\pi^{-2}) + 29t_0 + 693t_1 + 5418t_2 = 0$ with the precission used.

Suitable combinatorial coefficients

What do coefficients such that

$$A_n = \binom{2n}{n}^5, \; B_n = \binom{2n}{n}^4 \binom{4n}{2n}, \; C_n = \frac{(6n)!}{(n!)^6}, \; D_n = \frac{(8n)!(2n)!}{(4n)!(n!)^6}$$

have in common?

ANSWER: They satisfy similar recurrences:

$$(n+1)^5 A_{n+1} = 32(2n+1)(2n+1)(2n+1)(2n+1)(2n+1)A_n,$$
$$(n+1)^5 B_{n+1} = 32(2n+1)(2n+1)(2n+1)(4n+1)(4n+3)B_n,$$
$$(n+1)^5 C_{n+1} = 72(2n+1)(3n+1)(3n+2)(6n+1)(6n+5)C_n,$$
$$(n+1)^5 D_{n+1} = 32(2n+1)(8n+1)(8n+3)(8n+5)(8n+7)D_n.$$

Year 2009

Let $\sigma = 1$ if $z > 0$ and $\sigma = -1$ if $z < 0$, and consider the expansion

$$\sum_{n=0}^{\infty} \sigma^n A_{n+x} \left(a + b(n+x) + c(n+x)^2\right) (\sigma z)^{n+x}$$
$$= \frac{1}{\pi^2} + 0 \cdot x - \frac{k}{2!}x^2 + 0 \cdot x^3 + \frac{j}{4!}\pi^2 x^4 + \mathcal{O}(x^5).$$

I made the following conjecture:

Conjecture (G.)

1-) For the Ramanujan-like series for $1/\pi^2$ (that is when z, a, b, c are algebraic) the values of k and j are rational.

2-) If k is a rational number such that j is rational too, then z, a, b, c are algebraic (observe that k and j are not independent).

Year 2010

I proved by the WZ method the following formula

$$\sum_{n=0}^{\infty} \binom{2n}{n}^4 \binom{3n}{n} (74n^2+27n+3) \left(\frac{1}{16}\right)^{3n} = \frac{48}{\pi^2}, \qquad k \stackrel{?}{=} \frac{2}{3}.$$

Then, I simplified the system of equations, and taking $k = 8/3$, I got $j \stackrel{?}{=} 112$, and I could identify that $z_0 \stackrel{?}{=} (4\phi)^{-3}$, where ϕ denotes the fifth power of the golden ratio, and I discovered

$$\sum_{n=0}^{\infty} \binom{2n}{n}^4 \binom{3n}{n} \left[(32-\frac{216}{\phi})n^2 + (18-\frac{162}{\phi})n + (3-\frac{30}{\phi})\right] \left(\frac{1}{4\phi}\right)^{3n} \stackrel{?}{=} \frac{3}{\pi^2}.$$

It is the UNIQUE IRRATIONAL formula that I have found for $1/\pi^2$. The idea of writing it using ϕ instead of $\sqrt{5}$ was due to Zudilin.

Connections with the Calabi-Yau theory

Simplifying the system of equations I arrived to some formulas involving $T(z)$ and powers of $\log z + H(z)$, with exponents $1, 2, 3$, where $T(z)$ and $H(z)$ are holomorphic functions. Then I made the (natural) substitution

$$\log z + H(z) = \log q, \qquad q = ze^{H(z)} \;\Rightarrow\; z = z(q).$$

Gert observed the following wonderful connections with the CY theory: He proved that $z(q)$ is the mirror map of a Calabi-Yau differential equation, and that

$$\left(q\frac{d}{dq}\right)^3 T(q) = 1 - K(q),$$

where $K(q)$ is the corresponding Yukawa coupling.

Collaboration

Gert and I begun collaboration in 2010. Our project consisted in:

1. Giving explicit formulas for k, j, a, b and c using the language (functions) of the Calabi-Yau theory (that Gert knew well).
2. Making searches for a big range of rational values of k, trying to find all the convergent solutions.
3. Discovering similar series for $1/\pi^2$ of non-hypergeometric type. Gert had discovered many promising coefficients A_n.

But before talking about our joint work I will give an introduction to the Calabi-Yau theory.

PART 2

CALABI-YAU OPERATORS.

We will give an introduction to the Calabi-Yau theory from the point of view of number theory (Zudilin, Almkvist, Yang, etc).

The quintic threefolds (1)

In 1991 Candelas, De la Ossa, Green and Parkes studied the operator

$$\mathcal{D} = \theta^4 - 5z\,(5\theta+1)\,(5\theta+2)\,(5\theta+3)\,(5\theta+4)\,, \quad \theta = z\frac{d}{dz}.$$

(1) The fundamental solutions of $\mathcal{D}y = 0$ are of the following form:

$$y_i(z) = \sum_{j=0}^{i} a_j(z)\frac{\log^{i-j}(z)}{(i-j)!}, \; i = 0, 1, 2, 3, \quad y_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n.$$

(2) The expansions of

$$y_0(z), \quad q = q(z) := \exp\left(\frac{y_1}{y_0}\right) \quad \& \quad z = z(q) \quad \text{(the mirror map)}$$

in powers of z and q have integer coefficients.

The quintic threefolds (2)

(3) The instanton numbers n_d of the Yukawa coupling expanded in Lambert series:

$$K(q) := \theta_q^2 \left(\frac{y_2}{y_0} \right) = \sum_{d=1}^{\infty} n_d \frac{d^3 q^d}{1-q^d}, \qquad \theta_q = q \frac{d}{dq},$$

are integers.

SURPRISE They calculated some of the first instanton numbers:

$$n_1 = 2875, \quad n_2 = 609250, \quad n_3 = 317206375, \ldots .$$

and observed that n_d counted the number of rational curves of degree d of a generic quintic threefold (enumerative property). The explanation of why n_d count those curves was given later by the theory of the Gromov-Witten invariants.

Calabi-Yau operators (Definition)

In a very interesting joint paper of Gert Almkvist and Wadim Zudilin (2004), a definition of Calabi–Yau operator was formulated.

Definition (Almkvist–Zudilin)

CALABI-YAU OPERATORS share the properties (1), (2), (3) of the quintic and the coefficients satisfy a certain special relation, which is equivalent to

$$\frac{d^2}{dz^2}\frac{y_3}{y_0} = z\frac{d^2}{dz^2}\frac{y_2}{y_0},$$

where y_0, y_1, y_2, y_3 are the solutions of $\mathcal{D}y = 0$.

The last property of the definition causes the second order Wronskians to satisfy a fifth order (rather than six) differential equation. The fourth and fifth order operator determine each other.

Operators of order five. The pullback

We can go backwards, and determine when a certain operator of order 5 comes from a Calabi-Yau operator of order 4.

Definition (Almkvist-Zudilin)

Differential equations of order 5 having a CY pullback of order 4 are known as *Calabi-Yau differential equations of order* 5.

Some examples given by Almkvist and Zudilin are

$$\mathcal{D} = \theta^5 - 6z(6\theta+1)(6\theta+2)(6\theta+3)(6\theta+4)(6\theta+5),$$

$$\mathcal{D} = \theta^5 - 3z(2\theta+1)(3\theta^2+3\theta+1)(15\theta^2+15\theta+4) - 3z^2(\theta+1)^3(3\theta+2)(3\theta+4).$$

The holomorphic solution of the first one is

$$w_0 = \sum_{n=0}^{\infty} \frac{(6n)!}{(n!)^6} z^n.$$

PART 3

OUR JOINT WORK

Gert and I wrote three joint papers. This part is based on these two:

- Ramanujan-like series for $1/\pi^2$ and string theory, Exp. Math. **21** (2012),
- Ramanujan-Sato-like series, Number Theory & related fields, Springer Proc. in Math. and Stat., **43** (2013),

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A related and very inspirational paper was written in 2008 by Yifan Yang and Wadim Zudilin.

Functions $\alpha(q)$ and $\tau(q)$

Definition (Almkvist, G.)

Let $\mathcal{D}w = 0$, where $\mathcal{D}$ is a CY operator of order 5. We define

$$\alpha(q) = \frac{\frac{1}{6}\log^3|q| - T(q) - h\zeta(3)}{\pi^2 \ln|q|},$$

$$\tau(q) = \frac{\frac{1}{2}\log^2|q| - (\theta_q T)(q)}{\pi^2} - \alpha(q),$$

where $T(q)$ is holomorphic, $\theta_q^3 T(q) = 1 - K(q)$, and $T(0) = 0$.

Theorem (Almkvist, G.)

Let z_c be the convergence radius of the series. We have

$$k = 2(\alpha - \alpha_c), \qquad j = 3(4\tau^2 - k^2 - 8\alpha_c k - 4\tau_c^2).$$

$c(q)$, τ_c, α_c, h

Let $P(z)$ be the coefficient of θ^5 in $\mathcal{D}$, then

$$c(q) = \tau(q)\sqrt{P(z)}, \qquad b(q) = \cdots, \qquad a(q) = \cdots.$$

The convergence radius z_c of $w_0(z) = \sum_{n=0}^{\infty} A_n z^n$ (the holomorphic solution of $\mathcal{D}w = 0$) is the smallest root of $P(z)$. We observe that $k_c = j_c = 0$, and that $a_c = b_c = c_c = 0$. We have

$$\frac{1}{\pi^2} = \lim_{z \to z_c} \sum_{n=0}^{\infty} A_n(a + bn + cn^2)z^n = \lim_{z \to z_c} c(z) \sum_{n=0}^{\infty} A_n \, n^2 z^n$$
$$= \tau_c \lim_{z \to z_c} \sqrt{P(z)} \sum_{n=0}^{\infty} A_n \, n^2 z^n = \tau_c f(q_c).$$

We get τ_c, then α_c and finally h.

Our method

We wrote a Maple program wchich follows these steps:

1. Determine τ_c, α_c, and h.
2. Let k take values in a big range of rational values.
3. Use the relation $k_0 = 2(\alpha_0 - \alpha_c)$ to get α_0.
4. Calculate the corresponding value q by solving $\alpha(q) = \alpha_0$.
5. Calculate j_0 and if it looks rational, then
6. Evaluate z_0, c_0, b_0, a_0 to get the Ramanujan-Sato series
$$\sum_{n=0}^{\infty} A_n (c_0 n^2 + b_0 n + a_0) z_0^n \stackrel{?}{=} \frac{1}{\pi^2}.$$

As all the values we get are approximations, the algebraic numbers we guess remain a mistery.

Example 1 - The sextic

The periods of the family of sextic fourfolds parameterized with a complex variable z are the solutions of $\mathcal{D}w = 0$, where

$$\mathcal{D} = \theta^5 - 6z(6\theta+1)(6\theta+2)(6\theta+3)(6\theta+4)(6\theta+5),$$

The holomorphic solution w_0 is

$$w_0 = \sum_{n=0}^{\infty} A_n z^n, \quad A_n = \frac{(6n)!}{(n!)^6} = 1,\ 720,\ 7484400, \dots$$

We have $P(z) = 1 - 6^6 z, \quad z_c = 1/6^6$.

$$z(q) = q - 16344q^2 + 123097644q^3 - \cdots,$$
$$K(q) = 1 - 10080q - 90720000q^2 - \cdots.$$

We proved that $\tau_c^2 = 16/3, \quad \alpha_c = 5/2, \quad h = 70$.

Example 1 - The sextic (The solutions)

For $k = 5/3$, we get $j = 84.999999999999999$, so $j \stackrel{?}{=} 85$

$$\sum_{n=0}^{\infty} \frac{(6n)!}{(n!)^6}(1930n^2 + 549n + 45)\frac{(-1)^n}{8^{6n}} \stackrel{?}{=} \frac{384}{\pi^2}.$$

For $k = 15$, we identify $j \stackrel{?}{=} 2661$, and

$$\sum_{n=0}^{\infty} \frac{(6n)!}{(n!)^6}(5418n^2 + 693n + 29)\frac{(-1)^n}{2880^{3n}} \stackrel{?}{=} \frac{128\sqrt{5}}{\pi^2}.$$

For $k = 8/3$, we identify $j \stackrel{?}{=} 160$, and

$$\sum_{n=0}^{\infty} \frac{(6n)!}{(n!)^6}(532n^2 + 126n + 9)\frac{1}{10^{6n}} \stackrel{?}{=} \frac{375}{4\pi^2}.$$

Non-hypergeometric coefficients

There exist non-hypergeometric coefficients A_n such that the corresponding w_0 is the holomorphic solution of a Calabi-Yau differential equation of order 5. Gert discovered many of them like

$$\#60: \qquad A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k} \binom{n+k}{n} \binom{2n-k}{n}$$

$$= 1, \ 8, \ 216, \ 9440, \ 525400, \ldots .$$

$$\#189: \qquad A_n = \binom{2n}{n} \sum_{j,k} \binom{n}{j}^2 \binom{n}{k}^2 \binom{j+k}{n}^2$$

$$= 1, \ 12, \ 756, \ 78960, \ 10451700, \ldots .$$

using the Zeilberger's algorithm for finding recurrences. They are in the Big Tables (Gert Almkvist, Christian van Enckevort, Duco van Straten, and Wadim Zudilin).

Example 2

$$\textit{Let} \qquad A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k} \binom{n+k}{n} \binom{2n-k}{n}.$$

Then w_0 is the holomorphic solution of

$$\begin{aligned}\mathcal{D} =&\theta^5 - 2z(2\theta+1)(31\theta^4+62\theta^3+54\theta^2+23\theta+4) \\ &+ 12z^2(\theta+1)(3\theta+2)(3\theta+4)(4\theta+3)(4\theta+5).\end{aligned}$$

We have $P(z) = (1-16z)(1-108z), \quad z_c = 1/108,$

$$z(q) = q - 32q^2 + 356q^3 - 5528q^4 + 43410q^5 - \cdots,$$
$$K(q) = 1 - 10q - 530q^2 - 23500q^3 - 890450q^5 - \cdots$$

We identify the invariants $\tau_c^2 \stackrel{?}{=} 4/23$, $\alpha_c \stackrel{?}{=} 1/3$, $\;h \stackrel{?}{=} 50/23$.

Example 2 (The solutions)

For $k = 8/23$, we get

$$\sum_{n=0}^{\infty} A_n(40n^2 + 20n + 3) \left(\frac{1}{6}\right)^{3n} \stackrel{?}{=} \frac{69}{\pi^2},$$

For $k = 16/23$, we get

$$\sum_{n=0}^{\infty} A_n(616n^2 + 282n + 40) \left(\frac{1}{2^2 \cdot 5^3}\right)^{n} \stackrel{?}{=} \frac{25 \cdot 23}{\pi^2},$$

For $k = 43/23$, we get

$$\sum_{n=0}^{\infty} A_n(16380n^2 + 5895n + 706) \left(\frac{-1}{72}\right)^{2n} \stackrel{?}{=} \frac{2^5 \cdot 3^2 \cdot 23}{\pi^2}.$$

Example 3

Let

$$A_n = \binom{2n}{n} \sum_{j,k} \binom{n}{j}^2 \binom{n}{k}^2 \binom{j+k}{n}^2,$$

then w_0 is the holomorphic solution of $\mathcal{D}w = 0$, where

$$\mathcal{D} = \theta^5 - 2z(2\theta+1)(65\theta^4 + 130\theta^3 + 105\theta^2 + 40\theta + 6) \\ + 16z^2(\theta+1)(2\theta+1)(2\theta+3)(4\theta+3)(4\theta+5).$$

For this family we have $\alpha_c \stackrel{?}{=} \frac{1}{2}$, $\tau_c^2 \stackrel{?}{=} \frac{8}{21}$, and $h \stackrel{?}{=} \frac{30}{7}$, and the series

$$\sum_{n=0}^{\infty} A_n (680n^2 + 328n + 48) \left(\frac{1}{18}\right)^{2n} \stackrel{?}{=} \frac{3^5 \cdot 7}{\pi^2}, \qquad k \stackrel{?}{=} \frac{4}{21},$$

and three more corresponding to $k \stackrel{?}{=} \frac{8}{7}$, $k \stackrel{?}{=} \frac{19}{21}$, and $k \stackrel{?}{=} \frac{139}{21}$.

Example 4

I wanted to give an example of complex series for $1/\pi^2$, and Gert used the transformation

$$\sum_{n=0}^{\infty} A_n z^n = \frac{1}{1-z} \sum_{n=0}^{\infty} a_n \left[u \left(\frac{z}{1-z} \right)^m \right]^n, \quad A_n = \sum_{k=0}^{n} u^k \binom{n}{mk} a_k,$$

with $u = -1$ and $m = 4$ to translate the hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(3n)!(4n)!}{n!^7} (252n^2 + 63n + 5)(-1)^n \left(\frac{1}{24} \right)^{4n} \stackrel{?}{=} \frac{48}{\pi^2},$$

into the series

$$\sum_{n=0}^{\infty} A_n \left(9072n^2 + (9072 - 756i)n + (2875 - 516i) \right) \left(\frac{1}{1-24i} \right)^n$$

$$\stackrel{?}{=} \frac{27504 + 3454i}{\pi^2}, \qquad A_n = \sum_{k=0}^{n} (-1)^k \binom{n}{4k} \frac{(3k)!(4k)!}{k!^7}.$$

On the origin of the algebraicities

(1) We know that $K(q)$ (the Yukawa coupling) encapsulates important arithmetic information (instanton numbers).

(2) For some special values of q we have shown that the functions $z(q)$, $c(q)$, $b(q)$, $a(q)$ take algebraic values.

(3) We know that algebraic numbers are roots of polynomials with integer coefficients. However, the problem of figuring out the origin of the polynomials (algebraicities) corresponding to (2) is open.

(4) The conjecture staying that for the special values of q we have that $\alpha(q)$ and $\tau^2(q)$ are both rational is even more intriguing.

GERT

YEAR 2017

THANK YOU FOR YOUR ATTENTION